Lecture 2. Dipole Field

2.1. Review of Dipole Electric Field

An *electric dipole* consists of a pair of $+q$ and $-q$ charge particles, which are separated by a distance d. The magnitude of the *dipole moment* of this *electric dipole* is $p = qd$. The direction of the *electric dipole moment* is along the direction from the negative charge to the positive charge. Electric field generated by the pair of charge particles at a distance r, with $r >> d$, is called *dipole electric field*.

Exercise 2.1.

Based on the Poisson equation, show that a single charge particle q, located at the origin ($r = 0$), can result in an electric field at $\mathbf{r} = \hat{r}r$, which satisfies the following equation

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{q}{4\pi\varepsilon_0 r^3}\mathbf{r} \tag{2.1}$$

To find the general form of *dipole electric field*, let us consider a charge $+q$ located at $(x, y, z) = (0, 0, d/2)$, and another charge $-q$ located at $(x, y, z) = (0, 0, -d/2)$. From Eq. (2.1), it yields electric field at $\mathbf{r} = \hat{r}r$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r_{+q}^3}\mathbf{r}_{+q} + \frac{-q}{4\pi\varepsilon_0 r_{-q}^3}\mathbf{r}_{-q} \tag{2.2}$$

where

$$\mathbf{r}_{+q} = \mathbf{r} - \hat{z}\frac{d}{2} = \hat{x}\sin\theta + \hat{z}\cos\theta - \hat{z}\frac{d}{2}, \tag{2.3}$$

$$\mathbf{r}_{-q} = \mathbf{r} - (-\hat{z})\frac{d}{2} = \hat{x}\sin\theta + \hat{z}\cos\theta + \hat{z}\frac{d}{2}, \tag{2.4}$$

$$r_{+q} = \sqrt{r^2\sin^2\theta + (r\cos\theta - \frac{d}{2})^2} = r\sqrt{1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}},$$

and

$$r_{-q} = \sqrt{r^2\sin^2\theta + (r\cos\theta + \frac{d}{2})^2} = r\sqrt{1 + \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}$$

For $r >> d$, it yields

$$r_{+q}^{-3} = r^{-3}\left(1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)^{-\frac{3}{2}} \approx r^{-3}(1 + \frac{3}{2}\frac{d}{r}\cos\theta) \tag{2.5}$$

and

$$r_{-q}^{-3} = r^{-3}\left(1 + \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)^{-\frac{3}{2}} \approx r^{-3}(1 - \frac{3}{2}\frac{d}{r}\cos\theta) \tag{2.6}$$

Substituting Eqs. (2.3), (2.4), (2.5) and (2.6) into Eq. (2.2), it yields,

$$\begin{aligned}
\mathbf{E}_{dipole}(\mathbf{r}) &= \frac{q}{4\pi\varepsilon_0 r^3}[(1 + \frac{3}{2}\frac{d}{r}\cos\theta)(\mathbf{r} - \hat{z}\frac{d}{2}) - (1 - \frac{3}{2}\frac{d}{r}\cos\theta)(\mathbf{r} + \hat{z}\frac{d}{2})] \\
&= \frac{q}{4\pi\varepsilon_0 r^3}[\mathbf{r}3\frac{d}{r}\cos\theta - \hat{z}d + \hat{z}\frac{3d^2}{2r}\cos\theta] \\
&\approx \frac{qd}{4\pi\varepsilon_0 r^3}[\hat{r}3\cos\theta - \hat{z}]
\end{aligned}$$

For $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ and $p_z = qd$, we can rewrite the *dipole electric field* to the following form:

$$\mathbf{E}_{dipole}(\mathbf{r}) = \frac{p_z}{4\pi\varepsilon_0 r^3}[\hat{r}2\cos\theta + \hat{\theta}\sin\theta] \tag{2.7}$$

2.2. Summary of Dipole Electric Field and Dipole Magnetic Field

Dipole electric field obtained in section 2.1 is

$$\mathbf{E}_{dipole}(r,\theta) = (\frac{p_z}{4\pi\varepsilon_0})\frac{1}{r^3}(\hat{r}2\cos\theta + \hat{\theta}\sin\theta) \tag{2.7}$$

where $\mathbf{p} = \hat{z}\, p_z = \hat{z}\, qd$ is the *electric dipole moment*, d is the distance between the $+q$ and $-q$ charge particles, which are located near $r = 0$, and r in Eq. (2.7) satisfies $r >> d$.

