Lecture 6. Linear Waves in Magnetohydrodynamic Plasma

6.0. How to Linearize a Nonlinear Equation

We shall use the mass continuity equation as an example to demonstrate how to linearize a nonlinear equation. Let A_0 denotes a background state and A_1 denotes a small perturbation, where $O(A_1) = O(\varepsilon)O(A_0)$. Then, A can be written as

$$A = A_0 + A_1 + O(\varepsilon^2)O(A_0) \approx A_0 + A_1 \tag{6.0.1}$$

Substituting equation (6.0.1) into the mass continuity equation, it yields

$$[\frac{\partial}{\partial t} + (\mathbf{V}_0 + \mathbf{V}_1)\cdot\nabla](\rho_0 + \rho_1) = -(\rho_0 + \rho_1)\nabla\cdot(\mathbf{V}_0 + \mathbf{V}_1) \tag{6.0.2}$$

The equilibrium state of continuity equation is

$$(\mathbf{V}_0\cdot\nabla)\rho_0 = -\rho_0\nabla\cdot\mathbf{V}_0 \tag{6.0.3}$$

Subtracting equation (6.0.3) from equation (6.0.2) yields

$$(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\nabla)\rho_1 + \mathbf{V}_1\cdot\nabla\rho_0 + \mathbf{V}_1\cdot\nabla\rho_1 = -\rho_0\nabla\cdot\mathbf{V}_1 - \rho_1\nabla\cdot\mathbf{V}_0 - \rho_1\nabla\cdot\mathbf{V}_1 \tag{6.0.4}$$

where $\mathbf{V}_1\cdot\nabla\rho_1$ and $\rho_1\nabla\cdot\mathbf{V}_1$ are of the order of $O(\varepsilon^2)$. Ignoring these nonlinear second-order small terms, equation (6.0.4) is reduced to a linearized equation,

$$(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\nabla)\rho_1 + \mathbf{V}_1\cdot\nabla\rho_0 = -\rho_0\nabla\cdot\mathbf{V}_1 - \rho_1\nabla\cdot\mathbf{V}_0 \tag{6.0.5}$$

The linearized equation shown in equation (6.0.5) can be used to study linear waves in a non-uniform background medium with either density gradient or velocity shear.

6.1. Linear Plane Waves in Uniform MHD Plasma

Magnetohydrodynamic (MHD) plasma is a plasma model under long wavelength and low frequency limit, in which the time scale and spatial scale of the MHD plasma phenomena are much longer than the ions' time scale and spatial scale, respectively. Lecture 4 shows that the MHD Ohm's law can lead to frozen-in flux, which is an important characteristic of MHD plasma. In addition to the characteristics of frozen-in conditions, MHD linear wave modes are also important characteristics of the MHD plasma. Governing equations of MHD plasma with isotropic pressure and zero heat flux are listed in Column (1) of Table 6.1.

Table 6.1. Governing equations of MHD plasma with isotropic pressure and zero heat flux

(1) MHD equations in (t, $\mathbf{x}$) domain	(2) linearized MHD equations in ($\omega, \mathbf{k}$) domain
Mass continuity equation $(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\rho = -\rho\nabla\cdot\mathbf{V}$	Mass continuity equation $(-i\omega)\tilde{\rho}_1 = -\rho_0(i\mathbf{k})\cdot\tilde{\mathbf{V}}_1$ (6.1)
MHD momentum equation $\rho(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p + \mathbf{J}\times\mathbf{B}$	MHD momentum equation $\rho_0(-i\omega)\tilde{\mathbf{V}}_1 = -(i\mathbf{k})\tilde{p}_1 + \tilde{\mathbf{J}}_1\times\mathbf{B}_0$ (6.2)
MHD energy equation $\frac{3}{2}[(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\ln(p\rho^{-5/3})] = 0$	MHD energy equation $(-i\omega)\tilde{p}_1 = \frac{\gamma p_0}{\rho_0}(-i\omega)\tilde{\rho}_1$ (6.3)
MHD charge continuity equation $\nabla\cdot\mathbf{J} = 0$	MHD charge continuity equation $(i\mathbf{k})\cdot\tilde{\mathbf{J}}_1 = 0$ (6.4)
MHD Ohm's law $\mathbf{E} + \mathbf{V}\times\mathbf{B} = 0$	MHD Ohm's law $\tilde{\mathbf{E}}_1 + \tilde{\mathbf{V}}_1\times\mathbf{B}_0 = 0$ (6.5)
Maxwell's equations: $\nabla\cdot\mathbf{E} = 0$ $\nabla\cdot\mathbf{B} = 0$ $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$	Maxwell's equations: $(i\mathbf{k})\cdot\tilde{\mathbf{E}}_1 = 0$ (6.6) $(i\mathbf{k})\cdot\tilde{\mathbf{B}}_1 = 0$ (6.7) $(i\mathbf{k})\times\tilde{\mathbf{E}}_1 = i\omega\tilde{\mathbf{B}}_1$ (6.8) $(i\mathbf{k})\times\tilde{\mathbf{B}}_1 = \mu_0\tilde{\mathbf{J}}_1$ (6.9)

