**Lecture 4. Frozen-in Flux in Magnetohydrodynamic Plasma**

Ideal magnetohydrodynamic (MHD) plasma model is applicable to study plasma phenomena in low-frequency and long-wavelength limit.

Ohm's Law in MHD limit (low-frequency, long-wavelength limit) can be written as $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$.

**Exercise 4.1.**

For steady state ($\partial / \partial t = 0$) plasma, we have $\mathbf{E} = -\nabla\Phi$. Show that electrostatic potential is constant along streamline and magnetic filed line *in steady state MHD plasma*. (i.e., Constant potential surface is determined by a set of streamlines and magnetic field lines *in steady state MHD plasma*.)

Using two different approaches, we are going to show in this lecture that if the plasma fluid satisfies $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$ then the magnetic flux is frozen-in the plasma, i.e.,

$$\frac{d\Phi_B}{dt} = \frac{d}{dt} \int\int_{S(t)} \mathbf{B}(\mathbf{x},t) \cdot d\mathbf{a} = 0 . \qquad (4.1)$$

where $d/dt$ is a physical notation (but not a mathematical notation) of time derivatives along the path of a fluid element.

The following three equations are the sufficient conditions of Eq.(4.1).

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \qquad \text{(MHD Ohm's Law, or MHD approximation)} \qquad (4.2)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(No magnetic monopole)} \qquad (4.3)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law)} \qquad (4.4)$$

**4.1. Proof of Frozen-in Flux (Method 1)**

By definition, variation of magnetic flux along the path lines of fluid elements is

$$\frac{d\Phi_B}{dt} = \frac{d}{dt} \int\int_{S(t)} \mathbf{B} \cdot d\mathbf{a} = \lim_{\Delta t \to 0} \frac{\int\int_{S(t+\Delta t)} \mathbf{B}(x, t + \Delta t) \cdot d\mathbf{a} - \int\int_{S(t)} \mathbf{B}(x,t) \cdot d\mathbf{a}}{\Delta t}$$

From $\nabla \cdot \mathbf{B} = 0$ yields

$$0 = \iiint \nabla \cdot \mathbf{B} d^3x = \oint\!\!\oint \mathbf{B}(x,t+\Delta t) \cdot d\mathbf{a}$$
$$= \int\int_{S(t+\Delta t)} \mathbf{B}(x,t+\Delta t) \cdot d\mathbf{a} - \int\int_{S(t)} \mathbf{B}(x,t+\Delta t) \cdot d\mathbf{a} + \oint \mathbf{B}(x,t+\Delta t) \cdot (d\mathbf{l} \times \mathbf{V}\Delta t)$$
$$= \int\int_{S(t+\Delta t)} \mathbf{B}(x,t+\Delta t) \cdot d\mathbf{a} - \int\int_{S(t)} \mathbf{B}(x,t+\Delta t) \cdot d\mathbf{a} + \oint d\mathbf{l} \cdot [\mathbf{V}\Delta t \times \mathbf{B}(x,t+\Delta t)]$$

or

$$\int\int_{S(t+\Delta t)} \mathbf{B}(x,t+\Delta t) \cdot d\mathbf{a} = \int\int_{S(t)} \mathbf{B}(x,t+\Delta t) \cdot d\mathbf{a} - \oint d\mathbf{l} \cdot [\mathbf{V}\Delta t \times \mathbf{B}(x,t+\Delta t)]$$

Thus

$$\frac{d\Phi_B}{dt} = \lim_{\Delta t \to 0} \frac{\int\int_{S(t+\Delta t)} \mathbf{B}(x,t+\Delta t) \cdot d\mathbf{a} - \int\int_{S(t)} \mathbf{B}(x,t) \cdot d\mathbf{a}}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{[\int\int_{S(t)} \mathbf{B}(x,t+\Delta t) \cdot d\mathbf{a} - \oint d\mathbf{l} \cdot [\mathbf{V}\Delta t \times \mathbf{B}(x,t+\Delta t)]] - \int\int_{S(t)} \mathbf{B}(x,t) \cdot d\mathbf{a}}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\int\int_{S(t)} [\mathbf{B}(x,t+\Delta t) - \mathbf{B}(x,t)] \cdot d\mathbf{a}}{\Delta t} - \oint d\mathbf{l} \cdot (\mathbf{V} \times \mathbf{B})$$
$$= \int\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} - \oint d\mathbf{l} \cdot (\mathbf{V} \times \mathbf{B})$$
$$= \int\int_S (-\nabla \times \mathbf{E}) \cdot d\mathbf{a} - \oint d\mathbf{l} \cdot (\mathbf{V} \times \mathbf{B})$$
$$= \oint d\mathbf{l} \cdot (-\mathbf{E}) - \oint d\mathbf{l} \cdot (\mathbf{V} \times \mathbf{B})$$
$$= \oint d\mathbf{l} \cdot (\mathbf{V} \times \mathbf{B}) - \oint d\mathbf{l} \cdot (\mathbf{V} \times \mathbf{B})$$
$$= 0$$

We have proved that

$$\frac{d\Phi_B}{dt} = \frac{d}{dt} \int\int_{S(t)} \mathbf{B} \cdot d\mathbf{a} = 0$$

## 4.2. Proof of Frozen-in Flux (Method 2)

Since $\nabla \cdot \mathbf{B} = 0$, we can let $\mathbf{B} = \nabla \times \mathbf{A}$. Thus Eq. (4.4) becomes

$$\mathbf{E}^{EM} = -\frac{\partial \mathbf{A}}{\partial t}$$

Since $\mathbf{E}^{ES} = -\nabla \Phi$, we have $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi$, or $\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} - \nabla \Phi$