Likewise, dipole magnetic field can be written as

$$\mathbf{B}_{dipole}(r,\theta) = (\frac{\mu_0\mu_z}{4\pi})\frac{1}{r^3}(\hat{r}2\cos\theta + \hat{\theta}\sin\theta) \tag{2.8}$$

where $\boldsymbol{\mu} = \hat{z}\,\mu_z = \hat{z}\, I_\phi A_z$ is the *magnetic dipole moment*, A_z is the area enclosed by the electric current loop I_ϕ, which are located near $r = 0$, and r in Eq. (2.8) satisfies $r >> \sqrt{A_z}$.

For convenience, we can rewrite Earth's dipole magnetic field in the following form

$$\mathbf{B}_{dipole}(r,\theta) = \frac{-M_E}{r^3}(\hat{r}\,2\cos\theta + \hat{\theta}\sin\theta) \tag{2.9}$$

where $-M_E = \mu_0\mu_z / 4\pi$, the z-axis is along the Earth dipole axis but pointed to the northern hemisphere, θ is the magnetic co-latitude and r is the radial distance from the center of Earth. During the quite time period, the magnetic field in the inner magnetosphere ($1R_E < r < 6R_E$) is close to a dipole magnetic field. But the dipole axis of the Earth is not constant with time. Moreover, due to the volcano activities under the Atlantic ocean, which recorded the polarity changes of the Earth dipole axis in the past billion years, the Earth's magnetic field is not a perfect dipole field near the Earth surface ($r \approx 1R_E$).

Figure 2.1 shows the pole-ward movement of the Earth's magnetic dipole axis on the surface of Earth in the northern hemisphere during the past 170 years. Based on the aurora display recorded in the Chinese history, I believe that the Earth's dipole axis swapped back and forth between the western and eastern hemispheres with a period around 700~800 years in the last 5000 years.

Table 2.1 shows the estimated location (the geographic latitude and the geographic longitude) of the magnetic dipole axis on the surface of Earth in the northern hemisphere in years 2001~2005.