For uniform background plasma, we can choose a moving frame such that $\mathbf{V}_0 = 0$. Substituting $\mathbf{V}_0 = 0$ into Ohm's law, it yields $\mathbf{E}_0 = 0$. Far from the source region, perturbations can be assumed in plane-wave format. A perturbation $A_1(\mathbf{x},t)$ can be written as

$$A_1(\mathbf{x},t) = \bar{A}_1(\mathbf{k},\omega)\cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi_A) = \mathrm{Re}\{\tilde{A}_1(\mathbf{k},\omega)\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]\}$$

where $\tilde{A}_1(\mathbf{k},\omega) = \bar{A}_1(\mathbf{k},\omega)e^{i\phi_A}$ is a complex number. The wave amplitude $\bar{A}_1(\mathbf{k},\omega)$ satisfies $O(\bar{A}_1) = O(\varepsilon)O(A_0)$, where A_0 denotes a background variable. Following the procedures described in equations (6.0.1)~(6.0.5), a set of linearized MHD equations in (ω, $\mathbf{k}$) domain are obtained and are listed in Column (2) of Table 6.1.

Our goal is to reduce the system equations, listed in Column (2) of Table 6.1, into a vector equation of $\tilde{\mathbf{V}}_1$. We shall focus on the momentum equation (6.2). In order to eliminate $\tilde{p}_1$ in equation (6.2), we substitute equation (6.1) into equation (6.3) to eliminate $\tilde{\rho}_1$, then substitute the resulting equation into equation (6.2) to eliminate $\tilde{p}_1$. Likewise, to eliminate $\tilde{\mathbf{J}}_1$ in equation (6.2), we substitute equation (6.5) into equation (6.8) to eliminate $\tilde{\mathbf{E}}_1$, then substitute the resulting equation into equation (6.9) to eliminate $\tilde{\mathbf{B}}_1$, and then substitute the resulting equation into equation (6.2) to eliminate $\tilde{\mathbf{J}}_1$.

Substituting equation (6.1) into equation (6.3) yields

$$\tilde{p}_1 = \frac{\gamma p_0}{\rho_0}\tilde{\rho}_1 = C_{S0}^2\tilde{\rho}_1 = C_{S0}^2\frac{\rho_0\mathbf{k}\cdot\tilde{\mathbf{V}}_1}{\omega} \tag{6.3'}$$

Substituting equation (6.5) into equation (6.8) to eliminate $\tilde{\mathbf{E}}_1$, then substituting the resulting equation into equation (6.9) to eliminate $\tilde{\mathbf{B}}_1$, it yields

$$\tilde{\mathbf{J}}_1 = \frac{i\mathbf{k}\times\tilde{\mathbf{B}}_1}{\mu_0} = \frac{i\mathbf{k}\times\dfrac{\mathbf{k}\times\tilde{\mathbf{E}}_1}{\omega}}{\mu_0} = \frac{i\mathbf{k}\times\dfrac{\mathbf{k}\times(-\tilde{\mathbf{V}}_1\times\mathbf{B}_0)}{\omega}}{\mu_0} = \frac{i\mathbf{k}\times[\mathbf{k}\times(\mathbf{B}_0\times\tilde{\mathbf{V}}_1)]}{\mu_0\omega} \tag{6.9'}$$

