Lecture 1. Background Knowledge for Studying Space Physics

1.1. Review of *Electromagnetics (E&M)*

Problem 1.1.

(a) Write down the differential form of the Maxwell's equations in the SI units in terms of electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, charge density ρ_c, and electric current density $\mathbf{J}$

(b) Write down the differential form of the Maxwell's equations in terms of $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$, as well as the charge density and current density.

Exercise 1.1.

Discuss the differences between the two types of Maxwell's equations given in **Problem 1.1.**

1.2. Review of *Plasma Physics*

Problem 1.2.

Let m_e be the electron mass. Let n_e and T_e be the number density and temperature of an electron fluid, respectively.

Let m_i be the proton mass. Let n_i and T_i be the number density and temperature of a proton fluid, respectively.

(a) Write down the definition of the electrons' plasma frequency ω_{pe}

(b) Write down the definition of the protons' plasma frequency ω_{pi}

(c) Write down the definition of the electrons' Debye length λ_{De}

(d) Write down the definition of the ions' Debye length λ_{Di}

(e) Determine the gyro frequency Ω_{ce} and gyro radius (Larmor radius) r_{Le} of an electron with initial velocity $\mathbf{v} = (v_\perp, 0, v_\parallel)$ moving in a uniform magnetic field $\mathbf{B} = (0,0,B)$.

(f) Determine the gyro frequency Ω_{ci} and gyro radius (Larmor radius) r_{Li} of a proton with initial velocity $\mathbf{v} = (0, v_\perp, v_\parallel)$ moving in a uniform magnetic field $\mathbf{B} = (0,0,B)$.

Exercise 1.2.

Verify your results of **Problem 1.2.**

Problem 1.3.

Let us consider an MHD plasma with mass density ρ and thermal pressure p in a uniform background magnetic field $\mathbf{B} = (0,0,B)$.

(a) Write down the definition of Alfvén speed of the medium

(b) Write down the definition of sound speed of the medium.

(c) MHD is the acronym of which word?

(d) Write down the Ohm's law of an ideal MHD plasma

1.3. Review of *Fluid Mechanics*

Problem 1.4.

Let us consider a linear wave with frequency f and wavelength λ.

(a) What is the unit of the frequency f in the SI system?

(b) Determine the angular frequency ω of the wave.

(c) Determine the wave number k of the wave

(d) Determine the phase velocity v_{ph} of the linear wave

Problem 1.5.

Let us consider a gas (a neutral fluid) with mass density ρ, average velocity $\mathbf{V}$, and thermal pressure p.

(a) Write down the mass continuity equation of the neutral fluid.

(b) What is the condition for an incompressible fluid?

(c) What is the condition for a compressible fluid?

(d) Write down the definition of vorticity of the neutral fluid.

1.4. Review of *Vector Analysis*

Problem 1.6.

(a) Explain why an electrostatic electric field can be written as $\mathbf{E} = -\nabla\phi$.

(b) Explain why the magnetic field can be written as $\mathbf{B} = \nabla \times \mathbf{A}$.

(c) Explain why the magnetic field can be written as $\mathbf{B} = \nabla\varphi \times \nabla\psi$.

(d) $\mathbf{E} = -\nabla\phi$ is not applicable for electromagnetic electric field. Express the electric field $\mathbf{E}$ in terms of scalar potential ϕ and vector potential $\mathbf{A}$.

Problem 1.7.

Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be three independent vectors.

(a) Let $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = a_1\mathbf{A} + a_2\mathbf{B} + a_3\mathbf{C}$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = b_1\mathbf{A} + b_2\mathbf{B} + b_3\mathbf{C}$. Determine a_1, a_2, a_3, b_1, b_2, and b_3.

(b) Decompose $\nabla \times (\mathbf{V} \times \mathbf{B})$

Problem 1.8.