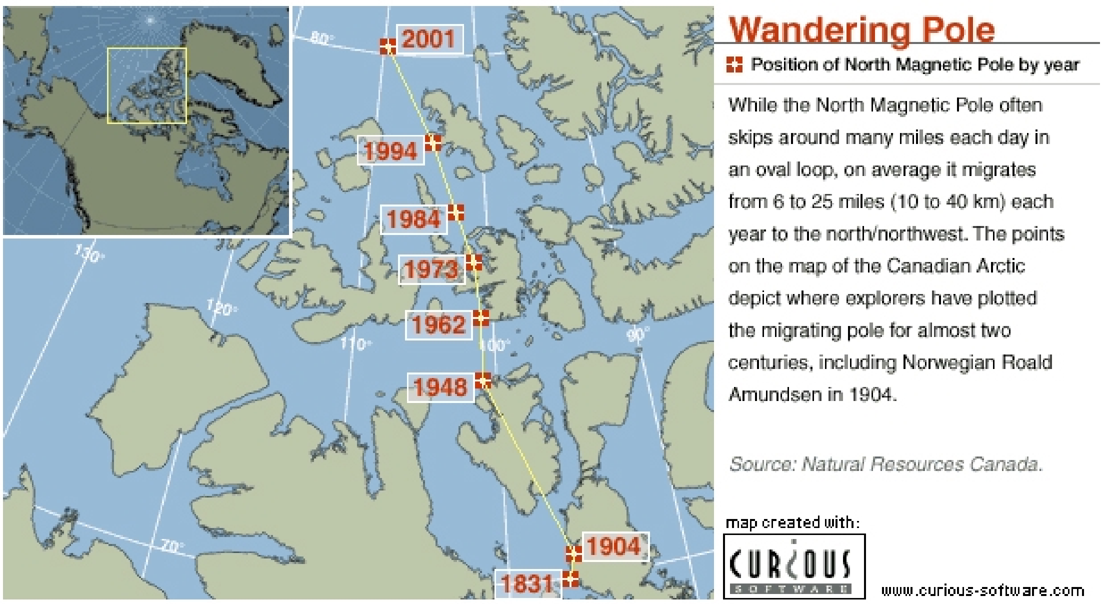

Figure 2.1. Wandering of magnetic pole in northern hemisphere from 1831-2001. (Source: http://www.cnn.com)

Table 2.1. Location of magnetic dipole in northern hemisphere on Earth surface

Year	Latitude	Longitude (W)
2001	81.3	110.8
2002	81.6	111.6
2003	82.0	112.4
2004	82.3	113.4
2005	82.7	114.4

(Source: Canadian Geologic Survey, http://www.ngdc.noaa.gov/seg/potfld/faqgeom.shtml)

Figures 2.2 shows contours of geomagnetic coordinates in year 2001. The contours of the geomagnetic latitudes and longitudes can be obtained based on the given geographic latitude and longitude of the dipole axis shown in Table 2.1.

Figure 2.3 shows contour plot of the total geomagnetic field strength on Earth surface in year 2001. As we can see that the minimum of the field strength is not parallel to the geomagnetic equator, which is the green curve in Figure 2.2. A minimum of the magnetic field strength can be found in the South American and the South Atlantic Ocean. Ionosphere plasma often shows south Atlantic anomaly (SSA) in this minimum B region.

Figure 2.4 shows the definitions of the declination angle, inclination angle, and the H, D, X, Z components of the geomagnetic field. Figure 2.5 shows the contour plots of declination angle and inclination angle of geomagnetic field in year 2001. The declination angle and inclination angle provide useful information for ground observations. It can help us to determine the average magnetic field-aligned direction at different locations.

Exercise 2.2.

From Figure 2.2 and Figure 2.5, estimate the geography latitude, the geomagnetic latitude, and the inclination angle of Taiwan in year 2001.

Exercise 2.3.

Do you expect that the poleward movement of the Earth's dipole axis may change the altitude, latitude, and longitude of the ionospheric targets observed by the Chung-li VHF Radar?

