Lecture 2.
Definitions of Fluid Variables in the Statistic Thermodynamics
統計熱力學中流體變數之定義
-- 流體變數，通常以場（Field）的方式呈現

Phase space density function

$f(x,y,z,v_x,v_y,v_z,t) = f(\mathbf{x},\mathbf{v},t)$.

The phase space density function is a distribution function of a gas in the phase space, where the phase space consists of the velocity space and the real space. The dimension of the phase space density is # / [L^3 $(L/T)^3$].

We can obtain the fluid variables, such as the mass density, the momentum density, the momentum-flux density, the energy density, and the energy-flux density, by integrating the phase space density multiplying the mass, the momentum, the momentum flux, the energy, and the energy flux over entire velocity space, respectively.

Fluid Fields (Variables)	
Number Density Field of the α th species $n_\alpha(\mathbf{x},t) = \iiint f_\alpha(\mathbf{x},\mathbf{v},t) d^3v$	$n_\alpha = \dfrac{N_\alpha}{Vol.}$
Mass Density Field $\rho(\mathbf{x},t) = \sum_\alpha m_\alpha n_\alpha(\mathbf{x},t) = \sum_\alpha \iiint m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t) d^3v$	
Charge Density Field $\rho_c(\mathbf{x},t) = \sum_\alpha e_\alpha n_\alpha(\mathbf{x},t) = \sum_\alpha \iiint e_\alpha f_\alpha(\mathbf{x},\mathbf{v},t) d^3v$	

第α種粒子在單位體積中的「動量」或「質量通量」

Momentum per unit volume of the α th species or **Mass Flux density (or per unit area ~~volume~~)** of the α th species $m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) = \iiint m_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v$
Total Momentum per unit volume of the system $\rho(\mathbf{x},t)\mathbf{V}(\mathbf{x},t) = \sum_\alpha m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) = \sum_\alpha \iiint m_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v$

Average Velocity (Bulk Velocity) Field of the α th species

$$\mathbf{V}_\alpha(\mathbf{x},t)=\frac{\iiint \mathbf{v}\, f_\alpha(\mathbf{x},\mathbf{v},t)d^3v}{\iiint f_\alpha(\mathbf{x},\mathbf{v},t)d^3v}$$

Center of Mass Average Velocity Field 質心速度場

$$\mathbf{V}(\mathbf{x},t)=\frac{\sum_\alpha \iiint m_\alpha \mathbf{v}\, f_\alpha(\mathbf{x},\mathbf{v},t)d^3v}{\sum_\alpha \iiint m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t)d^3v}=\frac{\sum_\alpha m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)}{\sum_\alpha m_\alpha n_\alpha(\mathbf{x},t)}$$

where $n_\alpha(\mathbf{x},t)$, $\mathbf{V}_\alpha(\mathbf{x},t)$ are the number density field and average velocity field of the α th species, respectively.

比較力學中 $\mathbf{V}(t)=\frac{\sum_{k=1}^{N} m_k \mathbf{v}_k(t)}{\sum_{k=1}^{N} m_k}$

The $\mathbf{v}_k(t)$ is the velocity of the k th particle. The $\mathbf{V}(t)$ is the average of the N particles

Thermal Pressure Tensor of the α th species

$$\mathbf{P}_\alpha(\mathbf{x},t)=\iiint m_\alpha[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)][\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

which is also the **momentum flux density (or per unit area ~~volume~~)** of the α th species **observed in its average velocity moving frame**.

From its definition, the **Thermal Pressure Tensor** of the α th species should be a **symmetric tensor**. For symmetric tensor, **the trace of the tensor** (矩陣的對角線和) is invariant after coordinate transformation.

第α種粒子的分壓

Thermal Pressure of the α th species

$$p_\alpha(\mathbf{x},t)=\frac{1}{3}\iiint m_\alpha[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]\cdot[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

or $p_\alpha=\frac{1}{3}\mathrm{trace}(\mathbf{P}_\alpha)$

比較力學中： $p_\alpha=\frac{1}{3}\sum_{k=1}^{N} m_\alpha(\mathbf{v}_k-\mathbf{V}_\alpha)\cdot(\mathbf{v}_k-\mathbf{V}_\alpha)$

For ideal gas, the **internal energy** per unit volume of the α th species (u_α) is the **kinetic energy** per unit volume of the α th species **observed in its average-velocity moving frame**, which is also called the **thermal energy**.

$$u_\alpha(\mathbf{x},t) = \iiint \frac{1}{2} m_\alpha [\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)] \cdot [\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t) d^3v$$

Thus, we have

$$u_\alpha(\mathbf{x},t) = \frac{1}{2}\text{trace}[\mathsf{P}_\alpha(\mathbf{x},t)] = \frac{3}{2} p_\alpha(\mathbf{x},t)$$

