Appendix D. Curvature Drift

Let us consider a charge particle with constant magnetic moment and non-zero velocity component parallel to the local magnetic field. If curvature of the magnetic field line is non-zero, then the particle's field-aligned moving frame will become a non-inertial frame. If the parallel speed $v_{\|}$ is nearly a constant, the *equation of motion* in the $v_{\|}$ non-inertial moving frame, can be approximately written as

$$m\frac{d\mathbf{v}}{dt} = \frac{\hat{R}_B m v_{\|}^2}{R_B} + q\mathbf{v}\times\mathbf{B} \tag{D.1}$$

where $\mathbf{v} = \mathbf{v}_{gyro} + \mathbf{v}_{drift}$, $\mathbf{v}_{gyro}$ is the high frequency gyro motion velocity, and $\mathbf{v}_{drift}$ is the low frequency (or time independent) drift velocity. Averaging Eq. (D.1) over one gyro period ($\tau = 2\pi/\Omega_c$, where $\Omega_c = |q|B/m$), we can obtain equation for low frequency guiding-center motion in the $v_{\|}$ non-inertial moving frame

$$\frac{\hat{R}_B m v_{\|}^2}{R_B} + q\mathbf{v}_{drift}\times\mathbf{B} = 0 \tag{D.2}$$

Solution of $\mathbf{v}_{drift}$ can be written as

$$\mathbf{v}_{drift} = \frac{\dfrac{\hat{R}_B m v_{\|}^2}{qR_B}\times\mathbf{B}}{B^2} = \frac{m v_{\|}^2}{qB^2}\left(\frac{\hat{R}_B}{R_B}\times\mathbf{B}\right) \tag{D.3}$$

Since

$$
\begin{aligned}
-\frac{\hat{R}_B}{R_B} &= \frac{d\hat{B}}{ds} = \hat{B}\cdot\nabla\hat{B} = \frac{\mathbf{B}}{B}\cdot\nabla\left(\frac{\mathbf{B}}{B}\right) \\
&= \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{B^2} - \frac{\mathbf{B}\mathbf{B}\cdot\nabla B}{B^3} \\
&= \frac{-\mathbf{B}\times(\nabla\times\mathbf{B}) + \nabla\dfrac{B^2}{2}}{B^2} - \frac{\hat{B}\hat{B}\cdot\nabla B}{B} \\
&= \frac{-\mathbf{B}\times(\nabla\times\mathbf{B})}{B^2} + \frac{\nabla B}{B} - \frac{\hat{B}\hat{B}\cdot\nabla B}{B} \\
&= \frac{-\mathbf{B}\times(\nabla\times\mathbf{B})}{B^2} + (\mathbf{1} - \hat{B}\hat{B})\cdot\frac{\nabla B}{B} \\
&= \frac{-\mathbf{B}\times(\nabla\times\mathbf{B})}{B^2} + \frac{\nabla_{\perp} B}{B}
\end{aligned}
$$

where $\mathbf{1} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$ is the unit tensor. For any given vector $\mathbf{A}$, we have $\mathbf{A} = \mathbf{A}\cdot\mathbf{1} = \mathbf{1}\cdot\mathbf{A}$.

If the magnetic field is along the z-direction, then $\hat{B}\hat{B} = \hat{z}\hat{z}$, and

$$\mathbf{1} - \hat{B}\hat{B} = \hat{x}\hat{x} + \hat{y}\hat{y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Any vector $\mathbf{A}$ can be decomposed into two parts, one is parallel to the local magnetic field ($\mathbf{A}_{\parallel}$), another is perpendicular to the local magnetic field ($\mathbf{A}_{\perp}$). Namely, $\mathbf{A} = \mathbf{A}_{\parallel} + \mathbf{A}_{\perp}$. It can be shown that

$$\boxed{\mathbf{A}_{\parallel} = \hat{B}\hat{B} \cdot \mathbf{A}}$$

and

$$\boxed{\mathbf{A}_{\perp} = (\mathbf{1} - \hat{B}\hat{B}) \cdot \mathbf{A}}$$

Likewise, we have

$$\nabla_{\perp} B = \nabla B - \hat{B}\hat{B} \cdot \nabla B = (\mathbf{1} - \hat{B}\hat{B}) \cdot \nabla B$$

Substituting

$$\boxed{-\frac{\hat{R}_B}{R_B} = \frac{-\mathbf{B} \times (\nabla \times \mathbf{B})}{B^2} + \frac{\nabla_{\perp} B}{B}}$$

into Eq. (D.3), yields

$$\begin{aligned}
\mathbf{v}_{drift} &= \frac{mv_{\parallel}^2}{qB^2}\left(\frac{\hat{R}_B}{R_B} \times \mathbf{B}\right) = \frac{-mv_{\parallel}^2}{qB^2}\left(\frac{-\hat{R}_B}{R_B} \times \mathbf{B}\right) \\
&= \frac{-mv_{\parallel}^2}{qB^2}\left[\left(\frac{-\mathbf{B} \times (\nabla \times \mathbf{B})}{B^2} + \frac{\nabla_{\perp} B}{B}\right) \times \mathbf{B}\right] \\
&= \frac{mv_{\parallel}^2}{qB^2}\left[\frac{B^2(\nabla \times \mathbf{B})}{B^2} - \frac{\mathbf{B}\mathbf{B} \cdot (\nabla \times \mathbf{B})}{B^2} - \frac{\nabla_{\perp} B}{B} \times \mathbf{B}\right] \\
&= \frac{mv_{\parallel}^2}{qB^2}\left[(\mathbf{1} - \hat{B}\hat{B}) \cdot (\nabla \times \mathbf{B}) - \frac{\nabla_{\perp} B}{B} \times \mathbf{B}\right] \\
&= \frac{mv_{\parallel}^2}{qB^2}\left[(\nabla \times \mathbf{B})_{\perp} - \frac{\nabla_{\perp} B}{B} \times \mathbf{B}\right]
\end{aligned}$$

Namely, curvature drift velocity can be written as

$$\boxed{\mathbf{v}_{drift} = \frac{mv_{\parallel}^2}{qB^2}\left(\frac{\hat{R}_B}{R_B} \times \mathbf{B}\right)}$$

or

$$\boxed{\mathbf{v}_{drift} = \frac{mv_{\parallel}^2}{qB^2}\left[(\nabla \times \mathbf{B})_{\perp} - \frac{\nabla_{\perp} B}{B} \times \mathbf{B}\right]}$$