Thermal Pressure Tensor
Scalar Thermal Pressure
Stress Tensor

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Pressure Tensor

$$\mathbf{P} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})f(\mathbf{v})d^3v$$

$$\begin{aligned} = \quad & \hat{x}\hat{x}\,P_{xx} + \hat{x}\hat{y}\,P_{xy} + \hat{x}\hat{z}\,P_{xz} \\ & + \hat{y}\hat{x}\,P_{yx} + \hat{y}\hat{y}\,P_{yy} + \hat{y}\hat{z}\,P_{yz} \\ & + \hat{z}\hat{x}\,P_{zx} + \hat{z}\hat{x}\,P_{zy} + \hat{z}\hat{z}\,P_{zz} \end{aligned}$$

where

$$P_{xx} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(v_x - V_x)^2 f(\mathbf{v})d^3v$$

$$P_{xy} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(v_x - V_x)(v_y - V_y)f(\mathbf{v})d^3v$$

and so on.

It can be easily shown that $P_{xy} = P_{yx}$, and so on.

So, $\mathbf{P} = \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix}$ is a symmetric matrix.

Pressure Tensor vs. Scalar Pressure

Since the thermal pressure tensor

$$\mathbf{P} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})f(\mathbf{v})d^3v$$

or

$$\mathbf{P} = \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix} \text{ is a symmetric matrix,}$$

we can define a scalar thermal pressure to be

$$p = \frac{1}{3}\mathrm{trace}(\mathbf{P}) = \frac{1}{3}(P_{xx} + P_{yy} + P_{zz})$$

Stress Tensor

Let $\mathbf{1} = (\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z})$ be the unit tensor.

The stress tensor is defined by

$\mathbf{\Pi} = \mathbf{P} - \mathbf{1}\,p$

$$
\begin{aligned}
= \quad & \hat{x}\hat{x}\left\{ P_{xx} - \frac{P_{xx} + P_{yy} + P_{zz}}{3} \right\} + \hat{x}\hat{y}\,P_{xy} + \hat{x}\hat{z}\,P_{xz} \\
& + \hat{y}\hat{x}\,P_{yx} + \hat{y}\hat{y}\left\{ P_{yy} - \frac{P_{xx} + P_{yy} + P_{zz}}{3} \right\} + \hat{y}\hat{z}\,P_{yz} \\
& + \hat{z}\hat{x}\,P_{zx} + \hat{z}\hat{y}\,P_{zy} + \hat{z}\hat{z}\left\{ P_{zz} - \frac{P_{xx} + P_{yy} + P_{zz}}{3} \right\}
\end{aligned}
$$

The corresponding matrix of the stress tensor is a traceless matrix.

What is Pressure?

力：單位時間動量的改變量
壓力：單位面積上所受的力
壓力：單位時間單位面積上動量的改變量
壓力：動量通量密度 mn**vv**

- 氣體壓力：單位時間單位面積上動量的改變量。因為

$$O(p) = \frac{O(mv)}{O(l^2)O(t)} = \frac{O(mv)O(l)}{O(l^3)O(t)} = \frac{O(mv^2)}{O(l^3)} = O(nmv^2)$$

- 故氣體壓力也可定義為：動量通量密度 n(m**v**)**v** 。例如：熱壓

$$\mathbf{P} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})f(\mathbf{v})d^3v$$

- 氣體總壓力(總動量通量密度)包括了氣體的動壓(dynamic pressure $nm\mathbf{VV}$)與熱壓(thermal pressure **P**)

$$\mathbf{P}_{total} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m\mathbf{vv}f(\mathbf{v})d^3v = nm\mathbf{VV} + \mathbf{P}$$

What is the Temperature of an Ideal Gas?

- 定義純量氣體壓力：

$$p = \frac{1}{3}\mathrm{trace}(\mathbf{P}) == \frac{1}{3}(P_{xx} + P_{yy} + P_{zz})$$

$$= \frac{1}{3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(\mathbf{v}-\mathbf{V})\cdot(\mathbf{v}-\mathbf{V})f(\mathbf{v})d^3v$$

因為“壓力張量”為一個“對稱矩陣”，因此其“對角線和”，是一個不會隨座標系的選取而改變的“不變量”。

- 因為理想氣體定律 $p = nk_BT$ 故

$$\frac{k_BT}{m} = \frac{p}{nm} = \frac{1}{3}\frac{1}{n}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\mathbf{v}-\mathbf{V})\cdot(\mathbf{v}-\mathbf{V})f(\mathbf{v})d^3v$$

- 所以理想氣體的溫度正比於其速度空間分布函數的變異數。