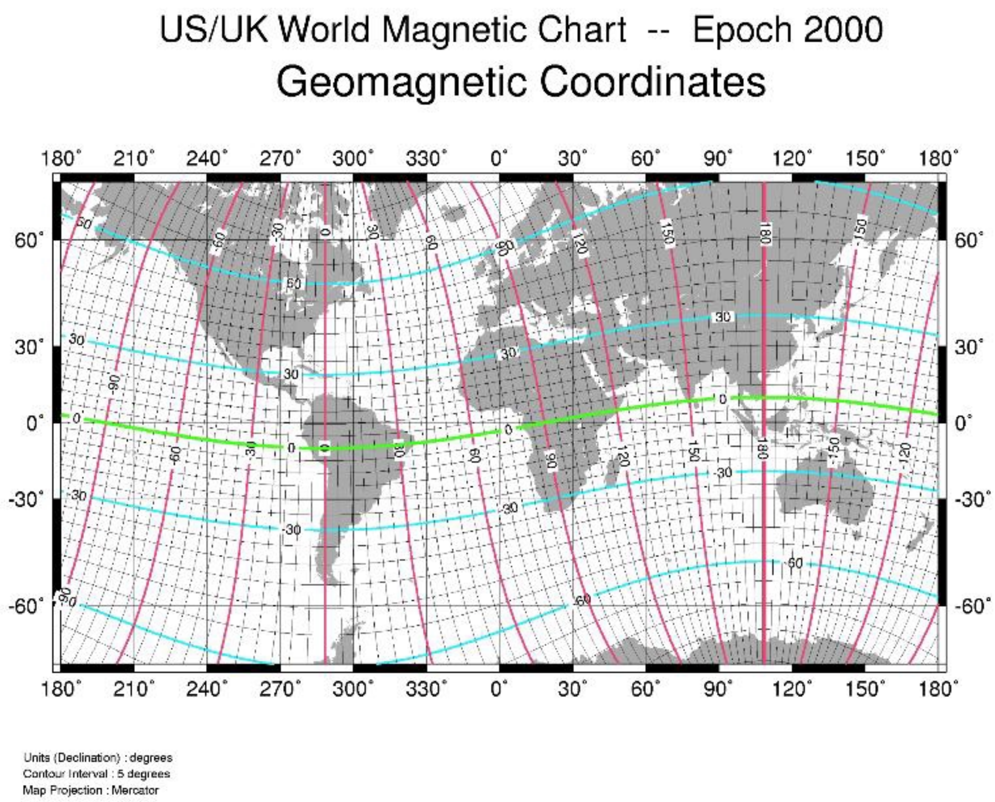

Figure 2.2. Contours of geomagnetic coordinates in year 2001. (Source: http://www.ngdc.noaa.gov/seg/potfld/faqgeom.shtml)

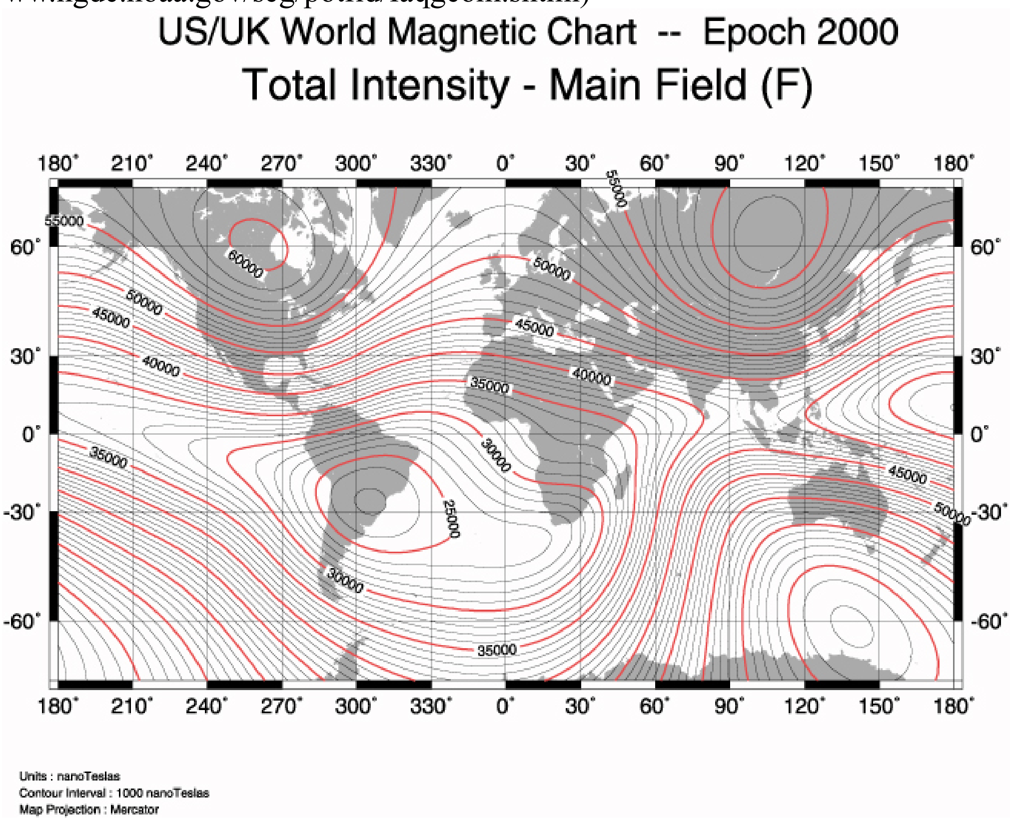

Figure 2.3. Contours of total geomagnetic field strength on Earth surface in year 2001. (Source: http://www.ngdc.noaa.gov/seg/potfld/faqgeom.shtml)

