# Lecture 1. Dipole Magnetic Field and Equations of Magnetic Field Lines

## 1.1. Dipole Magnetic Field

Since $\nabla \cdot \mathbf{B} = 0$ we can define

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1.1}$$

where $\mathbf{A}$ is called the vector potential. We use the bold face font to denote vector. For static magnetic field, we have

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{1.2}$$

Substituting Eq. (1.1) into Eq. (1.2) to eliminate $\mathbf{B}$, it yields

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J} \tag{1.3}$$

We choose the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, the equation (1.3) can be rewritten as

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{1.4}$$

Note that:

If we choose the Coulomb gauge: $\nabla \cdot \mathbf{A} = 0$, the scalar potential will contain no electromagnetic component.

If we choose the Lorentz gauge: $\frac{1}{c^2}\frac{\partial \phi(\mathbf{x},t)}{\partial t} + \nabla \cdot \mathbf{A}(\mathbf{x},t) = 0$, the scalar potential will contain an electromagnetic component.

Eq. (1.4) is similar to the Poisson equation of the electrostatic potential

$$\nabla^2 \phi = -\frac{\rho_c}{\varepsilon_0} \tag{1.5}$$

General solution of Eq. (1.5) can be written as

$$\phi(\mathbf{r}) = \int \frac{\rho_c(\mathbf{r}')}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' \tag{1.6}$$

Special Case:

The scalar potential create by a point charge $q$ is

$$\phi = \frac{q}{4\pi\varepsilon_0 r}$$

Likewise, the general solution of Eq. (1.4) can be written as

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu_0 \mathbf{J}(\mathbf{r}')}{4\pi |\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' \tag{1.7}$$

**Exercise 1.1.**

Let $f(x)=(1+x)^{\alpha}$. For $0<|x|<1$, please determine the approximate polynomial expression of $f(x)$. That is, $f(x)\approx a_0+a_1x+a_2x^2+a_3x^3$
where a term with magnitude of the order of $O(x^4)$ has been ignored.
(a) Write down the general form of $a_0$, $a_1$, $a_2$, $a_3$ for a given $\alpha$
(b) For $\alpha=1/2$, write down the approximate polynomial expression of $f(x)$.
(c) For $\alpha=-1/2$, write down the approximate polynomial expression of $f(x)$.
(d) For $\alpha=-1$, write down the approximate polynomial expression of $f(x)$.
(e) For $\alpha=-2$, write down the approximate polynomial expression of $f(x)$.

*Answesr of Exercise 1.1*
(a) $a_0=1$ $a_1=\alpha$ $a_2=\alpha(\alpha-1)/2!$ $a_3=\alpha(\alpha-1)(\alpha-2)/3!$
(b) $f(x)=\sqrt{1+x}\approx 1+\frac{1}{2}x+\frac{-1}{8}x^2+\frac{1}{16}x^3$
(c) $f(x)=\frac{1}{\sqrt{1+x}}\approx 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3$
(d) $f(x)=\frac{1}{1+x}\approx 1-x+x^2-x^3$
(e) $f(x)=\frac{1}{(1+x)^2}\approx 1-2x+3x^2-4x^3$

