Lecture 3: Fluid Equations

The purpose of this lecture is to introduce the concepts of thermal pressure, thermal energy, heat flux, and entropy from the fluid equation. The average flow velocity is usually ignored in the thermodynamics. But we have to consider the energy and momentum carried by the average flow velocity in the fluid dynamics.

3.1. Continuity Equations

The continuity equation （連續方程式） describe how the number density field $n(\mathbf{x},t)$ varies with time.

$$\frac{\partial n(\mathbf{x},t)}{\partial t} = +(\text{source per unit volume}) - (\text{sink per unit volume}) - \nabla \cdot [n(\mathbf{x},t)\mathbf{V}(\mathbf{x},t)] \qquad (3.1)$$

Integrating equation (3.1) over a finite volume *Vol.*, it yields

$$\iiint \frac{\partial n(\mathbf{x},t)}{\partial t} d^3x = \frac{\partial N}{\partial t}$$
$$= +\text{source} - \text{sink} - \iiint_{Vol.} \nabla \cdot [n(\mathbf{x},t)\mathbf{V}(\mathbf{x},t)] d^3x$$
$$= +\text{source} - \text{sink} - \oiint_{S(Vol.)} [n(\mathbf{x},t)\mathbf{V}(\mathbf{x},t)] \cdot d\mathbf{a}$$

That is,

$$\frac{\partial N}{\partial t} = +\text{source} - \text{sink} - \oiint_{S(Vol.)} [n(\mathbf{x},t)\mathbf{V}(\mathbf{x},t)] \cdot d\mathbf{a} \qquad (3.2)$$

where N denotes the number of gas particles in the volume (*Vol.*). The last term in Eq. (3.2) indicates that particles moving into the volume can increase the number of particles in this volume, but particles moving out of the volume can reduce the number of particles in this volume. The physical meaning of Eq. (3.2) can be understood from the following two examples.

範例一：

桃園縣總人口數隨時間的變化情形，與那段時間裡出生的人數（source）、死亡的人數（sink）、遷入的人數、以及遷出的人數，都有關係。

範例二：

一塊體積裡，中性氣體粒子的總數隨時間的變化，就與那段時間裡由四周移入這塊體積的氣體量、移出這塊體積的氣體量、由固態、液態、或電漿態轉為氣態的數量（source）、以及由氣態轉為固態、液態、或電漿態的數量（sink）有關。

If we ignore the source term and the sink term in the continuity equations (3.1) and (3.2), and consider the continuity equation of the α th species, it yields

$$\frac{\partial n_\alpha(\mathbf{x},t)}{\partial t} = -\nabla\cdot[n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)] \tag{3.3}$$

$$\frac{\partial N_\alpha}{\partial t} = -\oiint_{S(Vol.)}[n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)]\cdot d\mathbf{a} \tag{3.4}$$

From the definitions of $n_\alpha(\mathbf{x},t)$ and $\mathbf{V}_\alpha(\mathbf{x},t)$ discussed in Lecture 2, equation (3.3) and (3.4) can be rewritten as

$$\frac{\partial}{\partial t}\iiint f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = -\nabla\cdot[\iiint \mathbf{v}f_\alpha(\mathbf{x},\mathbf{v},t)d^3v] \tag{3.5}$$

$$\frac{\partial}{\partial t}\iiint_{Vol.}[\iiint f_\alpha(\mathbf{x},\mathbf{v},t)d^3v]d^3x = -\oiint_{S(Vol.)}[\iiint \mathbf{v}f_\alpha(\mathbf{x},\mathbf{v},t)d^3v]\cdot d\mathbf{a} \tag{3.6}$$

Multiplying Eqs. (3.5) and (3.6) by the mass of the α th species m_α, it yields

$$\frac{\partial}{\partial t}\iiint m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = -\nabla\cdot[\iiint m_\alpha \mathbf{v}f_\alpha(\mathbf{x},\mathbf{v},t)d^3v] \tag{3.7}$$

$$\frac{\partial}{\partial t}\iiint_{Vol.}[m_\alpha\iiint f_\alpha(\mathbf{x},\mathbf{v},t)d^3v]d^3x = -\oiint_{S(Vol.)}[m_\alpha\iiint \mathbf{v}f_\alpha(\mathbf{x},\mathbf{v},t)d^3v]\cdot d\mathbf{a} \tag{3.8}$$

