Appendix F. Deriving the Relativistic Vlasov Equation From the Relativistic Klimontovich Equation

F.1. Relativistic Klimontovich Equation

Let us define a relativistic microscopic distribution function of the αth species in six-dimensional phase space

$$N_\alpha(\mathbf{x},\mathbf{u},t) = \sum_{k=1}^{N_0} \delta[\mathbf{x} - \mathbf{x}_k(t)]\delta[\mathbf{u} - \mathbf{u}_k(t)] \tag{F.1}$$

where $\mathbf{x}_k(t)$ and $\mathbf{u}_k(t)$ satisfy the following relativistic equations of motion

$$\frac{d\mathbf{x}_k(t)}{dt} = \frac{\mathbf{u}_k(t)}{\sqrt{1 + \dfrac{\mathbf{u}_k(t)\cdot\mathbf{u}_k(t)}{c^2}}} \tag{F.2}$$

$$\frac{d\mathbf{u}_k(t)}{dt} = \frac{e_\alpha}{m_\alpha}\{\mathbf{E}^m[\mathbf{x}_k(t),t] + \frac{\mathbf{u}_k(t)}{\sqrt{1 + \dfrac{\mathbf{u}_k(t)\cdot\mathbf{u}_k(t)}{c^2}}} \times \mathbf{B}^m[\mathbf{x}_k(t),t]\} \tag{F.3}$$

in which $\mathbf{E}^m(\mathbf{x},t)$ and $\mathbf{B}^m(\mathbf{x},t)$ are the microscopic electric field and magnetic field, respectively. Following the same procedure as discussed in Chapter 2, relativistic Klimontovich equation can be obtained by evaluating time derivative of $N_\alpha(\mathbf{x},\mathbf{u},t)$.

$$\frac{\partial N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial t}=\frac{\partial}{\partial t}\sum_{k=1}^{N_0}\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{u}-\mathbf{u}_k(t)]$$

$$=\sum_{k=1}^{N_0}\{\frac{\partial}{\partial t}\delta[\mathbf{x}-\mathbf{x}_k(t)]\}\delta[\mathbf{u}-\mathbf{u}_k(t)]+\sum_{k=1}^{N_0}\delta[\mathbf{x}-\mathbf{x}_k(t)]\{\frac{\partial}{\partial t}\delta[\mathbf{u}-\mathbf{u}_k(t)]\}$$

$$=\sum_{k=1}^{N_0}\{\frac{\partial\delta[\mathbf{x}-\mathbf{x}_k(t)]}{\partial\mathbf{x}}\cdot[-\frac{d\mathbf{x}_k(t)}{dt}]\}\delta[\mathbf{u}-\mathbf{u}_k(t)]+\sum_{k=1}^{N_0}\delta[\mathbf{x}-\mathbf{x}_k(t)]\{\frac{\partial\delta[\mathbf{u}-\mathbf{u}_k(t)]}{\partial\mathbf{u}}\cdot[-\frac{d\mathbf{u}_k(t)}{dt}]\}$$

$$=\sum_{k=1}^{N_0}\{\frac{\partial}{\partial\mathbf{x}}\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{u}-\mathbf{u}_k(t)]\}\cdot[-\frac{\mathbf{u}_k(t)}{\sqrt{1+\frac{\mathbf{u}_k(t)\cdot\mathbf{u}_k(t)}{c^2}}}]$$

$$+\sum_{k=1}^{N_0}\{\frac{\partial}{\partial\mathbf{u}}\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{u}-\mathbf{u}_k(t)]\}\cdot[-\frac{e_\alpha}{m_\alpha}\{\mathbf{E}^m[\mathbf{x}_k(t),t]+\frac{\mathbf{u}_k(t)}{\sqrt{1+\frac{\mathbf{u}_k(t)\cdot\mathbf{u}_k(t)}{c^2}}}\times\mathbf{B}^m[\mathbf{x}_k(t),t]\}]$$

$$=\sum_{k=1}^{N_0}\{\frac{\partial}{\partial\mathbf{x}}\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{u}-\mathbf{u}_k(t)]\}\cdot[-\frac{\mathbf{u}}{\sqrt{1+\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}}}]$$

$$+\sum_{k=1}^{N_0}\{\frac{\partial}{\partial\mathbf{u}}\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{u}-\mathbf{u}_k(t)]\}\cdot[-\frac{e_\alpha}{m_\alpha}\{\mathbf{E}^m(\mathbf{x},t)+\frac{\mathbf{u}}{\sqrt{1+\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}}}\times\mathbf{B}^m(\mathbf{x},t)\}]$$

$$=[-\frac{\mathbf{u}}{\sqrt{1+\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}}}]\cdot\frac{\partial}{\partial\mathbf{x}}\sum_{k=1}^{N_0}\{\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{u}-\mathbf{u}_k(t)]\}$$

$$+[-\frac{e_\alpha}{m_\alpha}\{\mathbf{E}^m(\mathbf{x},t)+\frac{\mathbf{u}}{\sqrt{1+\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}}}\times\mathbf{B}^m(\mathbf{x},t)\}]\cdot\frac{\partial}{\partial\mathbf{u}}\sum_{k=1}^{N_0}\{\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{u}-\mathbf{u}_k(t)]\}$$

$$=-\frac{\mathbf{u}}{\sqrt{1+\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}}}\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{x}}-\frac{e_\alpha}{m_\alpha}\{\mathbf{E}^m(\mathbf{x},t)+\frac{\mathbf{u}}{\sqrt{1+\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}}}\times\mathbf{B}^m(\mathbf{x},t)\}\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{u}}$$

or

$$\boxed{\frac{\partial N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial t}+\mathbf{v}(\mathbf{u})\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{x}}+\frac{e_\alpha}{m_\alpha}\{\mathbf{E}^m(\mathbf{x},t)+\mathbf{v}(\mathbf{u})\times\mathbf{B}^m(\mathbf{x},t)\}\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{u}}=0}\qquad\text{(F.4)}$$

where

$$\mathbf{v}(\mathbf{u})\equiv\frac{\mathbf{u}}{\sqrt{1+\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}}}$$

Eq. (F.4) is the relativistic Klimontovich equation of the microscopic distribution function $N_\alpha(\mathbf{x},\mathbf{u},t)$.