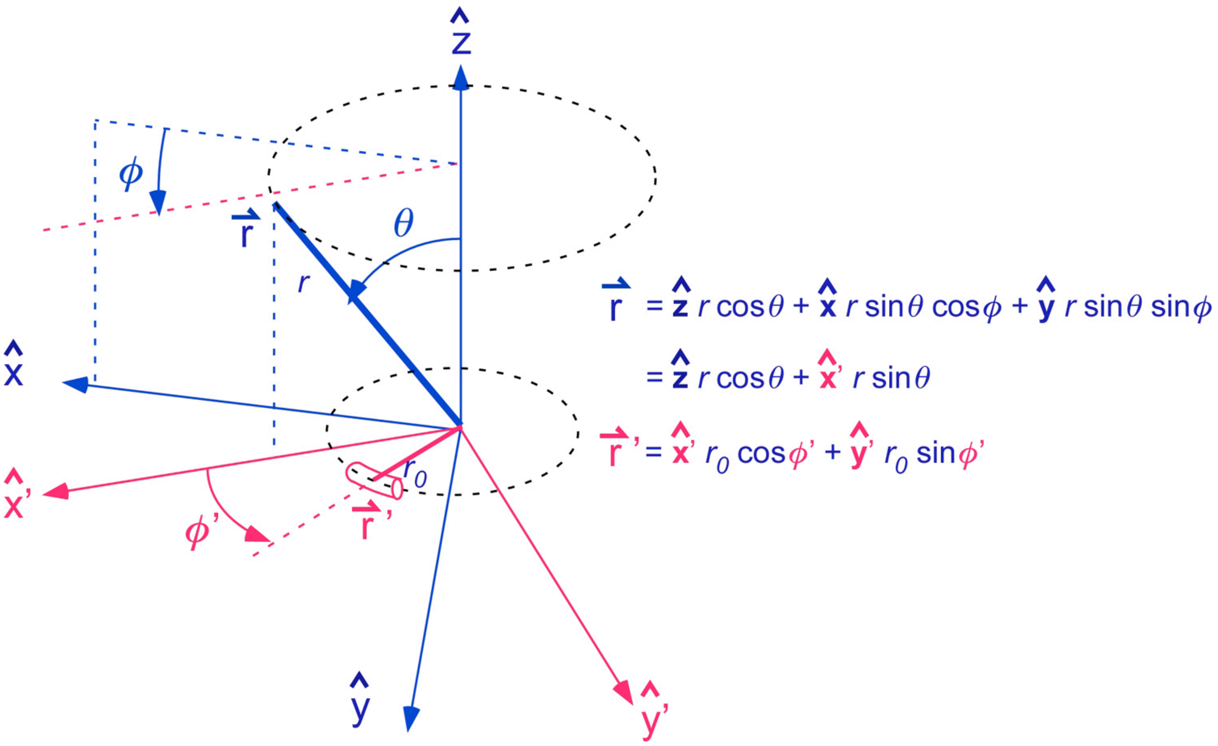


**Figure 1.1.** A coordinate system for the study of the field generate by a ring current

$$\mathbf{J}(\mathbf{r}')=\hat{\phi}'J_0\delta(r'-r_0)\delta(\theta'-\frac{\pi}{2})=(-\sin\phi'\hat{x}'+\cos\phi'\hat{y}')J_0\delta(r'-r_0)\delta(\theta'-\frac{\pi}{2})$$

Let us consider a coordinate system as illustrated in Figure 1.1, where

$$\mathbf{r} = r\cos\theta\hat{z} + r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} = r\cos\theta\hat{z} + r\sin\theta\hat{x}'$$

$$\mathbf{r}' = r_0\cos\phi'\hat{x}' + r_0\sin\phi'\hat{y}'$$

$$\hat{\phi}' = -\sin\phi'\hat{x}' + \cos\phi'\hat{y}'$$

$$\hat{\phi} = \hat{y}'$$

Given a ring current $\mathbf{J}(\mathbf{r}') = \hat{\phi}' J_0 \delta(r' - r_0)\delta(\theta' - \frac{\pi}{2}) = (-\sin\phi'\hat{x}' + \cos\phi'\hat{y}')J_0\delta(r' - r_0)\delta(\theta' - \frac{\pi}{2})$

The General solution of the vector potential $\mathbf{A}$ is (Jackson, section 5.5)

$$\begin{aligned}
&\mathbf{A}(\mathbf{x}) \\
&= A_\phi\hat{\phi} = A_\phi\hat{y}' \\
&= \int \frac{\mu_0 \mathbf{J}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' \\
&= \int_{r_0-\Delta r/2}^{r_0+\Delta r/2} dr' \int_{(\pi/2)-\Delta\theta/2}^{(\pi/2)+\Delta\theta/2} r' d\theta' \int_0^{2\pi} r'\sin\theta' d\phi' \frac{\mu_0 J_0 \delta(r'-r_0)\delta[\theta'-(\pi/2)]\hat{\phi}'}{4\pi\sqrt{r^2\cos^2\theta + (r\sin\theta - r_0\cos\phi')^2 + (r_0\sin\phi')^2}} \\
&= \frac{\mu_0 J_0 (\Delta r)(r_0\Delta\theta) r_0}{4\pi} \int_0^{2\pi} \frac{-\sin\phi'\hat{x}' + \cos\phi'\hat{y}'}{\sqrt{r^2 + r_0^2 - 2rr_0\sin\theta\cos\phi'}} d\phi'
\end{aligned} \tag{1.8}$$