Consider a scalar field f and a vector field $\mathbf{A}$. Let (x, y, z) be a coordinate of a point P in the Cartesian coordinate. We define (r, θ, ϕ) be the spherical coordinate of the point P, which satisfies the following coordinate transform: $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, and $y = r\sin\theta\sin\phi$.

(a) Write down the gradient f in the Cartesian coordinate (x, y, z) system.

(b) Write down the gradient f in the spherical coordinate (r, θ, ϕ) system.

(c) Write down the divergence of $\mathbf{A}$ in the Cartesian coordinate (x, y, z) system.

(d) Write down the divergence of $\mathbf{A}$ in the spherical coordinate (r, θ, ϕ) system.

(e) Write down the curl of $\mathbf{A}$ in the Cartesian coordinate (x, y, z) system.

(f) Write down the curl of $\mathbf{A}$ in the spherical coordinate (r, θ, ϕ) system.

(g) Write down $\nabla^2 f$ in the spherical coordinate (r, θ, ϕ) system.

Exercise 1.3.

Verify your results of **Problem 1.8.**

1.5. Coordinate Systems and Unit Vectors

Different notations have been used, in Math and in Physics, to represent the coordinates of a position vector $\mathbf{r}$ in the spherical coordinate system.

In advanced mathematics, such as *Calculus*, *Vector Analysis*, and *Linear Algebra*, the notations of the spherical coordinate system are commonly denoted as (r,ϕ,θ), where

r is the length of the position vector $\mathbf{r}$,

ϕ is the angle between the position vector $\mathbf{r}$ and the z-axis,

θ is the azimuthal angle measured from the x-axis.

In advanced physics, *Electromagnetics (E&M), Fluid Dynamics, Plasma Physics, Statistical Thermal Dynamics, Special Relativity, Quantum Physics, Classical Mechanics*, the notations of the spherical coordinate system are commonly denoted as (r,θ,ϕ), where

r is the length of the position vector $\mathbf{r}$,

θ is the angle between the position vector $\mathbf{r}$ and the z-axis

ϕ is the azimuthal angle measured from the x-axis.

Space Physics is an advanced physics. Thus, we shall use the notations (r,θ,ϕ) to represent the spherical coordinate system in the rest of this course.

Table 1.1 Notations for Different Coordinate Systems

3-D Coordinate System	Mathematics	Physics	2-D Coordinate System
Cartesian	(x,y,z)	(x,y,z)	Cartesian: (x,y)
Cylindrical	(r,θ,z)	(r,θ,z) or (ρ,ϕ,z)	Polar: (r,θ)
Spherical	(r,ϕ,θ)	(r,θ,ϕ)	N/A

Exercise 1.4.

Determine coordinate transformation between the Cartesian coordinate system (x,y,z) and the spherical coordinate system (r,θ,ϕ), where θ is the co-latitude (i.e., the angle measured from the z-axis), and ϕ is the azimuthal angle measured from the x-axis.

1.6. Coordinate Systems Commonly Used in Space Observations (Appendix 3 in the textbook by Kivelson and Russell, 1995)

1.6.1. Coordinate System Good for Studying Magnetospheric Physics

GSM Coordinate System (good for studying outer magnetosphere)
Geocentric Solar Magnetic System **(A.3.3.6)**
$x-z$ plane contains Earth magnetic dipole and Sun-Earth line
$\hat{x}$: A unit vector pointing toward the Sun from the Earth.
$\hat{z}$: A unit vector perpendicular to the x-axis, roughly in the northward direction.
$\hat{y}$: A dawn-to-dusk unit vector. ($\hat{y} = \hat{z} \times \hat{x}$)
Solar wind observed by Earth orbiting satellites is roughly in the $-x$ direction in the GSM coordinate system.
Earth magnetotail and plasmasheet is roughly parallel to the x-axis of the GSM coordinate system. But plasmasheet may not be in the z=0 plane.

