Lecture 2. Definitions of Fluid Variables in Statistic Thermal Dynamics

寫在前面

本講中所用的符號與物理量說明：

方程式中我們用斜體的 Times New Roman 字型或希臘字型，表示純量。

方程式中我們用粗體的 Times New Roman 字型或粗體的希臘字型，表示向量。

方程式中我們用粗體且上下一般粗細的字型，表示二階以上的張量。

純量 (scalar, a zeroth rank tensor) 是一個只具有大小但不具方向的物理量。

向量 (vector, a first rank tensor) 是一個具有大小有一個方向的物理量。

二階張量 (second rank tensor) 則是一個具有大小與“兩個”方向的物理量。

常見的二階張量：

- 壓力張量 (pressure tensor, P) 是單位面積上所受的力。
  壓力張量的兩個方向就是由受力面的方向與力的方向來決定。
- 轉動慣量張量 (inertial tensor, I) 決定外加力矩能造成什麼樣的角加速度 ($\boldsymbol{\tau} = \mathsf{I} \cdot \boldsymbol{\alpha}$) 。對於一個非球對稱的物體，其轉動慣量張量不會是一個均向性的張量，因此外加力矩 $\boldsymbol{\tau}$ 的方向與角加速度 $\boldsymbol{\alpha}$ 的方向就不一定相同。
  （對於一個非球對稱的物體，只有當外加力矩 $\boldsymbol{\tau}$ 的方向沿某些特定方向時，其角加速度 $\boldsymbol{\alpha}$ 的方向才會平行於外加力矩 $\boldsymbol{\tau}$ 的方向。請問該如何決定這些特定方向？）
- 因為物理上，我們把 I (i 的大寫) 保留給轉動慣量了。所以物理上，我們用 $\mathsf{1}$ (數目字 123 的 1) 來表示單位張量 (unit tensor) 。三度空間中的二階“單位張量”定義為 $\mathsf{1} = \mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z$ ，其中 $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ 分別表沿 x, y, z 方向的單位向量 (unit vector) 。

手寫時，向量的寫法為 ($\vec{V}$) ($\vec{V}$)，二階張量的寫法為 ($\bar{\bar{P}}$) ($\vec{\vec{P}}$) ，單位向量的寫法為 ($\hat{x}$) ($\hat{x}$)。如果各位看到第一個括號中出現亂碼，不要覺得奇怪，因為這就是為什麼在印刷體時，我們不用這些手寫符號表示向量與張量，因為這些字型隨電腦系統不同而有不同的定義！（若用蘋果電腦看這些符號，一定沒問題！）各位在手寫筆記與考試作答時，請依照第二個括號中手寫的方式，標示這些物理量。其他的手寫向量標示法包括了 $\underset{\sim}{V}$ 或 $\underline{V}$ 。其他的手寫二階張量標示法包括了 $\underset{\approx}{P}$ 或 $\underline{\underline{P}}$ 。

To the students who are major in space physics:
The space plasma is an ionized gas (or a partially ionized gas) with very high temperature. We cannot use the thermometer to measure the plasma temperature in space. So we need to know the general "definition of temperature" in order to find a way to measure the temperature of the plasma in space. We can use Langmuir probe to directly measure the electron temperature in the ionosphere by assuming that the electrons are in normal distribution in the velocity space. We can also use radar to measure the electron temperature and/or electron phase space distribution "indirectly." Indeed the radar observations of the sporadic-E (irregular plasma density distribution in the E-region ionosphere) indicate counter-streaming hot electrons in the sporadic-E regions.

In general, a distribution function of a gas is a function of position, velocity, and time. That is $f = f(x,y,z,v_x,v_y,v_z,t) = f(\mathbf{x},\mathbf{v},t)$. It is also called the phase space density, where the phase space consists of the velocity space and the real space. In the statistic thermal dynamics, we can define the fluid variables the number density, the mass density, the momentum density, the momentum-flux density, the energy density, and the energy-flux density by integrating the variables and the corresponding distribution functions over entire velocity space.