Let $I_0 = J_0(\Delta r)(r_0\Delta\theta)$, $\alpha = r^2 + r_0^2$, $\beta = 2rr_0\sin\theta$, Eq. (1.8) can be written as

$$\begin{aligned}
A_\phi\hat{y}' &= \frac{\mu_0 I_0 r_0}{4\pi}(\hat{x}'\int_0^{2\pi} \frac{-\sin\phi'}{\sqrt{\alpha - \beta\cos\phi'}} d\phi' + \hat{y}'\int_0^{2\pi} \frac{\cos\phi'}{\sqrt{\alpha - \beta\cos\phi'}} d\phi') \\
&= \frac{\mu_0 I_0 r_0}{4\pi}\hat{x}'[\int_{\phi'=0}^{\phi'=\pi} \frac{d\cos\phi'}{\sqrt{\alpha - \beta\cos\phi'}} + \int_{\phi'=\pi}^{\phi'=2\pi} \frac{d\cos\phi'}{\sqrt{\alpha - \beta\cos\phi'}}] \\
&\quad + \frac{\mu_0 I_0 r_0}{4\pi}\hat{y}'\int_0^{2\pi} \frac{\cos\phi'}{\sqrt{\alpha}}[1 - \frac{1}{2}(-\frac{\beta}{\alpha}\cos\phi') + O(\frac{\beta^2}{\alpha^2}\cos^2\phi')]d\phi' \\
&= \frac{\mu_0 I_0 r_0}{4\pi}\hat{x}'[\int_1^0 \frac{dx}{\sqrt{\alpha - \beta x}} + \int_0^1 \frac{dx}{\sqrt{\alpha - \beta x}}] \\
&\quad + \frac{\mu_0 I_0 r_0}{4\pi}\hat{y}'\int_0^{2\pi} \frac{\cos\phi'}{\sqrt{\alpha}}[1 + \frac{1}{2}\frac{\beta}{\alpha}\cos\phi' + O(\frac{\beta^2}{\alpha^2}\cos^2\phi')]d\phi' \\
&= \frac{\mu_0 I_0 r_0}{4\pi}\hat{y}'\int_0^{2\pi} \frac{\cos\phi'}{\sqrt{\alpha}}[1 + \frac{1}{2}\frac{\beta}{\alpha}\cos\phi' + O(\frac{\beta^2}{\alpha^2}\cos^2\phi')]d\phi'
\end{aligned} \tag{1.9}$$

For $r \gg r_0$, i.e., $\alpha \gg \beta$, we can ignore the small second-order term $O(\frac{\beta^2}{\alpha^2}\cos^2\phi')$ in Eq. (1.9). It yields

$$
\begin{aligned}
A_\phi \hat{y}' &\approx \frac{\mu_0 I_0 r_0}{4\pi} \hat{y}' \int_0^{2\pi} \frac{\cos\phi'}{\sqrt{\alpha}} [1 + \frac{1}{2}\frac{\beta}{\alpha}\cos\phi'] d\phi' \\
&= \hat{y}' \frac{\mu_0 I_0 r_0}{4\pi} [\int_0^{2\pi} \cos\phi' d\phi' + \frac{\beta}{2\alpha^{3/2}} \int_0^{2\pi} \cos^2\phi' d\phi'] \\
&= \hat{y}' \frac{\mu_0 I_0 r_0}{4\pi} [0 + \frac{\beta}{2\alpha^{3/2}} \pi] \\
&\approx \hat{y}' \frac{\mu_0 I_0 r_0}{4\pi} [\frac{2 r r_0 \sin\theta}{2 r^3} \pi] \\
&= \hat{y}' \frac{\mu_0 (I_0 \pi r_0^2)}{4\pi} \frac{\sin\theta}{r^2}
\end{aligned}
\tag{1.10}
$$

Using the definition of magnetic moment $M = I_0 \pi r_0^2$, Eq. (1.10) can be written as

$$
A_\phi = \frac{\mu_0 M}{4\pi} \frac{\sin\theta}{r^2} \tag{1.11}
$$

**Exercise 1.2.**

Please determine the dipole magnetic field $\mathbf{B}$ from the vector potential given in Eq. (1.11)

*Solution of Exercise 1.2:*

Since $\mathbf{B} = \nabla \times \mathbf{A}$, it yields

$$
\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta \frac{\mu_0 M}{4\pi}\frac{\sin\theta}{r^2} \end{vmatrix}
$$

$$
\begin{aligned}
&= \frac{\mu_0 M}{4\pi} \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \frac{\sin^2\theta}{r} \end{vmatrix} \\
&= \frac{\mu_0 M}{4\pi} \frac{1}{r^2 \sin\theta} [\hat{r} \frac{\partial}{\partial\theta} (\frac{\sin^2\theta}{r}) - r\hat{\theta} \frac{\partial}{\partial r} (\frac{\sin^2\theta}{r})] \\
&= \frac{\mu_0 M}{4\pi} \frac{1}{r^2 \sin\theta} [\hat{r} (\frac{2\sin\theta\cos\theta}{r}) + r\hat{\theta} (\frac{\sin^2\theta}{r^2})] \\
&= \frac{\mu_0 M}{4\pi} \frac{1}{r^3} [\hat{r}(2\cos\theta) + \hat{\theta}(\sin\theta)]
\end{aligned}
\tag{1.12}
$$