Substituting equations (6.3') and (6.9') into equation (6.2) yields

$$\rho_0(-i\omega)\tilde{\mathbf{V}}_1 = -i\mathbf{k}C_{S0}^2\frac{\rho_0\mathbf{k}\cdot\tilde{\mathbf{V}}_1}{\omega} + \frac{i\mathbf{k}\times[\mathbf{k}\times(\mathbf{B}_0\times\tilde{\mathbf{V}}_1)]}{\mu_0\omega}\times\mathbf{B}_0 \tag{6.2'}$$

Multiplying equation (6.2') by $-i\omega/\rho_0 k^2$ yields

$$-\frac{\omega^2}{k^2}\tilde{\mathbf{V}}_1 = -C_{S0}^2\hat{k}\hat{k}\cdot\tilde{\mathbf{V}}_1 - \frac{B_0^2}{\mu_0\rho_0}\{\hat{k}\times[\hat{k}\times(\hat{B}_0\times\tilde{\mathbf{V}}_1)]\}\times\hat{B}_0$$

or

$$\frac{\omega^2}{k^2}\tilde{\mathbf{V}}_1 = C_{S0}^2\hat{k}\hat{k}\cdot\tilde{\mathbf{V}}_1 + C_{A0}^2\hat{B}_0\times\{\hat{k}\times[\hat{k}\times(\hat{B}_0\times\tilde{\mathbf{V}}_1)]\} \tag{6.2''}$$

where $C_{A0} \equiv B_0/\sqrt{\mu_0\rho_0}$ is called Alfvén speed, and $C_{S0} \equiv \sqrt{\gamma p_0/\rho_0}$ is called sound speed.

As a result, we can obtain a set of equations for flow velocity $\tilde{\mathbf{V}}_1$, which can be written as

$$\mathbf{D}\cdot\tilde{\mathbf{V}}_1 = 0 \tag{6.10}$$

where

$$\mathsf{D} = [\frac{\omega^2}{k^2} - C_{A0}^2(\hat{B}_0 \cdot \hat{k})^2]\mathbf{1} - (C_{A0}^2 + C_{S0}^2)\hat{k}\hat{k} + C_{A0}^2(\hat{B}_0 \cdot \hat{k})(\hat{B}_0\hat{k} + \hat{k}\hat{B}_0) \quad (6.11)$$

For convenience, we can choose a coordinate system such that background magnetic field is along the $\hat{z}$-axis, and wave number $\mathbf{k}$ lies on x- z plane. Namely,

$$\mathbf{B}_0 = \hat{z}B_0 \quad (6.12)$$

and

$$\mathbf{k} = k(\hat{z}\cos\theta + \hat{x}\sin\theta) \quad (6.13)$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$. Substituting equations (6.12) and (6.13) into equations (6.10) and (6.11) yields

$$\begin{pmatrix} (\omega^2/k^2) - \alpha & 0 & -\delta \\ 0 & (\omega^2/k^2) - C_{A0}^2\cos^2\theta & 0 \\ -\delta & 0 & (\omega^2/k^2) - \beta \end{pmatrix} \begin{pmatrix} \tilde{V}_{1x} \\ \tilde{V}_{1y} \\ \tilde{V}_{1z} \end{pmatrix} = 0 \quad (6.14)$$

where

$$\alpha = C_{A0}^2\cos^2\theta + (C_{A0}^2 + C_{S0}^2)\sin^2\theta = C_{A0}^2 + C_{S0}^2\sin^2\theta \quad (6.15)$$

$$\beta = C_{A0}^2\cos^2\theta + (C_{A0}^2 + C_{S0}^2)\cos^2\theta - 2C_{A0}^2\cos^2\theta = C_{S0}^2\cos^2\theta \quad (6.16)$$

$$\delta = C_{S0}^2\cos\theta\sin\theta \quad (6.17)$$

Another way to obtain equation (6.14):

Since

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_z & +a_y \\ +a_z & 0 & -a_x \\ -a_y & +a_x & 0 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} -a_z b_y + a_y b_z \\ +a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{pmatrix}$$

