Appendix B. Grad-B Drift

Consider a charge particle moving in a system with non-uniform magnetic field $\mathbf{B}(\mathbf{r})$. However, the non-uniformity of the magnetic field is small enough that we can approximately use Taylor expansion to estimate the magnetic field seen by the charge particle along its gyro orbit in terms of the magnetic field observed at its guiding center location. Namely, for $\mathbf{r} = \mathbf{r}_{g.c.} + \mathbf{r}_{gyro}$, we have

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{r}_{g.c.}) + \mathbf{r}_{gyro} \cdot \nabla\mathbf{B} + \cdots \tag{B.1}$$

If this particle has no velocity component parallel to the local magnetic field and magnetic momentum of this particle is conserved then we can decompose its velocity $\mathbf{v}$ into

$$\mathbf{v} = \mathbf{v}_{gyro} + \mathbf{v}_{drift}$$

where $\mathbf{v}_{gyro}$ is the high frequency gyro motion velocity and $\mathbf{v}_{drift}$ is a nearly time-independent guiding center drift velocity. The *equation of motion* of this charge particle is

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \approx q(\mathbf{v}_{gyro} + \mathbf{v}_{drift}) \times [\mathbf{B}(\mathbf{r}_{g.c.}) + \mathbf{r}_{gyro} \cdot \nabla\mathbf{B}] \tag{B.2}$$

The *low frequency guiding-center equation of motion* becomes

$$\mathbf{v}_{drift} \times \mathbf{B}(\mathbf{r}_{g.c.}) + \left\langle \mathbf{v}_{gyro} \times (\mathbf{r}_{gyro} \cdot \nabla\mathbf{B}) \right\rangle = 0 \tag{B.3}$$

where the notation $\langle f \rangle$ denotes the time average value of f. Now let us consider a simplified case. Let $\mathbf{B} = B(x,y)\hat{z}$, then we have

$$\nabla\mathbf{B} = \nabla(B\hat{z}) = \frac{\partial B(x,y)}{\partial x}\hat{x}\hat{z} + \frac{\partial B(x,y)}{\partial y}\hat{y}\hat{z}$$

Equation of gyro motion component with respect to guiding center is

$$m\frac{d\mathbf{v}_{gyro}}{dt} = q\mathbf{v}_{gyro} \times \mathbf{B} \tag{B.4}$$

$$\frac{d\mathbf{r}_{gyro}}{dt} = \mathbf{v}_{gyro} \tag{B.5}$$

Solution of gyro motion Eq. (B.4) and Eq. (B.5) can be written as

$$\mathbf{v}_{gyro} = v_{gyro}[\hat{x}\cos(\Omega t + \varphi) - \frac{|q|}{q}\hat{y}\sin(\Omega t + \varphi)] \tag{B.6}$$

$$\mathbf{r}_{gyro} = \frac{v_{gyro}}{\Omega}[\hat{x}\sin(\Omega t + \varphi) + \frac{|q|}{q}\hat{y}\cos(\Omega t + \varphi)] \tag{B.7}$$

Thus,

$$\mathbf{r}_{gyro} \cdot \nabla \mathbf{B} = \frac{v_{gyro}}{\Omega} \sin(\Omega t + \varphi) \frac{\partial B(x,y)}{\partial x} \hat{z} + \frac{v_{gyro}}{\Omega} \frac{|q|}{q} \cos(\Omega t + \varphi)] \frac{\partial B(x,y)}{\partial y} \hat{z}$$

$$\mathbf{v}_{gyro} \times \mathbf{r}_{gyro} \cdot \nabla \mathbf{B}$$

$$= \frac{v_{gyro}^2}{\Omega} \cos(\Omega t + \varphi) \sin(\Omega t + \varphi) \frac{\partial B(x,y)}{\partial x} (\hat{x} \times \hat{z}) - \frac{v_{gyro}^2}{\Omega} \frac{|q|}{q} \sin^2(\Omega t + \varphi) \frac{\partial B(x,y)}{\partial x} (\hat{y} \times \hat{z})$$

$$+ \frac{v_{gyro}^2}{\Omega} \frac{|q|}{q} \cos^2(\Omega t + \varphi) \frac{\partial B(x,y)}{\partial y} (\hat{x} \times \hat{z}) - \frac{v_{gyro}^2}{\Omega} \sin(\Omega t + \varphi) \cos(\Omega t + \varphi) \frac{\partial B(x,y)}{\partial y} (\hat{y} \times \hat{z})$$

where $\Omega = |q| B / m$, thus

$$\frac{v_{gyro}^2}{\Omega} \frac{|q|}{q} = \frac{m v_{gyro}^2}{qB}$$

Since

$$\left\langle \sin(\Omega t + \varphi) \cos(\Omega t + \varphi) \right\rangle = 0$$

$$\left\langle \cos^2(\Omega t + \varphi) \right\rangle = \left\langle \sin^2(\Omega t + \varphi) \right\rangle = \frac{1}{2} \tag{B.8}$$

we have

$$\left\langle \mathbf{v}_{gyro} \times \mathbf{r}_{gyro} \cdot \nabla \mathbf{B} \right\rangle = -\frac{m v_{gyro}^2}{2qB} \frac{\partial B(x,y)}{\partial x} (\hat{y} \times \hat{z}) + \frac{m v_{gyro}^2}{2qB} \frac{\partial B(x,y)}{\partial y} (\hat{x} \times \hat{z})$$

$$= -\frac{m v_{gyro}^2}{2qB} [\frac{\partial B(x,y)}{\partial x} \hat{x} + \frac{\partial B(x,y)}{\partial y} \hat{y}] = \frac{m v_{gyro}^2}{2qB} (-\nabla_\perp B) \tag{B.9}$$

Substituting Eq. (B.9) into Eq. (B.3), it yields

$$\mathbf{v}_{drift} \times \mathbf{B}(\mathbf{r}_{g.c.}) + \frac{m v_{gyro}^2}{2qB} (-\nabla_\perp B) = 0 \tag{B.10}$$

Solution of $\mathbf{v}_{drift}$ in Eq. (B.10) is the grad-B drift velocity in the first-order approximation. It can be written as

$$\mathbf{v}_{drift} = \frac{m v_{gyro}^2}{2qB} \frac{(-\nabla_\perp B) \times \mathbf{B}}{B^2} \tag{B.11}$$

For $v_{drift} << v_{gyro}$, the perpendicular speed, $v_\perp$, of the charge particle is approximately equal to v_{gyro}. Thus, it is commonly using the following expression to denote the first-order *grad-B drift velocity.*

$$\mathbf{v}_{drift} = \frac{m v_\perp^2}{2qB} \frac{(-\nabla_\perp B) \times \mathbf{B}}{B^2} \tag{B.12}$$