Lecture 3: Instabilities in the Solar Interior and in the Solar Atmosphere

3.1. Review of the convectional instability in a hydrodynamic equilibrium medium

Let us consider a hydrodynamic equilibrium background medium, with background gas pressure $p_0(z)$, gas temperature $T_0(z)$, average mass density $\rho_0(z)$, and a relative uniform gravitational field $g(-\hat{z})$. The hydrodynamic equilibrium state yields

$$-\frac{dp_0(z)}{dz} - \rho_0(z)g = 0 \tag{3.1}$$

For ideal gas, we have

$$p_0(z) = \rho_0(z)RT_0(z) \tag{3.2}$$

Equation (3.2) yields,

$$\frac{1}{p_0}\frac{dp_0}{dz} = \frac{1}{\rho_0}\frac{d\rho_0}{dz} + \frac{1}{T_0}\frac{dT_0}{dz} \tag{3.2a}$$

Let us consider an air parcel, which moves from $z(t=0) = z_0$ to $z(t) = z_0 + \Delta z(t)$, with thermal pressure $p(t)$, mass density $\rho(t)$, temperature $T(t)$, and initial conditions $p(t=0) = p_0(z_0)$, $\rho(t=0) = \rho_0(z_0)$, and $T(t=0) = T_0(z_0)$. The horizontal pressure balance yields the gas pressure of the air parcel $p(t)$ changes from $p_0(z_0)$ to $p_0(z_0 + \Delta z)$. If this process is fast enough, then the air parcel will follow the adiabatic equation of state. i.e.,

$$\frac{d}{dt}\ln(p\rho^{-\gamma}) = 0 \tag{3.3}$$

where γ is the specific heat. The specific heat γ satisfies $\gamma = (f+2)/2$, where f is the degree of freedom of the gas particles For the ideal gas, we have $p = \rho RT$. Thus, the adiabatic equation of state can also be written as

$$\frac{d}{dt}\ln(p^{1-\gamma}T^{\gamma}) = 0 \tag{3.4}$$

Namely, the changes of the gas density and gas temperature satisfy the adiabatic equation of state. i.e.,

$$\frac{1}{p}\frac{dp}{dt} = \frac{\gamma}{\rho}\frac{d\rho}{dt} \tag{3.3a}$$

$$\frac{(1-\gamma)}{p}\frac{dp}{dt} + \frac{\gamma}{T}\frac{dT}{dt} = 0 \tag{3.4a}$$

The equation of motion of the air parcel when it moves to $z = z_0 + \Delta z(t)$ is

$$\rho(t)\frac{d^2 z(t)}{dt^2} = \rho(t)\frac{d^2 \Delta z(t)}{dt^2} = -\left.\frac{dp_0}{dz}\right|_{z=z_0+\Delta z} - \rho(t)g \tag{3.5}$$

Equation (3.1) yields

$$\left.\frac{dp_0}{dz}\right|_{z=z_0+\Delta z} = -\rho_0(z_0+\Delta z)g \tag{3.1a}$$

Substituting equation (3.1a) into equation (3.5), it yields

$$\frac{d^2\Delta z(t)}{dt^2} = \left(\frac{\rho_0(z_0+\Delta z)}{\rho(t)} - 1\right)g \tag{3.6}$$

We can conclude from equation (3.6) that, if $\rho(t) > \rho_0(z_0+\Delta z)$, there will be a restoring force to move the air parcel back to z_0, and the system is stable to the convectional instability. Or, if $\rho(t) < \rho_0(z_0+\Delta z)$, the system would be unstable to the convectional instability.