Thus

$$\hat{\mathbf{B}}_0 \times = \hat{\mathbf{z}} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\mathbf{k}} \times = (\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta) \times = \begin{pmatrix} 0 & -\cos\theta & 0 \\ +\cos\theta & 0 & -\sin\theta \\ 0 & +\sin\theta & 0 \end{pmatrix}$$

Equation (6.2") can be rewritten as

$$\frac{\omega^2}{k^2}\tilde{\mathbf{V}}_1 = C_{A0}^2(\hat{\mathbf{B}}_0\times)(\hat{\mathbf{k}}\times)(\hat{\mathbf{k}}\times)(\hat{\mathbf{B}}_0\times)\cdot\tilde{\mathbf{V}}_1 + C_{S0}^2(\hat{\mathbf{k}}\hat{\mathbf{k}})\cdot\tilde{\mathbf{V}}_1$$

or

$$\frac{\omega^2}{k^2}\tilde{\mathbf{V}}_1 = C_{A0}^2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\cos\theta & 0 \\ +\cos\theta & 0 & -\sin\theta \\ 0 & +\sin\theta & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\cos\theta & 0 \\ +\cos\theta & 0 & -\sin\theta \\ 0 & +\sin\theta & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\tilde{\mathbf{V}}_1$$

$$+C_{S0}^2\begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}(\ \sin\theta \quad 0 \quad \cos\theta\ )\tilde{\mathbf{V}}_1$$

It yields

$$\frac{\omega^2}{k^2}\tilde{\mathbf{V}}_1 = \mathbf{M}\cdot\tilde{\mathbf{V}}_1 = \begin{pmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{pmatrix}\begin{pmatrix} \tilde{V}_{1x} \\ \tilde{V}_{1y} \\ \tilde{V}_{1z} \end{pmatrix}$$

where

$$\begin{pmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{pmatrix} = \begin{pmatrix} C_{A0}^2 + C_{S0}^2\sin^2\theta & 0 & C_{S0}^2\cos\theta\sin\theta \\ 0 & C_{A0}^2\cos^2\theta & 0 \\ C_{S0}^2\cos\theta\sin\theta & 0 & C_{S0}^2\cos^2\theta \end{pmatrix}$$

Note that for $\tilde{\mathbf{V}}_1 \neq 0$, solutions of ω^2/k^2 for different wave modes can be considered as eigen values of the following matrix

$$\begin{pmatrix} \alpha & 0 & \delta \\ 0 & C_{A0}^2\cos^2\theta & 0 \\ \delta & 0 & \beta \end{pmatrix}$$

Characteristics of different wave modes can be obtained from the corresponding eigen vectors.

Exercise 6.1.

Review eigen values and eigen vectors of a symmetric matrix. Determine eigen values λ_1, λ_2, λ_3, and the corresponding normalized eigen vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, of the following symmetric matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Show that these eigen vectors of the symmetric matrix form an orthogonal basis and after coordinate transformation, the representation of matrix M in this new basis $B' = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ becomes

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}_{B'}$$

6.2. Linear Wave Modes in MHD Plasma

Number of linearized equations with time derivative term can lead to the same number of linear wave modes. Amount the nine equations in Table 6.1, seven of them contain a time derivative term. It will be shown in this section that, for $\theta \neq 0$ and $\theta \neq \pi/2$, seven linear wave modes can be found in the MHD plasma. Three of them are forward propagating waves. Based on their wave speeds, these three wave modes are called fast-mode wave, intermediate-mode wave, and slow-mode wave. The intermediate mode wave is also called Alfven-mode wave or share-Alfven wave. The other four wave modes are backward propagating fast-mode wave, intermediate-mode wave, slow-mode wave, and a non-propagating entropy-mode wave. The fast mode, Alfven mode, and slow mode are eigen modes of equation (6.14). The entropy mode is an additional wave mode, which can be obtained from the continuity equation.

6.2.1. Entropy-Mode Wave

Entropy mode in MHD plasma is characterized by $\tilde{\rho}_1 \neq 0$, but $\tilde{V}_{1x} = \tilde{V}_{1y} = \tilde{V}_{1z} = 0$ and $\omega = 0$. For $\omega = 0$, the phase speed ω/k vanishes. Thus, entropy mode is frozen in the plasma flow. Contact Discontinuity (CD) can be considered as a nonlinear version of entropy mode in MHD plasma.