2.1. Number Density, Mass Density, and Charge Density

The number density in the real space can be obtained by integrating the phase space density over entire velocity space

$$n(\mathbf{x},t) = \iiint f(\mathbf{x},\mathbf{v},t)d^3v$$

Likewise, we can obtained the mass density from the following integration,

$$\rho(\mathbf{x},t) = mn(\mathbf{x},t) = \iiint mf(\mathbf{x},\mathbf{v},t)d^3v$$

Now, if there are more than one species in the gas or ionized gas, we can define the number density of the α th species to be

$$n_\alpha(\mathbf{x},t) = \iiint f_\alpha(\mathbf{x},\mathbf{v},t)d^3v \qquad (2.1)$$

where $f_\alpha(\mathbf{x},\mathbf{v},t)$ is the phase space density of the α th species. The average mass density of the multiple-species system is

$$\rho(\mathbf{x},t) = \sum_\alpha m_\alpha n_\alpha(\mathbf{x},t) = \sum_\alpha \iiint m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{2.2}$$

Likewise, the charge density of the multiple-species system is

$$\rho_c(\mathbf{x},t) = \sum_\alpha e_\alpha n_\alpha(\mathbf{x},t) = \sum_\alpha \iiint e_\alpha f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{2.3}$$

2.2. Momentum Density and the Average Velocity

The momentum density of the α th species can be obtained by integrating the momentum ($m_\alpha \mathbf{v}$) in the phase space over entire velocity space. We define

$$\iiint m_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v = m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) \tag{2.4}$$

where $\mathbf{V}_\alpha(\mathbf{x},t)$ is the average velocity of the α th species. The definition of $\mathbf{V}_\alpha(\mathbf{x},t)$ can be obtained from equation (2.4). That is

$$\mathbf{V}_\alpha(\mathbf{x},t) = \frac{1}{n_\alpha(\mathbf{x},t)} \iiint \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{2.5}$$

The total momentum density of the multiple-species system is

$$\rho(\mathbf{x},t)\mathbf{V}(\mathbf{x},t) = \sum_\alpha m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) = \sum_\alpha \iiint m_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{2.6}$$

where $\mathbf{V}(\mathbf{x},t)$ is the center-of-mass average velocity (or bulk velocity) of the multiple-species system. The definition of $\mathbf{V}(\mathbf{x},t)$ can be obtained from equation (2.6). That is

$$\mathbf{V}(\mathbf{x},t) = \frac{1}{\rho(\mathbf{x},t)} \sum_\alpha m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) = \frac{1}{\sum_\alpha m_\alpha n_\alpha(\mathbf{x},t)} \sum_\alpha m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) \tag{2.7}$$

or

$$\mathbf{V}(\mathbf{x},t) = \frac{\sum_\alpha \iiint m_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v}{\sum_\alpha \iiint m_\alpha f_\alpha(\mathbf{x},\mathbf{v},t) d^3v} \tag{2.7a}$$

Note that, based on the definitions given in equations (2.4) and (2.6), the "momentum density" can be considered as the "mass-flux density". Similarly, we will show in the next section that the "pressure density" is equal to the "momentum-flux density".

2.3. Thermal Pressure Tensor, Scalar Thermal Pressure, and Stress Tensor

The momentum-flux density of the α th species can be obtained by integrating the net momentum flux ($m_\alpha \mathbf{vv}$) in the phase space over entire velocity space.

Exercise 2.1. Show that

$$\begin{aligned}&\iiint m_\alpha \mathbf{vv} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v \\ &= m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) + \iiint m_\alpha[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)][\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v\end{aligned} \tag{2.8}$$

Exercise 2.2. Show that the "momentum-flux density" is equivalent to a "pressure tensor".

The first term on the right-hand side of equation (2.8) is called the dynamic pressure, which is a moving-frame dependent pressure tensor. The second term on the right-hand side of equation (2.8) is the thermal pressure tensor, which is a moving-frame independent variable. Since the second-rank thermal pressure tensor of the α th species is defined by

$$\mathsf{P}_\alpha(\mathbf{x},t) = \iiint m_\alpha[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)][\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v \tag{2.9}$$

equation (2.8) can be rewritten as

$$\iiint m_\alpha \mathbf{vv} f_\alpha(\mathbf{x},\mathbf{v},t)d^3v = m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) + \mathsf{P}_\alpha(\mathbf{x},t) \tag{2.10}$$