The numerator $\rho_0(z_0+\Delta z)$ in equation (3.6) can be written as

$$\boxed{\begin{aligned}
\rho_0(z_0+\Delta z) &= \rho_0(z_0) + \Delta z\left.\frac{d\rho_0(z)}{dz}\right|_{z=z_0} + O(\Delta z^2\frac{d^2\rho_0}{dz^2}) \\
&= \rho_0(z_0)\{1+\Delta z\frac{1}{\rho_0}\left.\frac{d\rho_0}{dz}\right|_{z=z_0} + O(\Delta z^2\frac{1}{\rho_0}\frac{d^2\rho_0}{dz^2})\} \\
&= \rho_0(z_0)\{1+\Delta z[\frac{1}{p_0}\frac{dp_0}{dz} - \frac{1}{T_0}\frac{dT_0}{dz}]_{z=z_0} + O(\Delta z^2\frac{1}{\rho_0}\frac{d^2\rho_0}{dz^2})\} \\
&= \rho_0(z_0)\{1+\Delta z\,[\frac{-g}{RT_0} - \frac{1}{T_0}\frac{dT_0}{dz}]_{z=z_0} + O(\Delta z^2\frac{1}{\rho_0}\frac{d^2\rho_0}{dz^2})\}
\end{aligned}} \tag{3.7}$$

where equation (3.1) has been used to eliminate $p_0(z_0)$ in equation (3.7).

Making use of equation (3.3), we can eliminate the denominator $\rho(t)$ in equation (3.6). Equation (3.3) yields $p(t)\rho(t)^{-\gamma} = (p\rho^{-\gamma})_{t=0}$, or

$$\boxed{\begin{aligned}
\rho(t) &= [\frac{p(t)}{(p\rho^{-\gamma})_{t=0}}]^{1/\gamma} \\
&= \rho(t=0)[\frac{p_0(z_0+\Delta z)}{p_0(z_0)}]^{1/\gamma} \\
&= \rho_0(z_0)[\frac{p_0(z_0)+\Delta z\left.\frac{dp_0}{dz}\right|_{z_0} + O(\Delta z^2\frac{d^2p_0}{dz^2})}{p_0(z_0)}]^{1/\gamma} \\
&= \rho_0(z_0)[1-\Delta z\frac{g}{RT_0(z_0)} + O(\Delta z^2\frac{1}{p_0}\frac{d^2p_0}{dz^2})]^{1/\gamma}
\end{aligned}} \tag{3.8}$$

where equation (3.1) has been used to eliminate $p_0(z_0)$ in equation (3.8). Substituting equations (3.7) and (3.8) into equation (3.6), and ignoring the second order and the higher order terms, it yields

$$\boxed{\begin{aligned}\frac{d^2\Delta z}{dt^2} &\approx g\{\frac{\rho_0(z_0)[1+\Delta z(\frac{-g}{RT_0}-\frac{1}{T_0}\frac{dT_0}{dz}\Big|_{z=z_0}]}{\rho_0(z_0)[1-\Delta z\frac{g}{RT_0(z_0)}]^{1/\gamma}}-1\} \\ &\approx g\{\,[1+\Delta z(\frac{-g}{RT_0}-\frac{1}{T_0}\frac{dT_0}{dz})\Big|_{z=z_0}\,][1+\frac{1}{\gamma}\Delta z\frac{g}{RT_0(z_0)}]-1\,\} \\ &\approx g\,\Delta z\{(\frac{1}{\gamma}-1)\frac{g}{RT_0(z_0)}-\frac{1}{T_0}\frac{dT_0}{dz}\Big|_{z=z_0}\,\}\end{aligned}} \quad (3.9)$$

If the background medium has an adiabatic-vertical-temperature profile $T_{ad}(z)$, i.e.,

$p_0(z)^{1-\gamma}T_{ad}(z)^{\gamma} = \text{constant}$

then

$$(\frac{1}{\gamma}-1)\frac{g}{RT_0(z_0)} = \frac{1-\gamma}{\gamma}\frac{1}{p_0}\frac{dp_0}{dz} = (\frac{1}{T_{ad}}\frac{dT_{ad}}{dz}) \quad (3.10)$$

Thus, we can define an adiabatic-vertical-temperature-change rate

$$(\frac{1}{T}\frac{dT}{dz})_{ad} \equiv (\frac{1}{\gamma}-1)\frac{g}{RT_0(z_0)} \quad (3.11)$$