By definition, equation (3.7) yields the *mass continuity equation* of the α th species, i.e.,

$$\frac{\partial \rho_\alpha(\mathbf{x},t)}{\partial t} + \nabla\cdot[\rho_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)] = 0 \tag{3.9}$$

or

$$(\frac{\partial}{\partial t} + \mathbf{V}_\alpha\cdot\nabla)\rho_\alpha = -\rho_\alpha\nabla\cdot\mathbf{V}_\alpha \tag{3.10}$$

where

$$\rho_\alpha(\mathbf{x},t) = \iiint m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

$$\rho_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) = \iiint m_\alpha \mathbf{v}f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

Multiplying Eqs. (3.5) and (3.6) by the charge of the α th species e_α, it yields

$$\frac{\partial}{\partial t}\iiint e_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = -\nabla\cdot[\iiint e_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v] \quad (3.11)$$

$$\frac{\partial}{\partial t}\iiint_{Vol.}[e_\alpha \iiint f_\alpha(\mathbf{x},\mathbf{v},t)d^3v]d^3x = -\oiint_{S(Vol.)}[e_\alpha \iiint \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v]\cdot d\mathbf{a} \quad (3.12)$$

Summation over all species α, equation (3.11) yields the *charge continuity equation*,

$$\frac{\partial \rho_c(\mathbf{x},t)}{\partial t} + \nabla\cdot\mathbf{J}(\mathbf{x},t) = 0 \quad (3.13)$$

where the charge density $\rho_c(\mathbf{x},t)$ and current density $\mathbf{J}(\mathbf{x},t)$ are defined as

$$\rho_c(\mathbf{x},t) = \sum_\alpha \iiint e_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

$$\mathbf{J}(\mathbf{x},t) = \sum_\alpha \iiint e_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

3.2 Momentum Equation

在忽略 source term and sink term 的情況下， 純粹由移入移出的觀點，我們在方程式 (3.5) & (3.6) 中討論了如何估算“單位體積內粒子數量隨時間的改變量”，在方程式 (3.7) & (3.8) 中討論了如何估算“單位體積內質量隨時間的改變量”，在方程式 (3.11) ~ (3.13) 中討論了如何估算“單位體積內電荷量隨時間的改變量”。本節中，我們將依照同樣的道理，純粹由移入移出的觀點，去估算“單位體積內粒子總動量如何隨時間改變”，然後再把外力的效應加進去，也就是考慮單位體積內所受到的力，就可完整估算流塊的動量密度 $\rho_\alpha \mathbf{V}_\alpha$ (momentum density of the α th species) 如何隨時間改變。

Multiplying the continuity Eq. (3.5) by $m_\alpha \mathbf{v}$ and assuming that the time variations of the net momentum density is equal to the force per unit volume (i.e., the body force), it yields the momentum equation of the α th-species. Namely, the time variation of the momentum per unit volume of the α th species is

$$\frac{\partial}{\partial t}\iiint m_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = -\nabla\cdot[\iiint m_\alpha \mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v] + \begin{matrix}\text{all body forces}\\ \text{(forces per unit volume)}\end{matrix} \quad (3.14)$$

Likewise, by integrating equation (3.14) over a volume “Vol.”, we can determine the time variation of the net momentum in the given volume. That is

$$\frac{\partial}{\partial t}\iiint_{Vol.}[m_\alpha \iiint \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v]d^3x = -\oiint_{S(Vol.)}[m_\alpha \iiint \mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v]\cdot d\mathbf{a} + \text{Forces} \quad (3.14a)$$

