Appendix H. Special Functions for Studying Linear Waves in Kinetic Plasmas

Suggested Readings:
(1) Section 7-2 in Mathews and Walker (1970)
(1) Section 3-5 in Mathews and Walker (1970)

H.1. Bessel Function

Bessel function is a useful function in studying kinetic linear waves in a magnetized plasma.

H.1.1. Definition of Bessel Function

Bessel function is a solution of the following Bessel's differential equation with a real variable x

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

If n is an integer $J_{-n}(x) = (-1)^n J_n(x)$, where

$$J_n(x) = \frac{x^n}{2^n n!}\{1 - \frac{x^2}{2^2 \cdot 1!(n+1)} + \frac{x^4}{2^4 \cdot 2!(n+1)(n+2)} - \frac{x^6}{2^6 \cdot 3!(n+1)(n+2)(n+3)} + \cdots\}$$

H.1.2. Generating Function of Bessel Function

The generating function of Bessel function is

$$\boxed{\sum_{n=-\infty}^{+\infty} h^n J_n(z) = \exp[\frac{z}{2}(h - \frac{1}{h})]} \qquad \text{(H.1)}$$

Let $h = e^{i\theta}$. Then, we have

$$\frac{1}{2}(h - \frac{1}{h}) = \frac{e^{i\theta} - e^{-i\theta}}{2} = i\sin\theta$$

Thus, Eq. (H.1) can be rewritten as

$$\boxed{\sum_{n=-\infty}^{+\infty} e^{in\theta} J_n(z) = \exp[iz\sin\theta]} \qquad \text{(H.2)}$$

H.1.3. Recursion Relations of Bessel Function

From Eq. (H.2), we can easily obtain recursion relations of Bessel Function. Differentiate Eq. (H.2) with respect to z, yields

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} e^{in\theta} J_n'(z) &= \frac{\partial}{\partial z}\exp[iz\sin\theta] \\
&= i\sin\theta\exp[iz\sin\theta] \\
&= \frac{e^{i\theta}-e^{-i\theta}}{2}\sum_{n=-\infty}^{+\infty} e^{in\theta} J_n(z) \\
&= \frac{1}{2}\{[\sum_{n=-\infty}^{+\infty} e^{i(n+1)\theta} J_n(z)] - [\sum_{n=-\infty}^{+\infty} e^{i(n-1)\theta} J_n(z)]\} \\
&= \frac{1}{2}\{[\sum_{n=-\infty}^{+\infty} e^{in\theta} J_{n-1}(z)] - [\sum_{n=-\infty}^{+\infty} e^{in\theta} J_{n+1}(z)]\} \\
&= \sum_{n=-\infty}^{+\infty} e^{in\theta}[\frac{J_{n-1}(z)-J_{n+1}(z)}{2}]
\end{aligned}$$

Comparing coefficients of $e^{in\theta}$ on two sides of this equation yields,

$$\boxed{2J_n'(z) = J_{n-1}(z) - J_{n+1}(z)} \tag{H.3}$$

Differentiate Eq. (H.2) with respect to θ, yields

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} ine^{in\theta} J_n(z) &= \frac{\partial}{\partial \theta}\exp[iz\sin\theta] \\
&= iz\cos\theta\exp[iz\sin\theta] \\
&= iz\frac{e^{i\theta}+e^{-i\theta}}{2}\sum_{n=-\infty}^{+\infty} e^{in\theta} J_n(z) \\
&= \frac{iz}{2}\{[\sum_{n=-\infty}^{+\infty} e^{i(n+1)\theta} J_n(z)] + [\sum_{n=-\infty}^{+\infty} e^{i(n-1)\theta} J_n(z)]\} \\
&= \frac{iz}{2}\{[\sum_{n=-\infty}^{+\infty} e^{in\theta} J_{n-1}(z)] + [\sum_{n=-\infty}^{+\infty} e^{in\theta} J_{n+1}(z)]\} \\
&= iz\sum_{n=-\infty}^{+\infty} e^{in\theta}[\frac{J_{n-1}(z)+J_{n+1}(z)}{2}]
\end{aligned}$$

Comparing coefficients of $e^{in\theta}$ on two sides of this equation yields,

$$\boxed{\frac{2n}{z}J_n(z) = J_{n-1}(z) + J_{n+1}(z)} \tag{H.4}$$

H.2. Error Function

Error function is a useful function in calculating plasma number density in kinetic plasma physics.