Substituting equation (3.11) into equation (3.9) it yields

$$\frac{d^2\Delta z}{dt^2} \approx \Delta z\{(\frac{1}{T}\frac{dT}{dz})_{ad} - \frac{1}{T_0}\frac{dT_0}{dz}\Big|_{z=z_0}\}g = -\Delta z N^2 \quad (3.12)$$

where we have defined

$$N^2 \equiv \{(-\frac{1}{T}\frac{dT}{dz})_{ad} - (-\frac{1}{T_0}\frac{dT_0}{dz}\Big|_{z=z_0})\}g \quad (3.13)$$

If $N^2 > 0$, the vertical displacement of the air parcel can lead to a vertical oscillation of the air parcel with angular frequency N, which is also called the Brunt-Väisällä frequency. Namely, when the temperature-decreasing rate of the background gas $-\frac{1}{T_0}\frac{dT_0}{dz}\Big|_{z=z_0}$ is less than the adiabatic-temperature-decreasing rate $(-\frac{1}{T}\frac{dT}{dz})_{ad}$ the system is stable to the convectional instability. The system is unstable to the convectional instability, if $N^2 < 0$ or if the temperature-decreasing rate of the background gas $-\frac{1}{T_0}\frac{dT_0}{dz}\Big|_{z=z_0}$ is greater than the adiabatic-temperature-decreasing rate $(-\frac{1}{T}\frac{dT}{dz})_{ad}$.

3.2. The convectional instability in the solar interior

If we consider the interior of the sun as a spherically symmetric hydrodynamic equilibrium background medium, with background plasma pressure $p_0(r)$, plasma temperature $T_0(r)$, plasma number density $n_0(r)$, average ion mass $\overline{\mu}(r)$. The average mass density is

$$\rho_0(r) = n_0(r)\overline{\mu}(r) \tag{3.14}$$

The gravitational field $\mathbf{g}(r) = -\hat{r}g(r)$ can be determined by

$$-\hat{r}g(r) = -\hat{r}4\pi \int_0^r r^2 \rho_0(r)\, dr \tag{3.15}$$

The hydrodynamic equilibrium state yields

$$-\frac{dp_0(r)}{dr} - \rho_0(r)g(r) = 0 \tag{3.16}$$

For ideal gas, we have

$$p_0(r) = n_0(r)k_B T_0(r) = \rho_0(r)\frac{k_B}{\overline{\mu}(r)}T_0(r) \tag{3.17}$$

Based on solar seismology, scientists can determine the radial distribution of the acoustic wave speed $C_{S0}(r)$ in solar interior, where

$$C_{S0}(r) = \sqrt{\frac{\gamma\, p_0(r)}{\rho_0(r)}} = \sqrt{\gamma\, k_B}\sqrt{\frac{T_0(r)}{\overline{\mu}(r)}} \tag{3.18}$$

The standard solar model (xxxx, 199x), provide an theoretical model for the average ion mass distribution in the solar interior $\overline{\mu}(r)$. Thus, the five equations (3.14), (3.15), (3.16), (3.17) and (3.18) together with a given $\overline{\mu}(r)$ model and observed $C_{S0}(r)$, we can determine the five unknowns $p_0(r)$, $T_0(r)$, $n_0(r)$, $\rho_0(r)$, and $g(r)$.

If we include the magnetic force in the convection zone, the equilibrium state along the radial direction becomes

$$-\frac{\partial p_0}{\partial r} - \rho_0(r)g(r) + (\mathbf{J}\times\mathbf{B})_r \approx -\frac{\partial p_0}{\partial r} - \rho_0(r)g(r) - \frac{\partial}{\partial r}\frac{B^2}{2\mu_0} + \frac{\mathbf{B}\cdot\nabla}{\mu_0}B_r = 0 \tag{3.16a}$$

Since the magnetic field cannot be spherical symmetry, if the magnetic force is included as described in equation (3.16a) the average ion mass $\overline{\mu}$ should become a spherical asymmetric function, i.e., $\overline{\mu} = \overline{\mu}(r,\theta)$. Namely, the spherical symmetric model of $\overline{\mu}(r)$ is no longer applicable to the plasma in the solar convection zone. Likewise, by including the magnetic force, it might lead to $p_0(r,\theta)$, $T_0(r,\theta)$, $n_0(r,\theta)$, in the solar convection zone. Thus, the convectional instability in the solar convection zone is far more complex than the convectional instability in the Earth's troposphere.