6.2.2. Alfven-Mode (or Intermediate-Mode) Wave

Alfven mode in the MHD plasma is characterized by $\tilde{V}_{1x} = \tilde{V}_{1z} = 0$ but $\tilde{V}_{1y} \neq 0$. For $\tilde{V}_{1x} = \tilde{V}_{1z} = 0$ but $\tilde{V}_{1y} \neq 0$, equation (6.14) yields

$$\frac{\omega^2}{k^2} = C_{A0}^2 \cos^2\theta \tag{6.18}$$

Equation (6.18) is the wave dispersion relation of Alfven-mode wave. Since the phase speed of Alfven mode is in between fast-mode and slow-mode wave speed, the Alfven mode is also called intermediate mode. It can be shown that Rotational Discontinuity (RD) can be considered as a nonlinear version of Alfven-mode wave in MHD plasma.

Characteristics of Alfven-mode wave:

From Alfven-mode wave dispersion relation $\omega = \pm k C_{A0} \cos\theta$, we can determine group velocity of Alfven mode to be

$$\mathbf{v}_g = \frac{d\omega}{d\mathbf{k}} = \hat{x}\frac{\partial\omega}{\partial k_x} + \hat{z}\frac{\partial\omega}{\partial k_z} = \pm\hat{z}C_{A0} = \pm\hat{B}_0 C_{A0}$$

6.2.3. Fast-Mode and Slow-Mode Wave

For $\tilde{V}_{1y} = 0$ but $\begin{pmatrix} \tilde{V}_{1x} \\ \tilde{V}_{1z} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

equation (6.14) yields

$$\det\begin{pmatrix} (\omega^2/k^2) - \alpha & -\delta \\ -\delta & (\omega^2/k^2) - \beta \end{pmatrix} = \left(\frac{\omega^2}{k^2}\right)^2 - \frac{\omega^2}{k^2}(\alpha+\beta) + \alpha\beta - \delta^2 = 0$$

where α, β, and δ are given in equations (6.15)~(6.17), which yields

$$\alpha + \beta = C_{A0}^2 + C_{S0}^2 \sin^2\theta + C_{S0}^2 \cos^2\theta = C_{A0}^2 + C_{S0}^2$$

and

$$\alpha\beta - \delta^2 = (C_{A0}^2 + C_{S0}^2 \sin^2\theta) C_{S0}^2 \cos^2\theta - C_{S0}^4 \cos^2\theta \sin^2\theta = C_{A0}^2 C_{S0}^2 \cos^2\theta$$

Thus, we have

$$\left(\frac{\omega^2}{k^2}\right)^2 - \frac{\omega^2}{k^2}(C_{A0}^2 + C_{S0}^2) + C_{A0}^2 C_{S0}^2 \cos^2\theta = 0 \tag{6.20}$$

Equation (6.20) has two roots of ω^2 / k^2. They are

$$\left(\frac{\omega^2}{k^2}\right)_{\substack{Fast\\Slow}} = (v_{ph}^2)_{\substack{Fast\\Slow}} = \frac{1}{2}\left\{(C_{A0}^2 + C_{S0}^2) \pm \sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2 \cos^2\theta}\right\} \tag{6.21}$$

where + sign equation is fast-mode wave dispersion relation and – sign equation is slow-mode wave dispersion relation.

Characteristics of Fast-mode and Slow-mode waves:

From Fast-mode and Slow-mode wave dispersion relation, we can determine group velocity of these two wave modes as

$$(\mathbf{v}_g)_{\substack{Fast\\Slow}} = \frac{\omega}{k}(\hat{k}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial\omega}{\partial\theta}) = \hat{k}(v_{ph})_{\substack{Fast\\Slow}} \pm \hat{\theta}\frac{1}{(v_{ph})_{\substack{Fast\\Slow}}}\frac{C_{A0}^2 C_{S0}^2 \cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2 \cos^2\theta}} \tag{6.22}$$

where $(v_{ph})_{\substack{Fast\\Slow}}$ is given in equation (6.21).