Based on the definitions of the fluid variables given in Lecture 2, the equation (3.14) can be rewritten as

$$\frac{\partial}{\partial t}\iiint m_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = \frac{\partial}{\partial t}(\rho_\alpha \mathbf{V}_\alpha)$$
$$= -\nabla\cdot[\iiint m_\alpha \mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v] + \text{Body Forces}$$
$$= -\nabla\cdot\{\iiint m_\alpha[(\mathbf{v}-\mathbf{V}_\alpha)+\mathbf{V}_\alpha][(\mathbf{v}-\mathbf{V}_\alpha)+\mathbf{V}_\alpha] f_\alpha(\mathbf{x},\mathbf{v},t)d^3v\} + \text{Body Forces}$$
$$= -\nabla\cdot\{\iiint m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha) f_\alpha(\mathbf{x},\mathbf{v},t)d^3v + \mathbf{V}_\alpha\mathbf{V}_\alpha \iiint m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v\} + \text{Body Forces}$$
$$= -\nabla\cdot\{\mathsf{P}_\alpha + \mathbf{V}_\alpha\mathbf{V}_\alpha\rho_\alpha\} + \text{Body Forces}$$

That is

$$\frac{\partial}{\partial t}(\rho_\alpha \mathbf{V}_\alpha) = -\nabla\cdot\{\mathsf{P}_\alpha + \rho_\alpha\mathbf{V}_\alpha\mathbf{V}_\alpha\} + \text{Body Forces} \tag{3.15}$$

where $\rho_\alpha\mathbf{V}_\alpha\mathbf{V}_\alpha$ is the dynamic pressure (動壓) of the α th species. The pressure tensor P_α of the α th species consists of the thermal pressure (熱壓) ($\mathbf{1}p_\alpha$) and the stress tensor (剪壓) ($\mathbf{\Pi}_\alpha = \mathsf{P}_\alpha - \mathbf{1}p_\alpha$), where, by definition, $p_\alpha = (1/3)\text{trace}(\mathsf{P}_\alpha)$.

For isotropic pressure ($\mathbf{\Pi}_\alpha = 0$), equation (3.15) can be rewritten as

$$\frac{\partial}{\partial t}(\rho_\alpha \mathbf{V}_\alpha) + \nabla\cdot(\rho_\alpha\mathbf{V}_\alpha\mathbf{V}_\alpha) = -\nabla p_\alpha + \text{Body Forces} \tag{3.16}$$

From (3.16)–$\mathbf{V}_\alpha$(3.9), it yields

$$\rho_\alpha(\frac{\partial}{\partial t} + \mathbf{V}_\alpha\cdot\nabla)\mathbf{V}_\alpha = -\nabla p_\alpha + \text{Body Forces} \tag{3.17}$$

Equations (3.15)~(3.17) are the momentum equations of the α th species.

Exercise 3.1.

想想看「負的壓力梯度」為什麼會被視為一種“力”？（提示：請由單位體積內動量隨時間的改變量，來體會這個“力”的貢獻。）

3.3. Energy Equation

在前兩節中，我們看到，除了 source and sink terms 外（對動量而言，其 source and sink terms 就是外力）， 系統還可藉著粒子們的運動，以移入移出的方式，來改變粒子數量密度、質量密度、電荷密度、以及動量密度。本節中，我們將依照同樣的方式，先考慮粒子們藉著運動將自身的動能移入移出系統，再考慮 source and sink terms 如：外力做功以及電磁波輻射，來造成系統能量的改變。

Multiplying the continuity Eq. (3.5) by $m_\alpha v^2/2$ and assuming that the time variations of the net kinetic energy density is equal to the net energy input (such as the work done by the force per unit volume and the radiation energy flux input per unit volume), it yields the energy equation of the α th-species. Namely, the time variation of the kinetic energy per unit volume of the α th species is

$$\frac{\partial}{\partial t}\iiint \frac{m_\alpha v^2}{2} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = -\nabla\cdot[\iiint \frac{m_\alpha v^2}{2}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v] + \begin{matrix}\text{net energy input}\\ \text{per unit volume}\end{matrix} \tag{3.18}$$