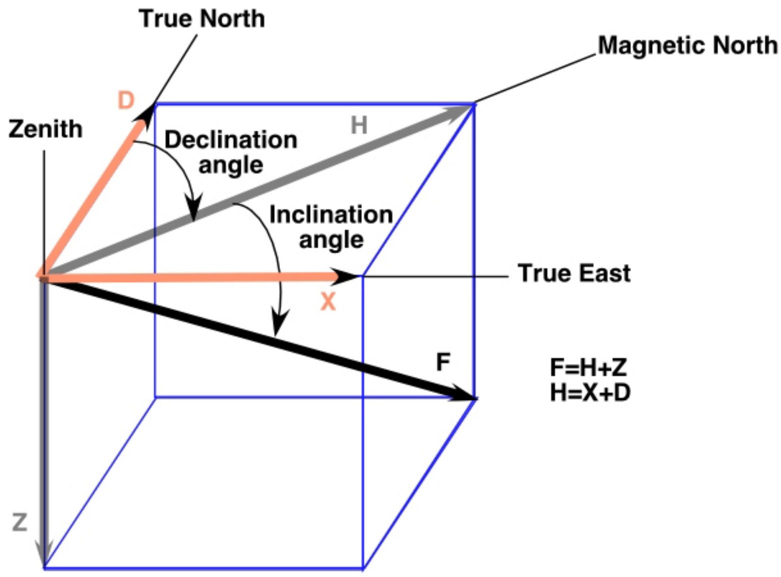

Figure 2.4. Illustration of seven geomagnetic parameters: total field strength (F), Z-component, H-component, D-component, X-component, inclination angle, and declination angle of the geomagnetic field.