Proof of equation (6.22):

By definition, group velocity is

$$\mathbf{v}_g = \frac{d\omega}{d\mathbf{k}} = \hat{k}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial\omega}{\partial\theta}$$

where

$$2\omega\frac{\partial\omega}{\partial k} = 2k\frac{\omega^2}{k^2}$$

and

$$2\omega\frac{\partial\omega}{\partial\theta} = k^2\frac{\partial}{\partial\theta}[\frac{1}{2}\left\{(C_{A0}^2 + C_{S0}^2) \pm \sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2 \cos^2\theta}\right\}]$$

$$= k^2(\frac{1}{2})(\pm\frac{1}{2})\frac{4\cdot 2C_{A0}^2 C_{S0}^2 \cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2 \cos^2\theta}}$$

Thus, we have

$$v_{ph}\mathbf{v}_g = \frac{\omega}{k}(\hat{k}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial\omega}{\partial\theta}) = \hat{k}\frac{\omega^2}{k^2} \pm \hat{\theta}\frac{C_{A0}^2 C_{S0}^2 \cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2 \cos^2\theta}}$$

$$= \hat{k}[\frac{1}{2}\left\{(C_{A0}^2 + C_{S0}^2) \pm \sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2 \cos^2\theta}\right\}] \pm \hat{\theta}\frac{C_{A0}^2 C_{S0}^2 \cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2 \cos^2\theta}}$$

or

$$(\mathbf{v}_g)_{\substack{Fast\\Slow}} = \frac{\omega}{k}(\hat{k}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial\omega}{\partial\theta}) = \hat{k}(v_{ph})_{\substack{Fast\\Slow}} \pm \hat{\theta}\frac{1}{(v_{ph})_{\substack{Fast\\Slow}}}\frac{C_{A0}^2 C_{S0}^2 \cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2 \cos^2\theta}}$$

6.2.4. Fredrick's Diagrams of MHD Waves' Phase Velocity and Group Velocity

Exercise 6.2.

(1) Ignoring entropy mode, plot phase velocity of three MHD wave modes on Fredrick's diagram, where polar coordinate $(r,\theta) = (\omega / k, \theta_{\mathbf{k},\mathbf{B}_0})$.

(2) Plot group velocity of three MHD wave modes on Fredrick's diagram, where polar coordinate $(r,\theta) = (v_g, \theta_{\mathbf{v}_g\mathbf{B}_0})$.

Exercise 6.3.

Consider MHD plane waves with $\mathbf{B}_0 = \hat{z}B_0$ and $\mathbf{k} = k(\hat{z}\cos\theta + \hat{x}\sin\theta)$

(1) Show that $\tilde{\mathbf{B}}_1 = B_0 k[\hat{x}(-\tilde{V}_{1x}\cos\theta) + \hat{y}(-\tilde{V}_{1y}\cos\theta) + \hat{z}(\tilde{V}_{1x}\sin\theta)]$.

(2) Show that $\tilde{\rho}_1 = \rho_0 \frac{k}{\omega}\tilde{V}_{1x}\frac{(\frac{\omega}{k})^2 - C_{A0}^2}{C_{S0}^2 \sin\theta}$.

(3) Show that $\tilde{B}_1 = \tilde{B} - B_0 = \frac{\mathbf{B}_0 \cdot \tilde{\mathbf{B}}_1}{B_0} = B_0 \frac{\tilde{V}_{1x} k \sin\theta}{\omega}$

(4) Show that, for $\tilde{B}_1 \neq 0$, we have $\frac{\tilde{\rho}_1}{\tilde{B}_1} = \frac{(\frac{\omega}{k})^2 - C_{A0}^2}{C_{S0}^2 \sin^2\theta}\frac{\rho_0}{B_0}$

(5) Show that for Alfven wave $\tilde{\rho}_1 = 0$, $\tilde{p}_1 = 0$, and $\tilde{B}_1 = 0$.

(6) Show that ρ_1 and B_1 are in phase for the fast-mode wave, but out of phase for the slow-mode wave.

(7) Show that, for Alfven mode, variations of $\mathbf{B}_1$ and $\mathbf{V}_1$ are in phase if $\pi/2 < \theta < \pi$, but out-off phase if $0 < \theta < \pi/2$.