F.2. Relativistic Vlasov Equation

Let $f_\alpha(\mathbf{x},\mathbf{u},t)$, $\mathbf{E}(\mathbf{x},t)$, and $\mathbf{B}(\mathbf{x},t)$ be the ensemble average of $N_\alpha(\mathbf{x},\mathbf{u},t)$, $\mathbf{E}^m(\mathbf{x},t)$, and $\mathbf{B}^m(\mathbf{x},t)$, respectively. Let

$N_\alpha(\mathbf{x},\mathbf{u},t) = f_\alpha(\mathbf{x},\mathbf{u},t) + \delta N_\alpha(\mathbf{x},\mathbf{u},t)$

$\mathbf{E}^m(\mathbf{x},t) = \mathbf{E}(\mathbf{x},t) + \delta\mathbf{E}^m(\mathbf{x},t)$

$\mathbf{B}^m(\mathbf{x},t) = \mathbf{B}(\mathbf{x},t) + \delta\mathbf{B}^m(\mathbf{x},t)$

If we use $\langle A \rangle$ to denote ensemble average of A, then we have

$\langle N_\alpha(\mathbf{x},\mathbf{u},t)\rangle = f_\alpha(\mathbf{x},\mathbf{u},t)$

$\langle \mathbf{E}^m(\mathbf{x},t)\rangle = \mathbf{E}(\mathbf{x},t)$

$\langle \mathbf{B}^m(\mathbf{x},t)\rangle = \mathbf{B}(\mathbf{x},t)$

and

$\langle \delta N_\alpha(\mathbf{x},\mathbf{u},t)\rangle = 0$

$\langle \delta\mathbf{E}^m(\mathbf{x},t)\rangle = 0$

$\langle \delta\mathbf{B}^m(\mathbf{x},t)\rangle = 0$

Taking ensemble average of Eq. (F.4) yields,

$$\left\langle \frac{\partial N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial t} + \mathbf{v}(\mathbf{u})\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}\{\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}(\mathbf{u})\times\mathbf{B}^m(\mathbf{x},t)\}\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}} \right\rangle = 0$$

or

$$\boxed{\begin{aligned}&\frac{\partial f_\alpha(\mathbf{x},\mathbf{u},t)}{\partial t} + \mathbf{v}(\mathbf{u})\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}[\mathbf{E}(\mathbf{x},t) + \mathbf{v}(\mathbf{u})\times\mathbf{B}(\mathbf{x},t)]\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}}\\ &+\frac{e_\alpha}{m_\alpha}\left\langle [\delta\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}(\mathbf{u})\times\delta\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial \delta N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}}\right\rangle = 0\end{aligned}} \tag{F.5}$$

Let $Df_\alpha(\mathbf{x},\mathbf{u},t)/Dt$ denote the time derivative of the distribution function $f_\alpha(\mathbf{x},\mathbf{u},t)$ along its characteristic curve in the $(\mathbf{x},\mathbf{u})$ phase space, then Eq. (F.5) can be rewritten as

$$\boxed{\begin{aligned}&\frac{Df_\alpha(\mathbf{x},\mathbf{u},t)}{Dt} = \frac{\partial f_\alpha(\mathbf{x},\mathbf{u},t)}{\partial t} + \mathbf{v}(\mathbf{u})\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}[\mathbf{E}(\mathbf{x},t) + \mathbf{v}(\mathbf{u})\times\mathbf{B}(\mathbf{x},t)]\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}}\\ &= -\frac{e_\alpha}{m_\alpha}\left\langle [\delta\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}(\mathbf{u})\times\delta\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial \delta N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}}\right\rangle = \left.\frac{\delta f_\alpha(\mathbf{x},\mathbf{u},t)}{\delta t}\right|_{collision}\end{aligned}} \tag{F.6}$$

For

$$= -\frac{e_\alpha}{m_\alpha}\left\langle [\delta\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}(\mathbf{u}) \times \delta\mathbf{B}^m(\mathbf{x},t)] \cdot \frac{\partial \delta N_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}} \right\rangle = \left.\frac{\delta f_\alpha(\mathbf{x},\mathbf{u},t)}{\delta t}\right|_{collision} = 0$$

The Boltzmann equation Eq. (F.6) is reduced to Vlasov equation:

$$\boxed{\frac{\partial f_\alpha(\mathbf{x},\mathbf{u},t)}{\partial t} + \mathbf{v}(\mathbf{u}) \cdot \frac{\partial f_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}[\mathbf{E}(\mathbf{x},t) + \mathbf{v}(\mathbf{u}) \times \mathbf{B}(\mathbf{x},t)] \cdot \frac{\partial f_\alpha(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}}} \tag{F.7}$$

where

$$\mathbf{v}(\mathbf{u}) \equiv \frac{\mathbf{u}}{\sqrt{1 + \frac{\mathbf{u} \cdot \mathbf{u}}{c^2}}}$$