Based on the definition given in equation (2.9), the pressure tensor is a symmetric tensor. For symmetric matrix, the trace of the matrix is one of the three invariants of the matrix after an orthogonal transformation (e.g., a coordinate transformation such that after the transformation the lengths of vectors and the angles between vectors are preserved). Therefore, we can always define a scalar pressure of the α th species to be

$$p_\alpha = \frac{1}{3}\mathrm{trace}(\mathsf{P}_\alpha) = \frac{1}{3}\iiint m_\alpha[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]\cdot[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v \tag{2.11}$$

For ideal gas, we have $p(Volume) = Nk_BT$ and by definition the number density is $n = N / Volume$. Thus, the temperature of the α th species is

$$T_\alpha(\mathbf{x},t) = \frac{1}{3}\frac{1}{n_\alpha(\mathbf{x},t)k_B}\iiint m_\alpha[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]\cdot[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v \tag{2.12}$$

Note that, for isotropic pressure, the pressure tensor is reduced to $\mathsf{P}_\alpha = \mathbf{1}p_\alpha$, where $\mathbf{1}$ is the unit tensor. For anisotropic pressure tensor, we can define a stress tensor $\mathbf{\Pi}_\alpha$, such that

$$\mathbf{\Pi}_\alpha = \mathbf{P}_\alpha - \mathbf{1} p_\alpha \tag{2.13}$$

The net momentum flux (or total pressure) of a multiple-species system is

$$\sum_\alpha \iiint m_\alpha \mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v = \sum_\alpha [m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) + \mathbf{P}_\alpha(\mathbf{x},t)] \tag{2.14}$$

We can define a pressure tensor $\mathbf{P}(\mathbf{x},t)$ in the center-of-mass moving frame, such that

$$\sum_\alpha \iiint m_\alpha \mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v = \rho(\mathbf{x},t)\mathbf{V}(\mathbf{x},t)\mathbf{V}(\mathbf{x},t) + \mathbf{P}(\mathbf{x},t) \tag{2.15}$$

Equations (2.14) and (2.15) yields

$$\mathbf{P}(\mathbf{x},t) = \sum_\alpha [m_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t) + \mathbf{P}_\alpha(\mathbf{x},t)] - \rho(\mathbf{x},t)\mathbf{V}(\mathbf{x},t)\mathbf{V}(\mathbf{x},t) \tag{2.16}$$

Thus, unless all the species moves at the same speed, i.e., $\mathbf{V}(\mathbf{x},t) = \mathbf{V}_\alpha(\mathbf{x},t)$ for all the α th species, which yields $\mathbf{P}(\mathbf{x},t) = \sum_\alpha \mathbf{P}_\alpha(\mathbf{x},t)$, otherwise, in general, $\mathbf{P}(\mathbf{x},t) \neq \sum_\alpha \mathbf{P}_\alpha(\mathbf{x},t)$.

2.4. Kinetic Energy Density and Thermal Energy Density (Internal Energy)

The kinetic energy density of the α th species can be obtained by integrating the kinetic energy ($m_\alpha \mathbf{v}\cdot\mathbf{v}/2$) in the phase space over entire velocity space. Making use of equations (2.10) and (2.11), the kinetic energy density of the α th species can be written as

$$\iiint m_\alpha \frac{\mathbf{v}\cdot\mathbf{v}}{2} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v = \frac{1}{2} m_\alpha n_\alpha(\mathbf{x},t) V_\alpha^2(\mathbf{x},t) + \frac{3}{2} p_\alpha(\mathbf{x},t) \tag{2.17}$$

The first term on the right-hand side of equation (2.17) is the kinetic energy density due to bulk speed of the fluid element, which is a moving-frame dependent kinetic energy density. The second term on the right-hand side of equation (2.17) is the thermal energy density, which is a moving-frame independent variable.

Integrating the thermal energy density over a volume Vol. we can obtain the internal energy U_α of the α th species in the given Vol., that is

$$U_\alpha = \frac{3}{2} p_\alpha Vol. = \frac{3}{2} N_\alpha k_B T_\alpha \tag{2.18}$$

2.5. Kinetic Energy Flux Density and Heat Flux Density

The kinetic energy-flux density of the α th species can be obtained by integrating the kinetic energy flux ($m_\alpha \mathbf{v}\cdot\mathbf{v}\mathbf{v}/2$) in the phase space over entire velocity space.