SM Coordinate System (good for studying inner magnetosphere, i.e., plasmasphere)
Solar Magnetic Coordinates **(A.3.3.7)**
$x-z$ plane contains Earth magnetic dipole and Sun-Earth line
$\hat{z}$: A unit vector along Earth magnetic dipole, roughly in the northward direction.
$\hat{x}$: A unit vector perpendicular to the z-axis, roughly in the Earth-to-Sun direction.
$\hat{y}$: A dawn-to-dusk unit vector. ($\hat{y} = \hat{z} \times \hat{x}$)
Magnetic field in plasmasphere is nearly symmetric with respect to the z-axis.

1.6.2. Coordinate System Good for Ground Observations

Geographic Coordinates (GEO) **(A.3.3.2)**
$x-z$ plane contains Greenwich meridian and Earth rotation axis
$\hat{z}$: A unit vector along Earth rotation axis in the northward direction.
$\hat{x}$: A unit vector pointing toward the Greenwich meridian.
$\hat{y} = \hat{z} \times \hat{x}$

Geomagnetic Coordinates (MAG) **(A.3.3.3)**

$x - z$ plane contains Greenwich meridian and Earth magnetic dipole

$\hat{z}$: A unit vector along Earth magnetic dipole roughly in the northward direction.

$\hat{x}$: A unit vector pointing toward the Greenwich meridian.

$\hat{y} = \hat{z} \times \hat{x}$

Geocentric Equatorial Inertial System (GEI) **(A.3.3.1)**

(Good for Ground Observations of Sun and Starts) (**Astrophysics**)

$x - z$ plane contains Earth rotation axis and the intersection of equatorial plane and ecliptic plane

$\hat{z}$: A unit vector along Earth rotation axis in the northward direction.

$\hat{x}$: A unit vector pointing toward the vernal equinox from the Earth.

$\hat{y} = \hat{z} \times \hat{x}$

1.6.3. Coordinate System Good for Solar Wind and IMF Observations

GSE (Geocentric Solar Ecliptic System, **A.3.3.4**)

GEQ (Geocentric Solar Equatorial System, **A.3.3.5**)

1.7. Coordinates Transformation

Consider a basis $\mathcal{B} = \{\hat{e}_1,\ \hat{e}_2,\ \hat{e}_3\}$ and a new basis $\mathcal{B}' = \{\hat{e}_1',\ \hat{e}_2',\ \hat{e}_3'\}$. Each element in the new basis $\mathcal{B}'$ can be written as a linear combination of the elements in the old basis, i.e.,

$$\hat{e}_j' = \sum_{i=1}^{3} a_{ij}\hat{e}_i \tag{1.1}$$

Namely, the matrix representation of $\hat{e}_j'$ in the old basis $\mathcal{B}$ is

$$\left(\hat{e}_j'\right)_{\mathcal{B}} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{pmatrix} \tag{1.2}$$

For convenience, we shall use operating matrix A to denote the transformation matrix between bases $\mathcal{B}$ and $\mathcal{B}'$, i.e.,

$$\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \uparrow \\ \hat{e}_1' \\ \downarrow \end{pmatrix}_{\mathcal{B}} & \begin{pmatrix} \uparrow \\ \hat{e}_2' \\ \downarrow \end{pmatrix}_{\mathcal{B}} & \begin{pmatrix} \uparrow \\ \hat{e}_3' \\ \downarrow \end{pmatrix}_{\mathcal{B}} \end{pmatrix} \tag{1.3}$$

For any given vector $\mathbf{V}$, it can be written as a linear combination of the elements in the old basis, i.e.,

$$\mathbf{V} = \sum_{i=1}^{3} V_i\hat{e}_i \tag{1.4}$$

Thus, the matrix representation of vector $\mathbf{V}$ in the old basis $\mathcal{B}$ is

$$\left(\mathbf{V}\right)_{\mathcal{B}} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}. \tag{1.5}$$

Exercise 1.5.

