Appendix C. Magnetic Pressure Gradient Force and Magnetic Tension Force

We shall show in this appendix that the $\mathbf{J}\times\mathbf{B}$ magnetic force can be decomposed into magnetic pressure gradient force and magnetic tension force.

MHD Ampere's law yields

$\nabla\times\mathbf{B}=\mu_0\mathbf{J}$

Thus, the magnetic force can be rewritten as

$$\mathbf{J}\times\mathbf{B}=\frac{\nabla\times\mathbf{B}}{\mu_0}\times\mathbf{B}=-\nabla\frac{B^2}{2\mu_0}+\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0}=(-\vec{\nabla}_\perp\frac{B^2}{2\mu_0}-\vec{\nabla}_\parallel\frac{B^2}{2\mu_0})+\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0} \tag{C.1}$$

Let $\hat{R}_B$ denotes radial vector of a curved magnetic field line and let R_B denotes the radius of curvature of the magnetic field line. Since

$$-\frac{\hat{R}_B}{R_B}=\frac{d\hat{B}}{ds}=\hat{B}\cdot\nabla\hat{B}=\frac{\mathbf{B}}{B}\cdot\nabla(\frac{\mathbf{B}}{B})=\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{B^2}-\frac{\mathbf{BB}\cdot\nabla B}{B^3}=\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{B^2}-\frac{\hat{B}\hat{B}\cdot\nabla B}{B}=\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{B^2}-\frac{\vec{\nabla}_\parallel B}{B}$$

or

$$\mathbf{B}\cdot\nabla\mathbf{B}=\frac{\vec{\nabla}_\parallel B^2}{2}-B^2\frac{\hat{R}_B}{R_B} \tag{C.2}$$

Substituting (C.2) into (C.1) yields

$$\mathbf{J}\times\mathbf{B}=(-\vec{\nabla}_\perp\frac{B^2}{2\mu_0}-\vec{\nabla}_\parallel\frac{B^2}{2\mu_0})+(\frac{\vec{\nabla}_\parallel B^2}{2\mu_0}-\frac{B^2}{\mu_0}\frac{\hat{R}_B}{R_B})=-\vec{\nabla}_\perp\frac{B^2}{2\mu_0}-\frac{B^2}{\mu_0}\frac{\hat{R}_B}{R_B} \tag{C.3}$$

where $-\vec{\nabla}_\perp\frac{B^2}{2\mu_0}$ is the magnetic pressure gradient force and $-\frac{B^2}{\mu_0}\frac{\hat{R}_B}{R_B}$ is the restoring force due to the tension force of the magnetic field line. Namely, we can define the magnetic pressure to be

$$p_B=\frac{B^2}{2\mu_0} \tag{C.4}$$

and the magnetic tension to be

$$T_B=\frac{B^2}{\mu_0} \tag{C.5}$$

Note that the magnetic pressure gradient force and the magnetic tension force cancel each other in a dipole magnetic field environment.

Dr. Alfvén first proposed that the magnetic tension force provides a restoring force for the Alfvén wave. We can compare Alfvén wave with the wave propagating along a string. Consider a small transverse displacement $y(x,t)$ of a string with linear mass density σ. Wave equation of the transverse wave can be derived from equation of motion of an element on the string (e.g., Symon, 1971). As a result, the wave equation can be written as,

$$\frac{\partial^2 y}{\partial t^2} \approx \frac{T}{\sigma}\frac{\partial^2 y}{\partial x^2} \tag{C.6}$$

where T is the tension of the string. According to Eq. (C.6), the phase speed of the transverse wave propagating along a string is

$$v_{ph} = \sqrt{T/\sigma} \tag{C.7}$$

Let us compare Eq. (C.7) with the Alfvén speed. From Eq. (C.5), the Alfvén speed can be rewritten as

$$C_A = \sqrt{\frac{B^2}{\mu_0 \rho}} = \sqrt{\frac{T_B}{\rho}} \tag{C.8}$$

We have just verified the statement that was first proposed by Dr. Alfvén more than 50 years ago.

Reference

Symon, K. R., *Mechanics*, 3rd edition, Addison-Wesley Publishing Company, 1971.