(8) Determine the perturbation directions of $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{J}}_1$ for Alfven-mode, fast-mode, and slow-mode waves.

(9) Show that $\tilde{\mathbf{V}}_{1Fast} \cdot \tilde{\mathbf{V}}_{1Slow} = 0$.

Proof of $\tilde{\mathbf{V}}_{1Fast} \cdot \tilde{\mathbf{V}}_{1Slow} = 0$

equation (6.14) yields

$$(\tilde{V}_{1x})_{Fast}[(\omega^2 / k^2)_{Fast} - \alpha] - (\tilde{V}_{1z})_{Fast}\delta = 0$$

and

$$(\tilde{V}_{1x})_{Slow}[(\omega^2 / k^2)_{Slow} - \alpha] - (\tilde{V}_{1z})_{Slow}\delta = 0$$

Substituting the above two equations into $\tilde{\mathbf{V}}_{1Fast} \cdot \tilde{\mathbf{V}}_{1Slow}$, it yields

$$\begin{aligned}
\tilde{\mathbf{V}}_{1Fast} \cdot \tilde{\mathbf{V}}_{1Slow} &= (\tilde{V}_{1x})_{Fast}(\tilde{V}_{1x})_{Slow} + (\tilde{V}_{1z})_{Fast}(\tilde{V}_{1z})_{Slow} \\
&= (\tilde{V}_{1x})_{Fast}(\tilde{V}_{1x})_{Slow} + \{(\tilde{V}_{1x})_{Fast}[(\omega^2 / k^2)_{Fast} - \alpha] / \delta\}\{(\tilde{V}_{1x})_{Slow}[(\omega^2 / k^2)_{Slow} - \alpha] / \delta\} \\
&= (\tilde{V}_{1x})_{Fast}(\tilde{V}_{1x})_{Slow}\{1 + [(\omega^2 / k^2)_{Fast} - \alpha][(\omega^2 / k^2)_{Slow} - \alpha] / \delta^2\} \\
&= (\tilde{V}_{1x})_{Fast}(\tilde{V}_{1x})_{Slow}\{\delta^2 + (\omega^2 / k^2)_{Fast}(\omega^2 / k^2)_{Slow} - \alpha[(\omega^2 / k^2)_{Fast} + (\omega^2 / k^2)_{Slow}] + \alpha^2\} / \delta^2 \\
&= (\tilde{V}_{1x})_{Fast}(\tilde{V}_{1x})_{Slow}\{\delta^2 + \alpha^2 + \frac{1}{4}[b^2 - (b^2 - 4c)] - \alpha b\} / \delta^2 \\
&= (\tilde{V}_{1x})_{Fast}(\tilde{V}_{1x})_{Slow}\{\delta^2 + \alpha^2 + c - \alpha b\} / \delta^2
\end{aligned}$$

where

$$b = (\alpha + \beta)$$

$$c = \alpha\beta - \delta^2$$

Thus

$$\tilde{\mathbf{V}}_{1Fast} \cdot \tilde{\mathbf{V}}_{1Slow} = (\tilde{V}_{1x})_{Fast}(\tilde{V}_{1x})_{Slow}\{\delta^2 + \alpha^2 + (\alpha\beta - \delta^2) - \alpha(\alpha + \beta)\} / \delta^2 = 0$$

Students are encouraged to read the classical paper written by Kantrowitz and Petschek (1966) for detail discussion on MHD wave modes.

Reference

Kantrowitz, A., and H. E. Petschek, MHD characteristics and shock waves, in *Plasma Physics in Theory and Application,* edited by W. B. Kunkel, p. 148, McGraw-Hill Inc., New York, 1966.

Solutions of nonlinear MHD equilibrium states consist of Contact Discontinuity (CD), Tangential Discontinuity (TD), Rotational Discontinuity (RD), and Shock Waves.

Contact discontinuity (CD) is the nonlinear state of the entropy mode. Rotational discontinuity (RD) is the nonlinear state of the intermediate mode. Fast shock is the nonlinear state of the fast mode wave. Slow shock is the nonlinear state of the slow mode wave.

It can be shown that Tangential discontinuity (TD) can be considered as a nonlinear state of perpendicularly propagated Alfven-mode and/or slow-mode wave.