Determine the matrix representation of vector $\mathbf{V}$ in the new basis $\mathcal{B}'$,

i.e., $\left(\mathbf{V}\right)_{\mathcal{B}'} = \begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix} = ?$

Solution of Exercise 1.5:

Let $\mathbf{V} = \sum_{j=1}^{3} V'_j \hat{e}'_j$ (1.6)

Substituting Eq. (1.1) into Eq. (1.6) yields

$$\mathbf{V} = \sum_{j=1}^{3} V'_j \hat{e}'_j = \sum_{j=1}^{3} V'_j (\sum_{i=1}^{3} a_{ij}\hat{e}_i) = \sum_{i=1}^{3}\sum_{j=1}^{3} V'_j a_{ij}\hat{e}_i = \sum_{i=1}^{3} (\sum_{j=1}^{3} a_{ij} V'_j)\hat{e}_i \quad (1.7)$$

Comparing the coefficients of $\hat{e}_i$ in the last expression in Eq. (1.7) and in Eq. (1.4), it yields

$$V_i = \sum_{j=1}^{3} a_{ij} V'_j \quad (1.8)$$

Thus, we have

$$(\mathbf{V})_{\mathcal{B}} = \mathsf{A} \cdot (\mathbf{V})_{\mathcal{B}'} \quad (1.9)$$

Eq. (1.9) yields

$$(\mathbf{V})_{\mathcal{B}'} = \begin{pmatrix} V'_1 \\ V'_2 \\ V'_3 \end{pmatrix} = \mathsf{A}^{-1} \cdot (\mathbf{V})_{\mathcal{B}} = \left(\begin{pmatrix} \uparrow \\ \hat{e}'_1 \\ \downarrow \end{pmatrix}_{\mathcal{B}} \begin{pmatrix} \uparrow \\ \hat{e}'_2 \\ \downarrow \end{pmatrix}_{\mathcal{B}} \begin{pmatrix} \uparrow \\ \hat{e}'_3 \\ \downarrow \end{pmatrix}_{\mathcal{B}} \right)^{-1} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad (1.10)$$

Special case:

For $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ and $\hat{e}'_i \cdot \hat{e}'_j = \delta_{ij}$, we have $\mathsf{A}^{-1} = \mathsf{A}^t$. Thus,

$$(\mathbf{V})_{\mathcal{B}'} = \mathsf{A}^t \cdot (\mathbf{V})_{\mathcal{B}} \quad (1.10a)$$

i.e.,

$$\begin{pmatrix} V'_1 \\ V'_2 \\ V'_3 \end{pmatrix} = \begin{pmatrix} (\leftarrow & \hat{e}'_1 & \rightarrow)_{\mathcal{B}} \\ (\leftarrow & \hat{e}'_1 & \rightarrow)_{\mathcal{B}} \\ (\leftarrow & \hat{e}'_1 & \rightarrow)_{\mathcal{B}} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad (1.10b)$$

For any given second rank tensor P, it can be written as a linear combination of the *dyad product* of $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, i.e.,

$$\mathsf{P} = \sum_{j=1}^{3}\sum_{i=1}^{3} P_{ij} \hat{e}_i \hat{e}_j \quad (1.11)$$

Thus, the matrix representation of the second rank tensor P in the old basis $\mathcal{B}$ is

$$(\mathsf{P})_{\mathcal{B}} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \quad (1.12)$$

Exercise 1.6.

Determine the matrix representation of the second rank tensor $\mathbf{P}$ in the new basis $\mathcal{B}'$,

i.e., $$(\mathbf{P})_{\mathcal{B}'} = \begin{pmatrix} P'_{11} & P'_{12} & P'_{13} \\ P'_{21} & P'_{22} & P'_{23} \\ P'_{31} & P'_{32} & P'_{33} \end{pmatrix} = ?$$

Solution of Exercise 1.6.:

Let $$\mathbf{P} = \sum_{l=1}^{3}\sum_{k=1}^{3} P'_{kl}\hat{e}'_k\hat{e}'_l \qquad (1.13)$$