Definition of *Error function*

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\sqrt{2}\sigma x}^{\sqrt{2}\sigma x} \exp[-\frac{v^2}{2\sigma^2}]\, dv$$

Definition of *Complimentary error function*

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

For $x \le 1$, there is a useful expansion series for the *Error function*, i.e.,

$$\begin{aligned}
\operatorname{erf} x &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^x (1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots - \cdots)\, dt \\
&= \frac{2}{\sqrt{\pi}} (x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots - \cdots)
\end{aligned}$$

For large x, there is another useful expansion series for the *Error function*, or the *complimentary error function*, i.e.,

$$\begin{aligned}
\operatorname{erf} x = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2} [&\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \frac{1 \cdot 3 \cdot 5}{2^4 x^7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 x^9} - \cdots + \cdots \\
&+ (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n x^{2n-1}}] \\
&+ (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{e^{-t^2}}{t^{2n}} dt
\end{aligned}$$

or

$$\begin{aligned}
\operatorname{erfc} x = + \frac{2}{\sqrt{\pi}} e^{-x^2} [&\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \frac{1 \cdot 3 \cdot 5}{2^4 x^7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 x^9} - \cdots + \cdots \\
&+ (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n x^{2n-1}}] \\
&+ (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{e^{-t^2}}{t^{2n}} dt
\end{aligned}$$

H.3. Plasma Dispersion Function

Plasma dispersion function is a useful function in calculating linear wave dispersion relation in a plasma with *Maxwellian* distribution in the velocity space.

Plasma dispersion function is defined by (e.g., Fried and Conte, 1961)

$$Z(\xi)=\frac{1}{\sqrt{\pi}}\int_L \frac{e^{-u^2}}{u-\xi}du \tag{H.5}$$

where the Landau contour is defined by

$$\boxed{h(\xi)=\int_L \frac{g(u)}{u-\xi}du=\begin{cases}\int_{-\infty}^{+\infty}\frac{g(u)}{u-\xi}du & \text{if } \mathrm{Im}(\xi)>0\\ \wp\int_{-\infty}^{+\infty}\frac{g(u)}{u-\xi}du+\pi i g(\xi) & \text{if } \mathrm{Im}(\xi)\to 0\\ \int_{-\infty}^{+\infty}\frac{g(u)}{u-\xi}du+2\pi i g(\xi) & \text{if } \mathrm{Im}(\xi)<0\end{cases}}$$

where $\wp\int_{-\infty}^{+\infty}[g(u)/(u-\xi)]du$ denotes the principle value of the integration. Since the $Z(\xi)$ is also the Hilbert transform of a Gaussian, it has been well-examined. For instance, we can find an alternative expression of the plasma dispersion function in the following way.

The derivative of $Z(\xi)$ is

$$Z'(\xi)=\frac{1}{\sqrt{\pi}}\int_L \frac{e^{-u^2}}{(u-\xi)^2}du \tag{H.6}$$

On integration by parts, Eq.(2) yields

$$Z'(\xi)=\frac{1}{\sqrt{\pi}}\int_L \frac{-2u}{u-\xi}e^{-u^2}du \tag{H.7}$$

Since

$$\int_L \frac{-2u}{u-\xi}e^{-u^2}du=-2\int_L \frac{(u-\xi)+\xi}{u-\xi}e^{-u^2}du=-2\left[\int_{-\infty}^{+\infty}e^{-u^2}du+\xi\int_L \frac{e^{-u^2}}{u-\xi}du\right]$$

it yields,

$$Z'(\xi)=-2[1+\xi Z(\xi)] \tag{H.8}$$

Now, let us define a function $Y(x)$, such that

$$Y(x)=2i\exp(-x^2)\int_{-\infty}^{ix}\exp(-t^2)dt \tag{H.9}$$

Based on the Leibniz's rule, the derivative of $Y(x)$ is

$$Y'(x) = -2x\left[2i\exp(-x^2)\int_{-\infty}^{ix}\exp(-t^2)dt\right] + 2i\exp(-x^2)\exp[-(ix)^2]\frac{d(ix)}{dx} \qquad \text{(H.10)}$$