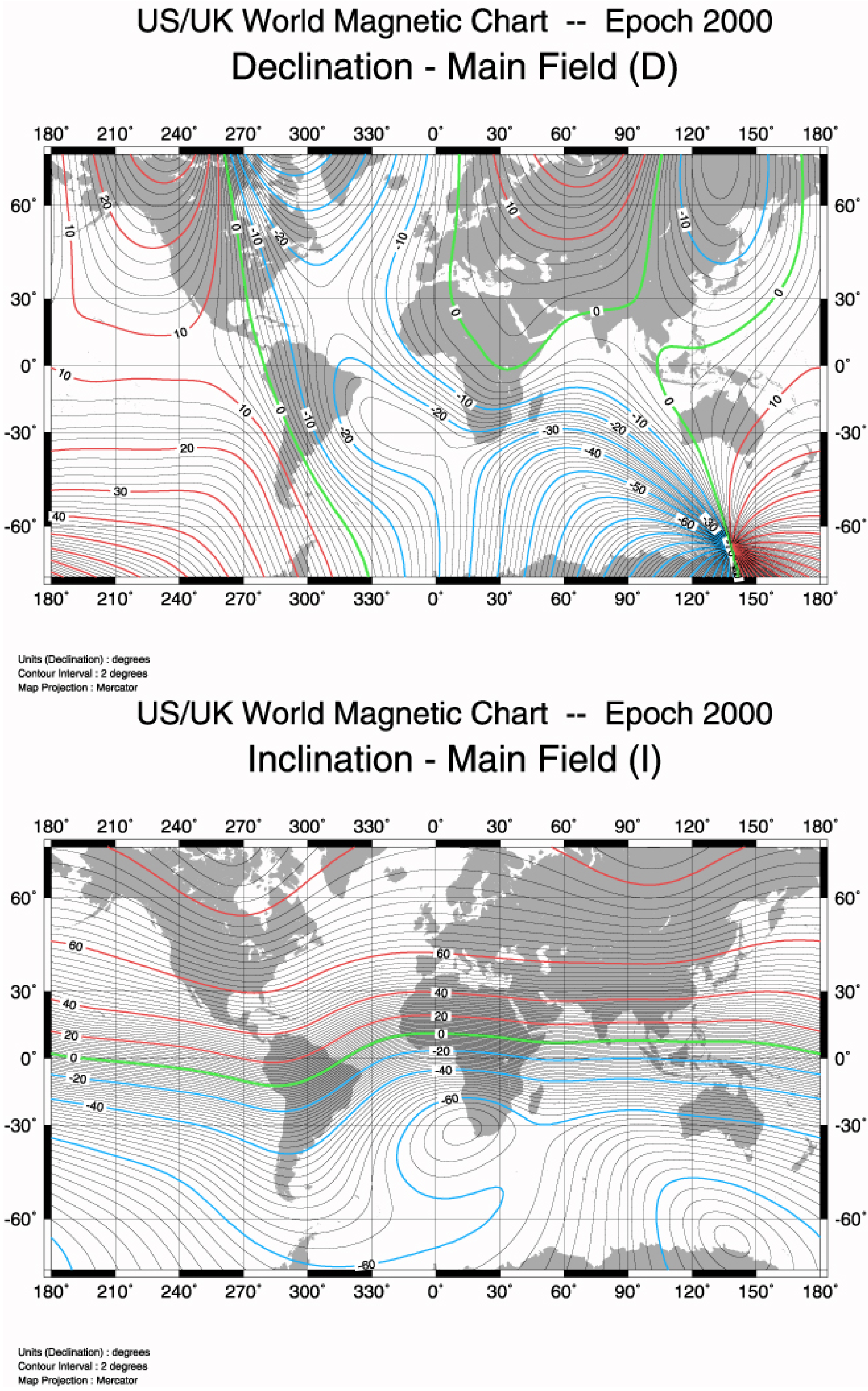

Figure 2.5. Contours of declination angle and inclination angle of geomagnetic field in year 2001. (Source: http://www.ngdc.noaa.gov/seg/potfld/faqgeom.shtml)

2.3. Field-line equation for the dipole magnetic field

Magnetic field is a vector field. Magnetic field line is the trace of magnetic field vectors. Let $d\mathbf{s} = \hat{r}\, dr + \hat{\theta}\, r d\theta$ to be the tangent vector of a magnetic field line. From the definition of magnetic field line, we have $d\mathbf{s} // \mathbf{B}$, or

$$\frac{dr}{B_r} = \frac{r d\theta}{B_\theta} = \frac{ds}{B} \tag{2.10}$$

For

$$\mathbf{B}(r,\theta) = \hat{r}\, B_r(r,\theta) + \hat{\theta}\, B_\theta(r,\theta) \tag{2.11}$$

Eq. (2.9) yields

$$B_r(r,\theta) = \frac{-M_E}{r^3} 2\cos\theta \tag{2.12}$$

$$B_\theta(r,\theta) = \frac{-M_E}{r^3} \sin\theta \tag{2.13}$$

Substituting Eqs. (2.12) and (2.13) into Eq. (2.10), yields

$$\frac{dr}{2\cos\theta} = \frac{r d\theta}{\sin\theta}$$

or

$$\frac{dr}{r} = \frac{2\cos\theta d\theta}{\sin\theta} = \frac{2 d\sin\theta}{\sin\theta}$$

or

$$d\ln r = d\ln(\sin^2\theta) \tag{2.14}$$

Integrating Eq. (2.14) once, it yields

$$r(\theta) = r_{eq}\sin^2\theta \tag{2.15}$$

where $r_{eq} = r(\theta = \pi/2)$ is the radial distance of the magnetic field line from the center of Earth on the magnetic equatorial plane. Eq. (2.15) is the equation of dipole magnetic field line. For convenience, space scientists assign an L-value to the magnetic field lines, which pass through magnetic equatorial plane at $r_{eq} = LR_E$, where R_E is the Earth's radius.

Exercise 2.4.

Determine the foot-point magnetic co-latitude θ on the Earth's surface of a dipole magnetic filed line with a given L value.

Exercise 2.5.

Based on results obtained in Exercise 2.4, determine the foot-point magnetic latitude ($90° - \theta$) of dipole magnetic filed line with L-value equal to 2, 3, 4, 5, 6, 7, 8, 9, 10, respectively.

Exercise 2.6.

For a given L-value, determine the length of the magnetic file line between two foot points on Earth's surface.