Show that if $\tilde{V}_{1x} = \tilde{V}_{1y} = \tilde{V}_{1z} = 0$, but $\tilde{\rho}_1 \neq 0$ and $\tilde{\mathbf{B}}_1 \neq 0$, then ω must be zero ($\omega = 0$), and $-(i\mathbf{k})\tilde{p}_1 + \tilde{\mathbf{J}}_1 \times \mathbf{B}_0 = 0$.

Proof:

For $\tilde{V}_{1x} = \tilde{V}_{1y} = \tilde{V}_{1z} = 0$, equation (6.10) or (6.14) is automatically fulfilled.

Substituting $\tilde{\mathbf{V}}_1 = 0$ into equation (6.5) yields $\tilde{\mathbf{E}}_1 = 0$.

Substituting $\tilde{\mathbf{V}}_1 = 0$ into equation (6.1) yields $\omega\tilde{\rho}_1 = 0$.

Substituting $\tilde{\mathbf{E}}_1 = 0$ into equation (6.8) yields $\omega\tilde{\mathbf{B}}_1 = 0$.

Thus, if $\rho_1 \neq 0$ and $\mathbf{B}_1 \neq 0$, we must have $\omega = 0$.

Substituting $\tilde{\mathbf{V}}_1 = 0$ into equation (6.2) yields

$$-(i\mathbf{k})\tilde{p}_1 + \tilde{\mathbf{J}}_1 \times \mathbf{B}_0 = 0 \tag{6.2a}$$

Substituting equation (6.9) into equation (6.2a) yields

$$-\mathbf{k}\tilde{p}_1 - \frac{\mathbf{k}(\tilde{\mathbf{B}}_1 \cdot \mathbf{B}_0)}{\mu_0} + \frac{(\mathbf{k} \cdot \mathbf{B}_0)\tilde{\mathbf{B}}_1}{\mu_0} = 0 \tag{6.2b}$$

For $\tilde{\mathbf{B}}_1 \neq 0$, equation (6.7) implies $\tilde{\mathbf{B}}_1 \perp \mathbf{k}$, thus equation (6.2b) can be decomposed into two parts. One of them is parallel or anti-parallel to the direction of $\mathbf{k}$. The other one is in the direction of $\tilde{\mathbf{B}}_1$. That is

$$-\mathbf{k}(\tilde{p}_1 + \frac{\tilde{\mathbf{B}}_1 \cdot \mathbf{B}_0}{\mu_0}) = 0 \tag{6.2c}$$

and

$$(\mathbf{k} \cdot \mathbf{B}_0)\tilde{\mathbf{B}}_1 = 0 \tag{6.2d}$$

Equation (6.2d) implies that if $\tilde{\mathbf{B}}_1 \neq 0$ then $\mathbf{k} \cdot \mathbf{B}_0 = 0$. (Likewise, if $\mathbf{k} \cdot \mathbf{B}_0 \neq 0$ then $\tilde{\mathbf{B}}_1 = 0$. This is the entropy mode discussed before.)

In summary, there are three types of non-propagating wave mode ($\omega = 0$) in MHD plasma :

1. Perpendicular-propagated Alfvén-mode wave, which is characterized by $\omega = 0$, $\tilde{\mathbf{B}}_1 \neq 0$, $\tilde{\rho}_1 = \tilde{p}_1 = 0$, $\tilde{\mathbf{B}}_1 \cdot \mathbf{B}_0 = 0$ and $\mathbf{k} \cdot \mathbf{B}_0 = 0$.
2. Perpendicular-propagated slow-mode wave, which is characterized by $\omega = 0$, $\tilde{\mathbf{B}}_1 \neq 0$, $\tilde{p}_1 \neq 0$, $\tilde{\mathbf{B}}_1 \cdot \mathbf{B}_0 \neq 0$, and $\mathbf{k} \cdot \mathbf{B}_0 = 0$.
3. Entropy mode, which is characterized by $\omega = 0$, $\tilde{\rho}_1 \neq 0$, $\tilde{p}_1 = 0$, $\tilde{\mathbf{B}}_1 = 0$, and $\mathbf{k} \cdot \mathbf{B}_0 \neq 0$.