For Earth dipole magnetic field

$$\mathbf{B} = \frac{\mu_0(-M_E)}{4\pi}\frac{1}{r^3}[\hat{r}(2\cos\theta) + \hat{\theta}(\sin\theta)] \tag{1.13}$$

It can be rewritten as

$$\mathbf{B} = \frac{-B_0}{(r/R_E)^3}[\hat{r}(2\cos\theta) + \hat{\theta}(\sin\theta)] \tag{1.14}$$

where $B_0 = |\mathbf{B}(r = R_E, \theta = \pi/2)| \approx 0.35G = 35000\gamma = 35000nT$ is the magnitude of magnetic field on the Earth' surface at the magnetic equator. The $\theta$ is called the co-latitude. The latitude is $\lambda = \left|\frac{\pi}{2} - \theta\right|$.

### 1.2. Differential Equations of the Magnetic Field Line

Let us consider a segment $d\mathbf{s}$ along the magnetic field line, where

$$d\mathbf{s} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

It yields

$$ds = |d\mathbf{s}| = \sqrt{dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2} = \sqrt{dx^2 + dy^2 + dz^2}$$

Let $\mathbf{B} = \hat{r}B_r + \hat{\theta}B_\theta + \hat{\phi}B_\phi = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$

Since $\mathbf{B} \parallel d\mathbf{s}$, it yields

$$\frac{dr}{B_r} = \frac{rd\theta}{B_\theta} = \frac{r\sin\theta d\phi}{B_\phi} = \frac{ds}{B} \tag{1.15}$$

Eq. (1.15) can be rewritten in the following system ordinary differential equations

$$\boxed{\begin{aligned} \frac{dr}{ds} &= \frac{B_r}{B} \\ \frac{rd\theta}{ds} &= \frac{B_\theta}{B} \\ \frac{r\sin\theta d\phi}{ds} &= \frac{B_\phi}{B} \end{aligned}} \tag{1.16}$$

Likewise, $\mathbf{B} \parallel d\mathbf{s}$ yields

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} = \frac{ds}{B} \tag{1.17}$$

Eq. (1.17) can be rewritten in the following system ordinary differential equations

$$\frac{dx}{ds} = \frac{B_x}{B}$$
$$\frac{dy}{ds} = \frac{B_y}{B} \tag{1.18}$$
$$\frac{dz}{ds} = \frac{B_z}{B}$$

Eqs. (1.16) and (1.18) can be solved by the 2nd order or the fourth order Runge-Kutta method.

## 1.3. Dipole Magnetic Field Line

The Earth dipole magnetic field is given in Eq. (1.14). From Eq. (1.15), the dipole magnetic field line should satisfy the following differential equation

$$\frac{dr}{2\cos\theta} = \frac{rd\theta}{\sin\theta} \tag{1.19}$$

Solving Eq. (1.19), it yields

$$\frac{dr}{r} = \frac{2\cos\theta d\theta}{\sin\theta} = 2\frac{d\sin\theta}{\sin\theta}$$

Integrating along the field line, it yields

$$\int_{r(\theta=\pi/2)}^{r(\theta)} d\ln r(\theta) = 2\int_{\sin(\theta=\pi/2)}^{\sin\theta} d\ln\sin\theta \tag{1.20}$$

Let $r(\theta = \pi/2) = r_{eq} = LR_E$ , Eq. (1.20) yields

$$r(\theta) = r_{eq}\sin^2\theta = LR_E\sin^2\theta \tag{1.21}$$

**Exercise 1.3.**

A dipole magnetic field line with a given $L$ value will intersect with the Earth's surface at latitude $\lambda_L$. For $L$ = 1, 2, 3, 4, 5, and 6, find the corresponding latitude $\lambda_L$ .

**Exercise 1.4.**

Plot the dipole magnetic field line with $L$ = 1, 2, 3, 4, 5, and 6.

**Exercise 1.5.**

Plot the dipole magnetic field lines, which intersect with the Earth's surface at $\lambda_L$ = 80º, 70º, 60º, 50º, 40º, 30º, 20º, and 10º. Estimate the corresponding $L$ values.