where

$$\iiint \frac{m_\alpha v^2}{2} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = \iiint \frac{1}{2}m_\alpha[(\mathbf{v}-\mathbf{V}_\alpha)+\mathbf{V}_\alpha]\cdot[(\mathbf{v}-\mathbf{V}_\alpha)+\mathbf{V}_\alpha]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

$$= \iiint \frac{1}{2}m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)\cdot(\mathbf{v}-\mathbf{V}_\alpha)f_\alpha(\mathbf{x},\mathbf{v},t)d^3v + \frac{1}{2}\mathbf{V}_\alpha\cdot\iiint(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

$$+\frac{1}{2}\iiint(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v\cdot\mathbf{V}_\alpha + \frac{1}{2}\mathbf{V}_\alpha\cdot\mathbf{V}_\alpha\iiint m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

Since $\iiint(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = 0$ and by definitions

$3p_\alpha = \iiint(\mathbf{v}-\mathbf{V}_\alpha)\cdot(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha d^3v$ and $\rho_\alpha = \iiint m_\alpha f_\alpha d^3v$,

it yields

$$\iiint \frac{m_\alpha v^2}{2} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = \frac{1}{2}\rho_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha \tag{3.19}$$

Likewise,

$$\iiint \frac{m_\alpha v^2}{2}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = \iiint \frac{1}{2}m_\alpha[(\mathbf{v}-\mathbf{V}_\alpha)+\mathbf{V}_\alpha]\cdot[(\mathbf{v}-\mathbf{V}_\alpha)+\mathbf{V}_\alpha][(\mathbf{v}-\mathbf{V}_\alpha)+\mathbf{V}_\alpha]f_\alpha d^3v$$

$$= \iiint \frac{1}{2}m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)\cdot(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)f_\alpha d^3v + \frac{1}{2}\mathbf{V}_\alpha\cdot\iiint(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha d^3v$$

$$+\frac{1}{2}\iiint(\mathbf{v}-\mathbf{V}_\alpha)\cdot\mathbf{V}_\alpha(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha d^3v + \frac{1}{2}\left(\iiint(\mathbf{v}-\mathbf{V}_\alpha)\cdot(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha d^3v\right)\mathbf{V}_\alpha$$

$$+\frac{1}{2}\mathbf{V}_\alpha\cdot\mathbf{V}_\alpha\iiint(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha d^3v + \frac{1}{2}\mathbf{V}_\alpha\cdot\left(\iiint(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha d^3v\right)\mathbf{V}_\alpha$$

$$+\frac{1}{2}\left(\iiint(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha d^3v\right)\cdot\mathbf{V}_\alpha\mathbf{V}_\alpha + \frac{1}{2}\mathbf{V}_\alpha\cdot\mathbf{V}_\alpha\mathbf{V}_\alpha\iiint m_\alpha f_\alpha d^3v$$

Since $\iiint(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = 0$ and by definition

$$\mathbf{q}_\alpha(\mathbf{x},t) = \iiint \frac{1}{2}m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)\cdot(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

$$\mathsf{P}_\alpha(\mathbf{x},t) = \iiint(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

$$3p_\alpha(\mathbf{x},t) = \iiint(\mathbf{v}-\mathbf{V}_\alpha)\cdot(\mathbf{v}-\mathbf{V}_\alpha)m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

$$\rho(\mathbf{x},t) = \iiint m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

it yields

$$\iiint \frac{m_\alpha v^2}{2}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = \mathbf{q}_\alpha + \mathbf{V}_\alpha\cdot\mathsf{P}_\alpha(\mathbf{x},t) + \frac{3}{2}p_\alpha\mathbf{V}_\alpha + \frac{1}{2}\rho_\alpha V_\alpha^2\mathbf{V}_\alpha \quad (3.20)$$

Substituting equations (3.19) and (3.20) into equation (3.18), it yields the energy equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha\right) = -\nabla\cdot\left[\mathbf{q}_\alpha + \mathbf{V}_\alpha\cdot\mathsf{P}_\alpha(\mathbf{x},t) + \frac{3}{2}p_\alpha\mathbf{V}_\alpha + \frac{1}{2}\rho_\alpha V_\alpha^2\mathbf{V}_\alpha\right] + \begin{matrix}\text{net energy input}\\ \text{per unit volume}\end{matrix} \quad (3.21)$$