Since

$$-2x\left[2i\exp(-x^2)\int_{-\infty}^{ix}\exp(-t^2)dt\right] + 2ie^{-x^2}e^{-(ix)^2}\frac{d(ix)}{dx} = -2x\left[2i\exp(-x^2)\int_{-\infty}^{ix}\exp(-t^2)dt\right] - 2$$

it yields,

$$Y'(x) = -2(xY+1) \qquad \text{(H.11)}$$

Eqs. (H.8) and (H.11) yield

$$Z(\xi) = Y(\xi) + C_0 \qquad \text{(H.12)}$$

where C_0 is a constant. Substituting $x = 0$ into Eq. (H.9), it yields

$$Y(0) = 2i\int_{-\infty}^{0}\exp(-t^2)dt = i\sqrt{\pi} \qquad \text{(H.13)}$$

Likewise, substituting $\xi = 0$ into Eq. (H.5) and integrating along the Landau contour, it yields

$$Z(0) = \frac{1}{\sqrt{\pi}}\int_L \frac{e^{-u^2}}{u}du = \frac{1}{\sqrt{\pi}}(0 + i\pi) = i\sqrt{\pi} \qquad \text{(H.14)}$$

Substituting Eqs. (H.13) and (H.14) into (H.12) for $\xi = 0$, it yields $C_0 = 0$. Thus, we have $Z(\xi) = Y(\xi)$. Namely, the plasma dispersion function has an alternative expression,

$$Z(\xi) = 2i\exp(-\xi^2)\int_{-\infty}^{i\xi}\exp(-u^2)du \qquad \text{(H.15)}$$

Based on the *Error function* and the *Complimentary error function* discussed in Section H.2, the plasma dispersion function can be written as

$$Z(\xi) = i\exp(-\xi^2)\sqrt{\pi}\,\mathrm{erfc}(-i\xi) \qquad \text{(H.16)}$$

Based on the results shown in Section H.2, for $|\xi| \ll 1$, $Z(\xi)$ can be approximately by

$$Z(\xi) = i\sqrt{\pi}e^{-\xi^2} - 2\xi + \frac{4}{3}\xi^3 - \frac{8}{15}\xi^5 + \cdots \qquad \text{(H.17a)}$$

or

$$Z(\xi) = i\sqrt{\pi}e^{-\xi^2} - \xi\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-\xi^2)^n}{\Gamma[n+(1/2)]} \qquad \text{(H.17b)}$$

For $|\xi| \gg 1$, $Z(\xi)$ can be approximately by

$$Z(\xi)=\begin{cases} -\dfrac{1}{\xi}-\dfrac{1}{2\xi^3}-\dfrac{3}{4\xi^5}-\dfrac{15}{8\xi^5}-\cdots & \text{for } \mathrm{Im}(\xi)>0 \\ -\dfrac{1}{\xi}-\dfrac{1}{2\xi^3}-\dfrac{3}{4\xi^5}-\dfrac{15}{8\xi^5}-\cdots+i\sqrt{\pi}e^{-\xi^2} & \text{for } \mathrm{Im}(\xi)\approx 0 \\ -\dfrac{1}{\xi}-\dfrac{1}{2\xi^3}-\dfrac{3}{4\xi^5}-\dfrac{15}{8\xi^5}-\cdots+i2\sqrt{\pi}e^{-\xi^2} & \text{for } \mathrm{Im}(\xi)<<0 \end{cases} \tag{H.18a}$$

or

$$Z(\xi)=\begin{cases} -\dfrac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\dfrac{\Gamma[n-(1/2)]}{\xi^{2n+1}} & \mathrm{Im}(\xi)>0 \\ -\dfrac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\dfrac{\Gamma[n-(1/2)]}{\xi^{2n+1}}+i\sqrt{\pi}e^{-\xi^2} & \mathrm{Im}(\xi)\approx 0 \\ -\dfrac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\dfrac{\Gamma[n-(1/2)]}{\xi^{2n+1}}+i2\sqrt{\pi}e^{-\xi^2} & \mathrm{Im}(\xi)<<0 \end{cases} \tag{H.18b}$$

Note that the plasma dispersion function is *not applicable* to a problem with non-Maxwellian distribution, such as a distribution with finite heat flux. On the other hand, we can also prove Eqs. (H.17a) and (H.18a) directly without the help from the complimentary error function.

Since $1+r+r^2+\cdots+r^n=\dfrac{1-r^n}{1-r}$

For $|\xi|>>1$, we have $\dfrac{1}{u-\xi}=\dfrac{1}{\xi}\left(\dfrac{1}{\frac{u}{\xi}-1}\right)=\dfrac{-1}{\xi}[1+\dfrac{u}{\xi}+(\dfrac{u}{\xi})^2+\cdots]$

For $|\xi|<<1$, we have $\dfrac{1}{u-\xi}=\dfrac{1}{u}\left(\dfrac{1}{1-\frac{\xi}{u}}\right)=\dfrac{1}{u}[1+\dfrac{\xi}{u}+(\dfrac{\xi}{u})^2+\cdots]=\dfrac{1}{u}+\dfrac{\xi}{u^2}+\dfrac{\xi^2}{u^3}+\cdots$