第α種粒子在單位體積中的「總動量通量」亦即
「總壓張量」＝「動壓張量」＋「熱壓張量」

Total Momentum Flux density (or per unit area ~~volume~~) of the α th species
= Dynamic pressure tensor + Thermal pressure tensor

$$\iiint m_\alpha \mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v = m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) + \mathsf{P}_\alpha(\mathbf{x},t)$$
$$= m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) + \mathbf{1} p_\alpha(\mathbf{x},t) + \Pi_\alpha(\mathbf{x},t)$$

Stress Tensor of the α th species （剪壓）

$$\Pi_\alpha = \mathsf{P}_\alpha - \mathbf{1} p_\alpha = \mathsf{P}_\alpha - \mathbf{1}(1/3)\text{trace}(\mathsf{P}_\alpha)$$

第α種粒子在單位體積中的「總動能」

Kinetic Energy per unit volume of the α th species
= mean flow kinetic energy per unit volume + thermal energy (internal energy) per unit volume

$$\iiint m_\alpha \frac{\mathbf{v}\cdot\mathbf{v}}{2} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v = \frac{1}{2} m_\alpha n_\alpha(\mathbf{x},t) V_\alpha^2(\mathbf{x},t) + \frac{3}{2} p_\alpha(\mathbf{x},t)$$
$$= \frac{1}{2} m_\alpha n_\alpha(\mathbf{x},t) V_\alpha^2(\mathbf{x},t) + \frac{D}{2} p_\alpha(\mathbf{x},t)$$
$$= \frac{1}{2} m_\alpha n_\alpha(\mathbf{x},t) V_\alpha^2(\mathbf{x},t) + u_\alpha(\mathbf{x},t)$$

Where D is the degrees of the freedom

第α種粒子在單位體積中的「動能通量」

Kinetic Energy Flux density (or per unit area ~~volume~~) of the α th species

$$\iiint m_\alpha \frac{\mathbf{v}\cdot\mathbf{v}}{2}\mathbf{v}\, f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

$$=[\frac{1}{2}m_\alpha n_\alpha(\mathbf{x},t)V_\alpha^2(\mathbf{x},t)+(1+\frac{3}{2})p_\alpha(\mathbf{x},t)]\mathbf{V}_\alpha(\mathbf{x},t)$$

$$+\mathbf{\Pi}_\alpha(\mathbf{x},t)\cdot\mathbf{V}_\alpha(\mathbf{x},t)+\mathbf{q}_\alpha(\mathbf{x},t)$$

兩種表示法

$$=[\frac{1}{2}m_\alpha n_\alpha(\mathbf{x},t)V_\alpha^2(\mathbf{x},t)+\frac{D+2}{D}u_\alpha(\mathbf{x},t)]\mathbf{V}_\alpha(\mathbf{x},t)$$

$$+\mathbf{\Pi}_\alpha(\mathbf{x},t)\cdot\mathbf{V}_\alpha(\mathbf{x},t)+\mathbf{q}_\alpha(\mathbf{x},t)$$

第α種粒子單位體積中的熱流量（一種向量）

Heat Flux density (or per unit area ~~volume~~) of the α th species

$$\mathbf{q}_\alpha(\mathbf{x},t)=\iiint\frac{1}{2}m_\alpha[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)][\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]\cdot[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$$

第α種粒子單位體積中的熵值（一種純量）

Entropy per unit volume of the α th species

$$s_\alpha(\mathbf{x},t)=-k_B\iiint[\ln f_\alpha(\mathbf{x},\mathbf{v},t)]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v\ (=\frac{3}{2}nk_B\ln(\frac{p_\alpha}{n_\alpha^{5/3}})+s_0)$$

(if $f_\alpha(\mathbf{x},\mathbf{v},t)$ is a normal distribution function)

傳統熱力學考慮發生在一個密閉容器中氣體的熱力過程。

容器體積為 Volume (假設容器內氣體密度、溫度均勻分布)

Number of particles in the given "Volume" is

$$N_\alpha=\iiint_{Volume} n_\alpha(\mathbf{x},t)d^3x \qquad (N_\alpha=n_\alpha Volume)$$

Internal Energy of the α th species in the given "Volume" is

$$U_\alpha=\iiint_{Volume} u_\alpha(\mathbf{x},t)d^3x \qquad (U_\alpha=\frac{3}{2}p_\alpha Volume=\frac{3}{2}N_\alpha k_B T_\alpha)$$

密閉容器中 N_α 為定值，U_α 只是溫度的函數。

Heat Flow of the α th species is the net heat flux entering the given "Volume"

第α種粒子的熱流就是流入體積 Volume 中的總熱流量

$$\delta Q_\alpha=-\oint\!\!\oint_{S(Volume)}\mathbf{q}_\alpha(\mathbf{x},t)\cdot d\mathbf{a}=-\iiint_{Volume}\nabla\cdot\mathbf{q}_\alpha(\mathbf{x},t)d^3x$$

Entropy of the α th species in the given "Volume" is

$$S_\alpha=\iiint_{Volume} s_\alpha(\mathbf{x},t)d^3x \qquad (S_\alpha\approx\frac{3}{2}N_\alpha k_B\ln[p_\alpha(Volume)^{5/3}]+S_0)$$