Exercise 2.7.

For a given L-value, determine the ratio of magnitudes of magnetic field on Earth's surface and at magnetic equatorial plane.

Exercise 2.8.

Based on the results obtained in Exercise 2.6 and Exercise 2.7, estimate plasma density profile as a function of height $h = r - R_E$ (or a function of r) in the inner magnetosphere at midnight magnetic equator (i.e., ignore photon ionization effect.)

2.4. Co-rotating E-field

A magnetohydodynamic (MHD) plasma is a simplified plasma model at low-frequency and long-wavelength limit. Consider time scale much longer than the Alfven wave traveling time along the magnetic field line, we can assume that a magnetized plasma system satisfies the MHD Ohm s law, that is

$\mathbf{E} = -\nabla\phi = -\mathbf{V}\times\mathbf{B}$.

Thus, for the MHD plasma, both magnetic field lines and streamlines are equal potential lines. Namely, if a perpendicular electric field (i.e., E-field which is perpendicular to the local magnetic field) is generated at one end of the magnetic field line, an Alfevn wave will carry this information and propagate along the magnetic field line to make the electric potential to be constant along the magnetic field line. Likewise, the fast mode wave in the MHD plasma will carry the electric field information along the streamlines to make the streamlines in the MHD plasma to be equal-potential lines.

We are going to show in this section that if the magnetic field in the inner magnetosphere of the Earth satisfies the dipole magnetic field model, then plasma confined by the dipole magnetic field will co-rotate with the Earth due to the presence of a co-rotating electric field. From the dipole field configuration, we can estimate the perpendicular electric field distribution along the magnetic field line based on the electric field generated in the ionosphere.

The equation for dipole magnetic field line obtained in Eq. (2.15) can be rewritten as

$$r = r_{eq}\sin^2\theta(r) \tag{2.16}$$

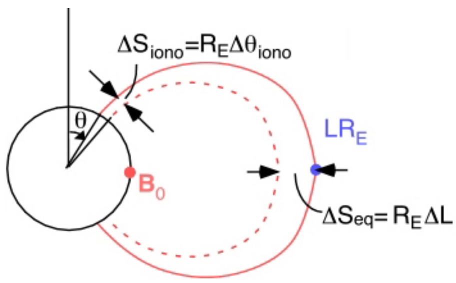

Figure 2.6. A sketch of the separation distance between two magnetic field lines along an *L*-value dipole magnetic field line.

Figure 2.6 sketches the separation distance between two magnetic field lines along an L-value dipole magnetic field line. The solid curve in Figure 2.6 is a dipole magnetic field line, which passes magnetic equatorial plane at $r_{eq} = LR_E$. Let this L-value field (solid curve) pass ionosphere at point $(r,\theta) \approx (R_E, \theta_{iono})$. Eq. (2.16) yields

$$R_E = LR_E \sin^2[\theta(r = R_E)] = LR_E \sin^2 \theta_{iono}$$

or

$$L \sin^2 \theta_{iono} = 1 \tag{2.17}$$

Differentiating Eq. (2.17) once, it yields,

$$\frac{\Delta L}{L} + \frac{2\Delta \sin\theta_{iono}}{\sin\theta_{iono}} = 0$$

or

$$\Delta L = -L \frac{2\cos\theta_{iono} \Delta\theta_{iono}}{\sin\theta_{iono}} \tag{2.18}$$

The dash curve in Figure 2.6 is a dipole magnetic field line, which passes magnetic equatorial plane at $r_{eq} = (L + \Delta L) R_E$, where $\Delta L < 0$. Let this field line pass ionosphere at $(r,\theta) = (R_E, \theta_{iono} + \Delta\theta_{iono})$. The distance between the solid curve and the dash curve at the equatorial plane is

$$\Delta S_{eq} = -R_E \Delta L \tag{2.19}$$

where $\Delta L < 0$. The distance between the solid curve and the dash curve in the ionosphere is approximately

$$\Delta S_{iono} \approx R_E \, \Delta\theta_{\,iono} \tag{2.20}$$

Substituting Eq. (2.18) into Eq. (2.19) to eliminate ΔL, then substituting Eq. (2.20) into the resulting equation to eliminate $\Delta\theta_{\,iono}$, it yields

