Appendix G. Functions of Complex Variable

Suggested Readings:
(1) Appendix in Mathews and Walker (1970)
(2) Appendix C in Nicholson (1983)

G.1. Analytic Function & Residue Theorem

This section provides background knowledge for studying inverse Laplace transfer and Landau contours.

If $f(z)$ be an analytic function in domain D, then for a closed loop $C \in D$, we have $\oint_C f(z)dz = 0$.

Let us consider the following integration

$$\oint_C \frac{f(z)}{z - z_0} dz$$

where $C: z = z_0 + \varepsilon e^{i\theta}$ for $\theta = 0 \to 2\pi$ and $\varepsilon \to 0$. Namely, for $z \in C$, we have $z - z_0 = \varepsilon e^{i\theta}$, $\lim_{\varepsilon \to 0} f(z) \approx f(z_0)$, and $dz = i\varepsilon e^{i\theta} d\theta$. Thus, the above integration becomes

$$\oint_C \frac{f(z)}{z - z_0} dz = \int_0^{2\pi} \frac{f(z_0)}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta = 2\pi i f(z_0)$$

or

$$f(\zeta) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \zeta} dz \tag{G.1}$$

If we define

$$g(z) = \frac{f(z)}{z - \zeta} \tag{G.2a}$$

then

$$\oint_C g(z)dz = 2\pi i[(z - \zeta)g(z)]_{z=\zeta} \tag{G.2b}$$

It is commonly called $[(z - \zeta)g(z)]_{z=\zeta}$ the *residue* of function $g(z)$ with a simple pole at $z = \zeta$.

Differentiating Eq. (G.1) with respect to ζ, yields

$$\frac{d}{d\zeta} f(\zeta) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-\zeta)^2} dz \tag{G.3}$$

If we define

$$g(z) = \frac{f(z)}{(z-\zeta)^2} \tag{G.4a}$$

then

$$\oint_C g(z)dz = 2\pi i \left(\frac{d}{dz}[(z-\zeta)^2 g(z)] \right)_{z=\zeta} \tag{G.4b}$$

Thus, $\left(\frac{d}{dz}[(z-\zeta)^2 g(z)] \right)_{z=\zeta}$ is the *residue* of function $g(z) = \frac{f(z)}{(z-\zeta)^2}$ at $z = \zeta$.

Likewise, Eq. (G.1) yields

$$\frac{d^{n-1}}{d\zeta^{n-1}} f(\zeta) = \frac{(n-1)!}{2\pi i} \oint_C \frac{f(z)}{(z-\zeta)^n} dz \tag{G.5}$$

If we define

$$g(z) = \frac{f(z)}{(z-\zeta)^n} \tag{G.6a}$$

then

$$\oint_C g(z)dz = 2\pi i \frac{1}{(n-1)!} \left(\frac{d^{n-1}}{dz^{n-1}}[(z-\zeta)^n g(z)] \right)_{z=\zeta} \tag{G.6b}$$

and $\frac{1}{(n-1)!} \left(\frac{d^{n-1}}{dz^{n-1}}[(z-\zeta)^n g(z)] \right)_{z=\zeta}$ is the *residue* of function $g(z) = \frac{f(z)}{(z-\zeta)^n}$ at $z = \zeta$.

G.2. Branch Point and Riemann Surface

This section provides background knowledge for stability analysis based on Nyquist method.

Definition of *branch point*

If a function $f(z)$ has a *branch point* at $z = z_0$, then $f(z_0 + \varepsilon e^{i\theta})$ is different from $f(z_0 + \varepsilon e^{i(\theta+2\pi)})$.

Definition of *Riemann surface*

Riemann surface is a surface of $z = z_0 + \varepsilon e^{i\theta}$ where $\varepsilon = 0 \to \infty$ and $\theta = 0 \to 2n\pi$, where n is large enough so that $f(z_0 + \varepsilon e^{i\theta}) = f(z_0 + \varepsilon e^{i[\theta + 2(n+1)\pi]})$. Definition of Riemann surface can be more complicated if a function $f(z)$ has more than one branch point.

Example 1:

Function $f(z)=\sqrt{z}$ has a branch point at $z=0$, because $f(re^{i\theta})\neq f(re^{i(\theta+2\pi)})$, but $z=re^{i\theta}=re^{i(\theta+2\pi)}$.

We can choose Riemann surface of $f(z)=\sqrt{z}$ to be a surface of $z=re^{i\theta}$ with $r=0\to\infty$ and $\theta=0\to 4\pi$.

Example 2:

Function $f(z)=\sqrt[3]{z-z_0}$ has a branch point at $z=z_0$, because $f(z_0+\varepsilon e^{i\theta})$, $f(z_0+\varepsilon e^{i(\theta+2\pi)})$, and $f(z_0+\varepsilon e^{i(\theta+4\pi)})$ are different from each other, but $z=z_0+\varepsilon e^{i\theta}=z_0+\varepsilon e^{i(\theta+2\pi)}=z_0+\varepsilon e^{i(\theta+4\pi)}$.

We can choose Riemann surface of $f(z)=\sqrt[3]{z-z_0}$ to be a surface of $z=z_0+\varepsilon e^{i\theta}$ with $\varepsilon=0\to\infty$ and $\theta=0\to 6\pi$.

Example 3:

Function $f(z)=\ln z$ has a branch point at $z=0$, because for $z=re^{i(\theta+2n\pi)}$, different n leads to the same z but different $f(z)=\ln z=\ln(r)+i(\theta+2n\pi)$.

We can choose Riemann surface of $f(z)=\ln z$ to be a surface of $z=re^{i\theta}$ with $r=0\to\infty$ and $\theta=-\infty\to+\infty$.

Consider integration along a closed loop C in the complex z-plane. If the loop C circles around point $z=0$ for N times, then

$$\oint_C d\ln z=\oint_C (d\ln r+i\,d\theta)=(2\pi i)N$$

Note that loop C is not a closed loop on the Riemann surface of $f(z)=\ln z$.

References

Mathews, J., and R. L. Walker (1970), *Mathematical Methods of Physics*, 2nd edition, Addison-Wesley, New York.

Nicholson, D. R. (1983), *Introduction to Plasma Theory*, John Wiley & Sons, New York.