It can be shown that the energy equation of a system with degrees of freedom f is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_\alpha V_\alpha^2 + \frac{f}{2}p_\alpha\right) = -\nabla\cdot\left[\mathbf{q}_\alpha + \mathbf{V}_\alpha\cdot\mathsf{P}_\alpha(\mathbf{x},t) + \frac{f}{2}p_\alpha\mathbf{V}_\alpha + \frac{1}{2}\rho_\alpha V_\alpha^2\mathbf{V}_\alpha\right] + \begin{matrix}\text{net energy input}\\ \text{per unit volume}\end{matrix} \quad (3.22)$$

For isotropic distribution, the equation (3.21) can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha\right) = -\nabla\cdot\left[\frac{5}{2}p_\alpha\mathbf{V}_\alpha + \frac{1}{2}\rho_\alpha V_\alpha^2\mathbf{V}_\alpha\right] + \begin{matrix}\text{net energy input}\\ \text{per unit volume}\end{matrix} \quad (3.23)$$

For isotropic distribution, the equation (3.22) can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_\alpha V_\alpha^2 + \frac{f}{2}p_\alpha\right) = -\nabla\cdot\left[\left(1+\frac{f}{2}\right)p_\alpha\mathbf{V}_\alpha + \frac{1}{2}\rho_\alpha V_\alpha^2\mathbf{V}_\alpha\right] + \begin{matrix}\text{net energy input}\\ \text{per unit volume}\end{matrix} \quad (3.24)$$

Exercise 3.3

(a) Show that equation (3.21) can be rewritten as

$$\frac{3}{2}(\frac{\partial}{\partial t}+\mathbf{V}_\alpha\cdot\nabla)[\ln(p_\alpha\rho_\alpha^{-5/3})]=-\frac{\nabla\cdot\mathbf{q}_\alpha+(\mathbf{\Pi}_\alpha\cdot\nabla)\cdot\mathbf{V}_\alpha}{p_\alpha}+\frac{\text{external energy input per unit volume}}{p_\alpha} \quad (3.25a)$$

(b) For isotropic distribution, show that equation (3.23) can be rewritten as

$$\frac{3}{2}(\frac{\partial}{\partial t}+\mathbf{V}_\alpha\cdot\nabla)[\ln(p_\alpha\rho_\alpha^{-5/3})]=\frac{\text{external energy input per unit volume}}{p_\alpha} \quad (3.25b)$$

(c) For isotropic distribution, show that equation (3.24) can be rewritten as

$$\frac{f}{2}(\frac{\partial}{\partial t}+\mathbf{V}_\alpha\cdot\nabla)[\ln(p_\alpha\rho_\alpha^{-(f+2)/f})]=\frac{\text{external energy input per unit volume}}{p_\alpha} \quad (3.26)$$

Equation (3.25a) can be written as

$$\frac{3}{2}(\frac{\partial}{\partial t}+\mathbf{V}_\alpha\cdot\nabla)[\ln(p_\alpha\rho_\alpha^{-\gamma})]=-\frac{\nabla\cdot\mathbf{q}_\alpha+(\mathbf{\Pi}_\alpha\cdot\nabla)\cdot\mathbf{V}_\alpha}{p_\alpha}+\frac{\text{external energy input per unit volume}}{p_\alpha} \quad (3.27)$$

where $\gamma = 5/3$. Note that, in general, $\gamma = (f+2)/f$.

If the terms on right-hand side of the equation (3.27) are small and negligible, the energy equation (3.27) is reduced to the adiabatic equation of state （絕熱方程式）. That is

$$(\frac{\partial}{\partial t}+\mathbf{V}_\alpha\cdot\nabla)[\ln(p_\alpha\rho_\alpha^{\gamma})]=0$$

Namely, $p_\alpha\rho_\alpha^{-5/3}$ is constant along the trajectory of a fluid element.