Exercise 2.3. Show that the kinetic energy-flux density of the α th species can be written as

$$\begin{aligned}&\iiint m_\alpha \frac{\mathbf{v}\cdot\mathbf{v}}{2}\mathbf{v}\, f_\alpha(\mathbf{x},\mathbf{v},t)d^3v \\ &= [\frac{1}{2}m_\alpha n_\alpha(\mathbf{x},t)V_\alpha^2(\mathbf{x},t)+\frac{3}{2}p_\alpha(\mathbf{x},t)]\mathbf{V}_\alpha(\mathbf{x},t)+\mathbf{P}_\alpha(\mathbf{x},t)\cdot\mathbf{V}_\alpha(\mathbf{x},t)+\mathbf{q}_\alpha(\mathbf{x},t)\end{aligned} \tag{2.19}$$

where the heat-flux density (a vector) is defined by

$$\mathbf{q}_\alpha(\mathbf{x},t)=\iiint\frac{1}{2}m_\alpha[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)][\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]\cdot[\mathbf{v}-\mathbf{V}_\alpha(\mathbf{x},t)]f_\alpha(\mathbf{x},\mathbf{v},t)d^3v \tag{2.20}$$

Substituting equation (2.13) into equation (2.19), it yields

$$\begin{aligned}&\iiint m_\alpha \frac{\mathbf{v}\cdot\mathbf{v}}{2}\mathbf{v}\, f_\alpha(\mathbf{x},\mathbf{v},t)d^3v \\ &= [\frac{1}{2}m_\alpha n_\alpha(\mathbf{x},t)V_\alpha^2(\mathbf{x},t)+\frac{5}{2}p_\alpha(\mathbf{x},t)]\mathbf{V}_\alpha(\mathbf{x},t)+\mathbf{\Pi}_\alpha(\mathbf{x},t)\cdot\mathbf{V}_\alpha(\mathbf{x},t)+\mathbf{q}_\alpha(\mathbf{x},t)\end{aligned} \tag{2.21}$$

The first three terms on the right-hand side of equation (2.21) are moving-frame dependent kinetic energy-flux density. The last term on the right-hand side of equation (2.21) is a moving-frame independent heat-flux density.

2.6. Fluid Variables of a Uniform Gas With Isotropic Normal Distribution in the Velocity Space

Let us consider an equilibrium ($\partial/\partial t = 0$) and uniform ($\partial/\partial x = \partial/\partial y = \partial/\partial z = 0$) gas with a normal distribution in the velocity space. The distribution function can be written as

$$f(v_x, v_y, v_z) = \frac{n}{(\sqrt{2\pi}\sigma)^3}\exp\left(-\frac{(v_x - V_x)^2 + (v_y - V_y)^2 + (v_z - V_z)^2}{2\sigma^2}\right) \tag{2.22}$$

where

$n = \iiint f(\mathbf{v})d^3v$ is the number density,

$\mathbf{V} = \frac{1}{n}\iiint \mathbf{v}\, f(\mathbf{v})d^3v$ is the average velocity, and $\mathbf{V} = \mathbf{e}_x V_x + \mathbf{e}_y V_y + \mathbf{e}_z V_z$,

$$\sigma^2 = k_B T / m = \frac{1}{3n}\iiint (\mathbf{v} - \mathbf{V})\cdot(\mathbf{v} - \mathbf{V}) f(\mathbf{v})d^3v$$
$$= \frac{1}{n}\iiint (v_x - V_x)^2 f(\mathbf{v})d^3v = \frac{1}{n}\iiint (v_y - V_y)^2 f(\mathbf{v})d^3v = \frac{1}{n}\iiint (v_z - V_z)^2 f(\mathbf{v})d^3v$$

is the variance of the distribution function in each velocity component. The standard deviation σ can be considered as the thermal speed of the gas. It can be shown that the thermal pressure tensor is

$$\mathbf{P} = \mathbf{1}p = \mathbf{1}nk_B T = \iiint m(\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V}) f(\mathbf{v})d^3v$$

Obviously, both the stress tensor $\mathbf{\Pi}$ and the heat-flux density $\mathbf{q}$ will vanish for an isotropic uniform gas (i.e., $\mathbf{\Pi} = 0$ and $\mathbf{q} = 0$ for an isotropic uniform gas).