Thus, for $\xi>>1$, $Z(\xi)$ is approximately

$$Z(\xi)=\frac{1}{\sqrt{\pi}}\int_L\frac{e^{-u^2}}{u-\xi}du$$

$$=\begin{cases} \dfrac{-1}{\xi}\dfrac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}[1+\dfrac{u}{\xi}+(\dfrac{u}{\xi})^2+\cdots]du & \text{for } \mathrm{Im}(\xi)>0 \\ \dfrac{-1}{\xi}\dfrac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}[1+\dfrac{u}{\xi}+(\dfrac{u}{\xi})^2+\cdots]du+i\sqrt{\pi}e^{-\xi^2} & \text{for } \mathrm{Im}(\xi)\approx 0 \\ \dfrac{-1}{\xi}\dfrac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}[1+\dfrac{u}{\xi}+(\dfrac{u}{\xi})^2+\cdots]du+i2\sqrt{\pi}e^{-\xi^2} & \text{for } \mathrm{Im}(\xi)<<0 \end{cases}$$

Since

$$\frac{-1}{\xi}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}[1+\frac{u}{\xi}+(\frac{u}{\xi})^2+\cdots]du = (\frac{-1}{\xi})[1+0+\frac{1}{\xi^2}(\frac{1}{\sqrt{2}})^2+0+\cdots]$$

$$= -\frac{1}{\xi}-\frac{1}{2\xi^3}-\frac{3}{4\xi^5}-\frac{15}{8\xi^5}-\cdots$$

It yields

$$Z(\xi)=\begin{cases} -\frac{1}{\xi}-\frac{1}{2\xi^3}-\frac{3}{4\xi^5}-\frac{15}{8\xi^5}-\cdots & \text{for } \mathrm{Im}(\xi)>0 \\ -\frac{1}{\xi}-\frac{1}{2\xi^3}-\frac{3}{4\xi^5}-\frac{15}{8\xi^5}-\cdots+i\sqrt{\pi}e^{-\xi^2} & \text{for } \mathrm{Im}(\xi)\approx 0 \\ -\frac{1}{\xi}-\frac{1}{2\xi^3}-\frac{3}{4\xi^5}-\frac{15}{8\xi^5}-\cdots+i2\sqrt{\pi}e^{-\xi^2} & \text{for } \mathrm{Im}(\xi)<<0 \end{cases}$$

Likewise, for $|\xi|<<1$, $Z(\xi)$ is approximately

$$Z(\xi)=\frac{1}{\sqrt{\pi}}\int_L \frac{e^{-u^2}}{u-\xi}du$$

$$=\frac{1}{\sqrt{\pi}}\int_L e^{-u^2}[\frac{1}{u}+\frac{\xi}{u^2}+\frac{\xi^2}{u^3}+\frac{\xi^3}{u^4}+\cdots]du$$

$$=\frac{1}{\sqrt{\pi}}(\pi i e^{-\xi^2}+\xi I_1+0+\xi^3 I_2+0+\cdots)$$

where $I_1=\int_L \frac{e^{-u^2}}{u^2}du$ and $I_2=\int_L \frac{e^{-u^2}}{u^4}du$. Integral by parts, it yields

$$I_1=\int_L \frac{e^{-u^2}}{u^2}du = -\left.\frac{e^{-u^2}}{u}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{2u\,e^{-u^2}}{u}du = 0-\int_{-\infty}^{\infty}2\,e^{-u^2}du = -2\sqrt{\pi}$$

$$I_2=\int_L \frac{e^{-u^2}}{u^4}du = -\left.\frac{1}{3}\frac{e^{-u^2}}{u^3}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{2u\,e^{-u^2}}{3u^3}du = 0-\int_{-\infty}^{\infty}\frac{2\,e^{-u^2}}{3u^2}du = -\frac{2}{3}I_1 = +\frac{4}{3}\sqrt{\pi}$$

Thus, for $|\xi|<<1$, $Z(\xi)$ is approximately

$$Z(\xi)=\frac{1}{\sqrt{\pi}}(\xi I_1+\xi^3 I_2+\cdots+\pi i e^{-\xi^2}) = -2\xi+\frac{4}{3}\xi^3-\frac{8}{15}\xi^5+\cdots+i\sqrt{\pi}\,e^{-\xi^2}$$

References:

Fried, B. F., and S. Conte (1961), *The Plasma Dispersion Function*, Academic, New York.

Mathews, J., and R. L. Walker (1970), *Mathematical Methods of Physics*, 2nd edition, Addison-Wesley, New York.