Mean, Variance, Standard Deviation of the Normal Distribution Random Numbers
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This Lecture consists of three Parts.

Part I. Normal Distribution Function

Normal Distribution Function

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp[\frac{-(x-\mu)^2}{2\sigma^2}]$$

can be served as a probability density function, because

$$\int_{-\infty}^{+\infty} f(x;\mu,\sigma)\,dx = 1 .$$

Proof (證明)：
Let

$$I = \int_{-\infty}^{+\infty} f(x;\mu,\sigma)\,dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx$$

Changing the integration variable *x* to *y*, the integration result will be the same. Thus, we have

$$I = \int_{-\infty}^{+\infty} f(y;\mu,\sigma)\,dy = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\exp[\frac{-(y-\mu)^2}{2\sigma^2}]dy$$

Multiplying the above two integrations, it yields a two-dimensional integration

$$I^2 = \int_{-\infty}^{+\infty} f(x;\mu,\sigma)\,dx \int_{-\infty}^{+\infty} f(y;\mu,\sigma)\,dy$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\exp[\frac{-(y-\mu)^2}{2\sigma^2}]dy$$

$$= (\frac{1}{\sqrt{2\pi}\sigma})^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp[\frac{-[(x-\mu)^2+(y-\mu)^2]}{2\sigma^2}]dx\,dy$$

Let us change variables
Let $x^* = x-\mu$ and $y^* = y-\mu$. It yields $dx^* = dx$ and $dy^* = dy$
For integration $x = (-\infty) \to (+\infty)$, the integration of x^* becomes $x^* = (-\infty-\mu) \to (+\infty-\mu) = (-\infty) \to (+\infty)$. Likewise, for integration $y = (-\infty) \to (+\infty)$, the integration of y^* becomes $y^* = (-\infty-\mu) \to (+\infty-\mu) = (-\infty) \to (+\infty)$. Thus, the two-dimensional integration can be rewritten as

$$I^2 = (\frac{1}{\sqrt{2\pi}\sigma})^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp[\frac{-[x^{*2}+y^{*2}]}{2\sigma^2}]dx^*\,dy^*$$

Let us change variables again.
Let $(r^*)^2 = (x^*)^2 + (y^*)^2$ and $\tan\theta = y^*/x^*$. It yields $dx^*dy^* = r^*dr^*d\theta$ and the integration domain $x^* = (-\infty) \to (+\infty)$ & $y^* = (-\infty) \to (+\infty)$ becomes $r^* = 0 \to (+\infty)$ & $\theta = 0 \to 2\pi$. Thus, the two-dimensional integration can be rewritten as

$$I^2 = (\frac{1}{\sqrt{2\pi}\sigma})^2 \int_0^{2\pi} \int_0^{\infty} \exp(\frac{-r^{*2}}{2\sigma^2})\, r^* \, dr^* \, d\theta$$

Since

$$\frac{d}{dr^*}[-\sigma^2 \exp(\frac{-r^{*2}}{2\sigma^2})] = \exp(\frac{-r^{*2}}{2\sigma^2})\, r^*$$

The two-dimensional integration I^2 can be rewritten as

$$I^2 = (\frac{1}{\sqrt{2\pi}\sigma})^2 2\pi \int_0^{\infty} \frac{d}{dr^*}[-\sigma^2 \exp(\frac{-r^{*2}}{2\sigma^2})]\, dr^*$$

$$= \frac{1}{\sigma^2}[-\sigma^2 \exp(\frac{-r^{*2}}{2\sigma^2})]_{r^*=0}^{r^*=\infty}$$

$$= -[\exp(-\infty) - \exp(0)]$$

$$= 1$$

Since $f(x;\mu,\sigma) > 0$ for all x, it yields

$$I = \int_{-\infty}^{+\infty} f(x;\mu,\sigma)\, dx > 0$$

Thus, $I^2 = 1$ yields $I = 1$.

Part II. Mean

Now, let us determine the mean of the normal distribution random numbers.

From the Table 6 in the Notes of "Probability vs. Probability Density Function", we have

If $\int_{-\infty}^{\infty} f(x)\,dx = 1$ then the mean of the random numbers x is $\bar{x} = \int_{-\infty}^{\infty} x\, f(x)\, dx$

Thus, the mean of the normal distribution random numbers is

$$\bar{x} = \int_{-\infty}^{+\infty} x\, f(x;\mu,\sigma)\, dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} x \exp[\frac{-(x-\mu)^2}{2\sigma^2}]\, dx$$

We can integrate the above integration to obtain the mean $\bar{x}$ directly. Or, we can take the advantage of the results of the following integration to find the mean $\bar{x}$.

Let

$$I_A = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} (x-\mu) \exp[\frac{-(x-\mu)^2}{2\sigma^2}]\, dx$$

Since $\exp[\frac{-(x-\mu)^2}{2\sigma^2}]$ is an even function of $x = \mu$ but $x - \mu$ is an odd function of $x = \mu$, the $(x-\mu)\exp[\frac{-(x-\mu)^2}{2\sigma^2}]$ should be an odd function of $x = \mu$. Therefore, the integration I_A is equal to zero. That is

$$I_A = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} (x-\mu) \exp[\frac{-(x-\mu)^2}{2\sigma^2}]\, dx = 0$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} x \exp[\frac{-(x-\mu)^2}{2\sigma^2}]\, dx - \mu \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]\, dx$$

$$= \bar{x} - \mu$$

where the result

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx = 1$$

obtained in **Part I** has been used. As a result, we have the mean $\bar{x} = \mu$ for normal distribution random numbers.

Part III. Variance

Now, let us determine the variance of the normal distribution random numbers.

From the Table 6 in the Notes of "Probability vs. Probability Density Function", we have

If $\int_{-\infty}^{\infty} f(x)dx = 1$ then the variance of the random numbers x is

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)dx$$

Thus, the variance of the normal distribution random numbers is

$$\text{Variance} = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x;\mu,\sigma)dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} (x-\mu)^2 \exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx$$

where the mean $\bar{x} = \mu$ obtained in **Part II** has been used.

We can integrate the above integration to obtain the "Variance" directly. Or, we can take the advantage of the results of the following derivatives to find the "Variance". Let

$$I_o(\sigma) = \int_{-\infty}^{+\infty} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx = \sqrt{2\pi}\sigma$$

where the result

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx = 1$$

obtained in **Part I** has been used.

Taking the derivative of $I_o(\sigma)$, it yields

$$\frac{d}{d\sigma} I_o(\sigma) = \int_{-\infty}^{+\infty} \frac{d}{d\sigma} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx = \frac{d}{d\sigma}(\sqrt{2\pi}\sigma) = \sqrt{2\pi}$$

Since

$$\int_{-\infty}^{+\infty} \frac{d}{d\sigma} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx = \int_{-\infty}^{+\infty} \frac{(x-\mu)^2}{\sigma^3} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx,$$

it yields

$$\int_{-\infty}^{+\infty} \frac{(x-\mu)^2}{\sigma^3} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx = \sqrt{2\pi}$$

Multiplying $\sigma^2 / \sqrt{2\pi}$ on both sides of the above equation, it yields the Variance of the normal distribution random numbers

$$\text{Variance} = \int_{-\infty}^{+\infty} \frac{(x-\mu)^2}{\sqrt{2\pi}\sigma} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]dx = \sigma^2$$

In summary, for a given set of mean μ and standard deviation σ , the normal distribution function can be written as

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp[\frac{-(x-\mu)^2}{2\sigma^2}]$$

Namely, the normal distribution function is uniquely determined for a given set of mean μ and standard deviation σ. But it is not always true for other probability functions.