Therefore, variation of magnetic flux along path lines of fluid elements becomes

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}\iint_{S(t)} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \frac{d}{dt}\oint_{c(S)} \mathbf{A} \cdot d\mathbf{l}$$
$$= \oint_c (\frac{\partial \mathbf{A}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_c \mathbf{A} \cdot \frac{d}{dt}[\mathbf{r}(s + \Delta s, t) - \mathbf{r}(s,t)]$$
$$= \oint_c (\frac{\partial \mathbf{A}}{\partial t}) \cdot d\mathbf{l} + \oint_c (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_c \mathbf{A} \cdot [\mathbf{V}(s + \Delta s, t) - \mathbf{V}(s,t)]$$
$$= \oint_c (-\mathbf{E} - \nabla\Phi) \cdot d\mathbf{l} + \oint_c (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_c \mathbf{A} \cdot \frac{[\mathbf{V}(s + \Delta s, t) - \mathbf{V}(s,t)]}{\Delta s}(\Delta s)$$
$$= \oint_c (\mathbf{V} \times \mathbf{B}) \cdot d\mathbf{l} + \oint_c (-\nabla\Phi) \cdot d\mathbf{l} + \oint_c (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_c \mathbf{A} \cdot [d\mathbf{l} \cdot (\nabla \mathbf{V})]$$
$$= \oint_c [\mathbf{V} \times (\nabla \times \mathbf{A})] \cdot d\mathbf{l} - \oint_c d\Phi + \oint_c (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_c d\mathbf{l} \cdot [\nabla(\mathbf{A}^c \cdot \mathbf{V})]$$
$$= \oint_c \{-\mathbf{V} \cdot \nabla \mathbf{A} + [\nabla(\mathbf{A} \cdot \mathbf{V}^c)]\} \cdot d\mathbf{l} + \oint_c (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_c d\mathbf{l} \cdot [\nabla(\mathbf{A}^c \cdot \mathbf{V})]$$
$$= \oint_c [d\mathbf{l} \cdot \nabla(\mathbf{A} \cdot \mathbf{V}^c)] + \oint_c d\mathbf{l} \cdot [\nabla(\mathbf{A}^c \cdot \mathbf{V})]$$
$$= \oint_c d\mathbf{l} \cdot \nabla(\mathbf{A} \cdot \mathbf{V})$$
$$= \oint_c d(\mathbf{A} \cdot \mathbf{V})$$
$$= 0$$

We have proved that

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}\iint_{S(t)} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \frac{d}{dt}\oint_{c(S)} \mathbf{A} \cdot d\mathbf{l} = 0$$

## 4.3. Conservation of Circulation vs. Frozen-in Flux in MHD Plasma

The idea of frozen-in flux of MHD plasma is adopt from conservation of circulation in an ideal fluid, where we define an ideal fluid is a non-viscous and isentropic fluid. (e.g., Landau and Lifshitz (Fluid Mechanics, 2nd ed. 1989)

Momentum equation of a non-viscous fluid is

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{\nabla p}{\rho} + \mathbf{g} \tag{4.5a}$$

or

$$\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V} \times (\nabla \times \mathbf{V}) + \nabla \frac{V^2}{2} = -\frac{\nabla p}{\rho} - \nabla \Phi_g \tag{4.5b}$$

Vorticity equation of a non-viscous fluid can be obtained from curl of the momentum equation (4.5b)

$$\frac{\partial \nabla \times \mathbf{V}}{\partial t} - \nabla \times [\mathbf{V} \times (\nabla \times \mathbf{V})] + \nabla \times [\nabla \frac{V^2}{2}] = -\nabla \times (\frac{\nabla p}{\rho}) - \nabla \times \nabla \Phi_g$$

Since $\nabla \times \nabla f = 0$, the above equation can be simplified as

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \nabla \times [\mathbf{V} \times \mathbf{\Omega}] = \frac{\nabla \rho \times \nabla p}{\rho^2} \tag{4.6}$$

where $\mathbf{\Omega} = \nabla \times \mathbf{V}$.

For an isentropic fluid,

$$\nabla w = \frac{\nabla p}{\rho}$$

Thus,

$$\nabla \times \frac{\nabla p}{\rho} = \nabla \times \nabla w = 0$$

Thus, for an ideal fluid, the vorticity equation (4.6) becomes

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \nabla \times [\mathbf{V} \times \mathbf{\Omega}] = 0 \tag{4.7}$$

where $\nabla \cdot \mathbf{\Omega} = 0$. From equation (4.7) one can show the conservation of circulation along the path line of an ideal fluid element

$$\frac{d\Gamma}{dt} = 0 = \frac{d}{dt}\oint \mathbf{V} \cdot d\mathbf{l} = \frac{d}{dt}\iint (\nabla \times \mathbf{V}) \cdot d\mathbf{a} = \frac{d}{dt}\iint \mathbf{\Omega} \cdot d\mathbf{a} \tag{4.8}$$

Equation (4.7) is similar to the combination of MHD Ohm's law and Faraday's law, i.e.,

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times [\mathbf{V} \times \mathbf{B}] = 0 \tag{4.9}$$

where $\nabla \cdot \mathbf{B} = 0$.

Likewise, equation (4.8) is similar to the result we obtained in the last two sections

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}\int\int_{S(t)} \mathbf{B} \cdot d\mathbf{a} = \frac{d}{dt}\oint_{c(S)} \mathbf{A} \cdot d\mathbf{l} = 0 \tag{4.10}$$