$$\Delta S_{eq} = -R_E \Delta L = -(-L \frac{2\cos\theta_{iono} \Delta\theta_{iono}}{\sin\theta_{iono}}) R_E = \frac{2\cos\theta_{iono}}{\sin^3 \theta_{iono}} \Delta S_{iono}$$

or

$$\frac{\Delta S_{eq}}{\Delta S_{iono}} = \frac{2\cos\theta_{iono}}{\sin^3 \theta_{iono}} \tag{2.21}$$

The electric field generated at the ionosphere is

$$\begin{aligned} \mathbf{E}_{iono} = -\mathbf{V}_{iono} \times \mathbf{B}_{iono} &= -(\omega_E R_E \sin\theta_{iono} \hat{\phi}) \times (-B_0 2\cos\theta_{iono} \hat{r} - B_0 \sin\theta_{iono} \hat{\theta}) \\ &= +\hat{\theta} \omega_E R_E \sin\theta_{iono} B_0 2\cos\theta_{iono} - \hat{r} \omega_E R_E \sin^2 \theta_{iono} B_0 \end{aligned} \tag{2.22}$$

where ω_E is the angular velocity of Earth rotating and B_0 is the strength of the dipole magnetic field at the equator of Earth's surface as indicated in Figure 2.6. The potential jump between the solid curve and the dash curve is

$$\Delta\phi = \mathbf{E}_{iono} \cdot (\hat{\theta}\Delta S_{iono}) = \omega_E R_E \sin\theta_{iono} B_0 2\cos\theta_{iono} \Delta S_{iono} \tag{2.23}$$

Since the magnetic field lines are equal potential lines, the potential jump between the solid curve and the dash curve can also be written as

$$\Delta\phi = \mathbf{E}_{eq} \cdot (-\hat{r})\Delta S_{eq}$$

i.e.,

$$\mathbf{E}_{eq} = (-\hat{r})\frac{\Delta\phi}{\Delta S_{eq}} \tag{2.24}$$

Substituting Eq. (2.23) into Eq. (2.24) to eliminate $\Delta\phi$, and then substituting Eq. (2.21) into the resulting equation to eliminate $\Delta S_{iono} / \Delta S_{eq}$, it yields

$$\mathbf{E}_{eq} = (-\hat{r})(\omega_E R_E \sin\theta_{iono} B_0 2\cos\theta_{iono}) / (\frac{2\cos\theta_{iono}}{\sin^3\theta_{iono}}) = (-\hat{r})\omega_E R_E B_0 \sin^4\theta_{iono} \tag{2.25}$$

Substituting Eq. (2.17) into Eq. (2.25) to eliminate $\sin^4\theta_{iono}$, it yields

$$\mathbf{E}_{eq} = -\hat{r}\frac{\omega_E R_E B_0}{L^2} \tag{2.26}$$

The plasma flow velocity in the equatorial plane can be estimated from the $\mathbf{E}\times\mathbf{B}$ drift velocity, i.e.,

$$\mathbf{V}_{eq} = \frac{\mathbf{E}_{eq}\times\mathbf{B}_{eq}}{(B_{eq})^2} = (-\hat{r}\frac{\omega_E R_E B_0}{L^2})\times(-\hat{\theta}\frac{B_0}{L^3})/(\frac{B_0}{L^3})^2 = \hat{\phi}\omega_E R_E L = \hat{\phi}\omega_E r_{eq} \tag{2.27}$$

Namely, plasma at the equatorial plane moves at the same angular velocity as the solid Earth. Electric field obtained in Eq. (2.26) is called the *co-rotating E-field* in the inner magnetosphere.

Exercise 2.9

It has been shown in this Lecture that the co-rotating electric field in the Earth's inner magnetosphere is perpendicular to the local magnetic field and is characterized by a negative radial component. Show that the co-rotating electric field in the Jovian magnetosphere is perpendicular to the local magnetic field and is characterized by a positive radial component.