Substituting Eq. (1.1) into Eq. (1.13) yields

$$\mathbf{P} = \sum_{l=1}^{3}\sum_{k=1}^{3} P'_{kl}\hat{e}'_k\hat{e}'_l = \sum_{l=1}^{3}\sum_{k=1}^{3} P'_{kl}(\sum_{i=1}^{3} a_{ik}\hat{e}_i)(\sum_{j=1}^{3} a_{jl}\hat{e}_j) = \sum_{i=1}^{3}\sum_{j=1}^{3}(\sum_{k=1}^{3}\sum_{l=1}^{3} a_{ik}P'_{kl}a_{jl})\hat{e}_i\hat{e}_j \qquad (1.14)$$

Comparing the coefficients of $\hat{e}_i\hat{e}_j$ in the last expression in Eq. (1.14) and in Eq. (1.11), it yields

$$P_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} a_{ik}P'_{kl}a_{jl} \qquad (1.15)$$

Thus, we have

$$(\mathbf{P})_{\mathcal{B}} = \mathbf{A}\cdot(\mathbf{P})_{\mathcal{B}'}\cdot\mathbf{A}^t \qquad (1.16)$$

Eq. (1.16) yields

$$(\mathbf{P})_{\mathcal{B}'} = \mathbf{A}^{-1}\cdot(\mathbf{P})_{\mathcal{B}}\cdot(\mathbf{A}^t)^{-1} \qquad (1.17)$$

i.e.,

$$\begin{pmatrix} P'_{11} & P'_{12} & P'_{13} \\ P'_{21} & P'_{22} & P'_{23} \\ P'_{31} & P'_{32} & P'_{33} \end{pmatrix} = \left(\begin{pmatrix}\uparrow\\ \hat{e}'_1\\ \downarrow\end{pmatrix}_{\mathcal{B}} \begin{pmatrix}\uparrow\\ \hat{e}'_2\\ \downarrow\end{pmatrix}_{\mathcal{B}} \begin{pmatrix}\uparrow\\ \hat{e}'_3\\ \downarrow\end{pmatrix}_{\mathcal{B}}\right)^{-1} \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} (\leftarrow & \hat{e}'_1 & \rightarrow)_{\mathcal{B}} \\ (\leftarrow & \hat{e}'_1 & \rightarrow)_{\mathcal{B}} \\ (\leftarrow & \hat{e}'_1 & \rightarrow)_{\mathcal{B}} \end{pmatrix}^{-1}$$

Special case:

For $\hat{e}_i\cdot\hat{e}_j = \delta_{ij}$ and $\hat{e}'_i\cdot\hat{e}'_j = \delta_{ij}$, we have $\mathbf{A}^{-1} = \mathbf{A}^t$. Thus,

$$(\mathbf{P})_{\mathcal{B}'} = \mathbf{A}^t\cdot(\mathbf{P})_{\mathcal{B}}\cdot\mathbf{A} \qquad (1.17a)$$

i.e.,

$$\begin{pmatrix} P'_{11} & P'_{12} & P'_{13} \\ P'_{21} & P'_{22} & P'_{23} \\ P'_{31} & P'_{32} & P'_{33} \end{pmatrix} = \begin{pmatrix} (\leftarrow & \hat{e}'_1 & \rightarrow)_{\mathcal{B}} \\ (\leftarrow & \hat{e}'_1 & \rightarrow)_{\mathcal{B}} \\ (\leftarrow & \hat{e}'_1 & \rightarrow)_{\mathcal{B}} \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \left(\begin{pmatrix}\uparrow\\ \hat{e}'_1\\ \downarrow\end{pmatrix}_{\mathcal{B}} \begin{pmatrix}\uparrow\\ \hat{e}'_2\\ \downarrow\end{pmatrix}_{\mathcal{B}} \begin{pmatrix}\uparrow\\ \hat{e}'_3\\ \downarrow\end{pmatrix}_{\mathcal{B}}\right) \qquad (1.17b)$$