Appendix B. Ohm's Law in One-Fluid Plasma

In addition to the various types of equation of state (energy equation), the Ohm's law in the one-fluid plasma is another equation that has many different approximations.

Substituting $\rho = n(m_i + m_e)$ in to Eq. (3.53), the generalized Ohm's law can be rewritten as

$$\frac{\partial}{\partial t}\mathbf{J} + \nabla\cdot\{\frac{e}{m_i}\mathsf{P}_i - \frac{e}{m_e}\mathsf{P}_e + \frac{\mathbf{VJ}+\mathbf{JV}-\rho_c\mathbf{VV}-\frac{\mathbf{JJ}}{en}[\frac{m_i-m_e}{m_i+m_e}+\frac{m_i m_e}{(m_i+m_e)^2}\frac{\rho_c}{en}]}{1-\frac{m_i-m_e}{m_i+m_e}\frac{\rho_c}{en}-\frac{m_i m_e}{(m_i+m_e)^2}(\frac{\rho_c}{en})^2}\}$$

$$-\frac{(m_i+m_e)ne^2}{m_i m_e}(\mathbf{E}+\mathbf{V}\times\mathbf{B}) + \frac{e(m_i-m_e)}{m_i m_e}(\rho_c\mathbf{E}+\mathbf{J}\times\mathbf{B}) = 0 \qquad \text{(B.1)}$$

where $n \equiv \rho/(m_i + m_e)$.

Multiplying Eq. (B.1) by $\mu_0 c^2/\omega_{pe}^2 (= m_e/ne^2)$ yields,

$$\frac{c^2}{\omega_{pe}^2}\frac{\partial}{\partial t}(\mu_0\mathbf{J}) + \frac{1}{ne}(\frac{m_e}{m_i}\nabla\cdot\mathsf{P}_i - \nabla\cdot\mathsf{P}_e)$$

$$+\frac{c^2}{\omega_{pe}^2}\nabla\cdot\{\frac{\mu_0\mathbf{VJ}+\mu_0\mathbf{JV}-\mu_0\rho_c\mathbf{VV}-\mu_0\frac{\mathbf{JJ}}{en}[\frac{m_i-m_e}{m_i+m_e}+\frac{m_i m_e}{(m_i+m_e)^2}\frac{\rho_c}{en}]}{1-\frac{m_i-m_e}{m_i+m_e}\frac{\rho_c}{en}-\frac{m_i m_e}{(m_i+m_e)^2}(\frac{\rho_c}{en})^2}\}$$

$$-\frac{(m_i+m_e)}{m_i}(\mathbf{E}+\mathbf{V}\times\mathbf{B}) + \frac{(m_i-m_e)}{m_i}(\frac{\rho_c}{ne}\mathbf{E}+\frac{\mathbf{J}}{ne}\times\mathbf{B}) = 0 \qquad \text{(B.2)}$$

For $m_e << m_i$ Eq. (B.2) can be approximated by

$$\boxed{\begin{aligned}&\frac{c^2}{\omega_{pe}^2}\frac{\partial}{\partial t}(\mu_0\mathbf{J}) - \frac{1}{ne}\nabla\cdot\mathsf{P}_e + \frac{c^2}{\omega_{pe}^2}\nabla\cdot\{(\mu_0\mathbf{VJ}+\mu_0\mathbf{JV}-\mu_0\rho_c\mathbf{VV}-\mu_0\frac{\mathbf{JJ}}{en})/(1-\frac{\rho_c}{en})\}\\ &\quad -(\mathbf{E}+\mathbf{V}\times\mathbf{B}) + (\frac{\rho_c}{ne}\mathbf{E}+\frac{\mathbf{J}}{ne}\times\mathbf{B}) = 0\end{aligned}} \qquad \text{(B.3)}$$

This is the first approximation of the generalized Ohm's law.

Ampere's law can be rewritten as

$$\mu_0\mathbf{J} = \nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \qquad \text{(B.4)}$$

Taking time derivative of Eq. (B.4), it yields

$$\frac{\partial}{\partial t}(\mu_0 \mathbf{J}) = \frac{\partial}{\partial t}(\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}) = \nabla \times (-\nabla \times \mathbf{E}) - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E} - \nabla\nabla \cdot \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{B.5}$$

where Faraday's law has been used in deriving Eq. (B.5). Substituting Eq. (B.5) into Eq. (B.3), it yields

$$\boxed{\begin{aligned}&\frac{c^2}{\omega_{pe}^2}(\nabla^2 \mathbf{E} - \nabla\nabla \cdot \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}) + \frac{c^2}{\omega_{pe}^2}\nabla \cdot \{(\mathbf{V}\mu_0\mathbf{J} + \mu_0\mathbf{J}\mathbf{V} - \mu_0\rho_c\mathbf{V}\mathbf{V} - \mu_0\frac{\mathbf{J}\mathbf{J}}{en})/(1-\frac{\rho_c}{en})\} \\ &\quad -(\mathbf{E} + \mathbf{V} \times \mathbf{B}) + \frac{\rho_c}{ne}\mathbf{E} + \frac{\mathbf{J}}{ne} \times \mathbf{B} - \frac{1}{ne}\nabla \cdot \mathbf{P}_e = 0\end{aligned}} \tag{B.6}$$

Thus, we can conclude that the first two terms in Eq. (B.6), i.e.,

$$\frac{c^2}{\omega_{pe}^2}(\nabla^2 \mathbf{E} - \nabla\nabla \cdot \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2})$$

and

$$\frac{c^2}{\omega_{pe}^2}\nabla \cdot \{(\mathbf{V}\mu_0\mathbf{J} + \mu_0\mathbf{J}\mathbf{V} - \mu_0\rho_c\mathbf{V}\mathbf{V} - \mu_0\frac{\mathbf{J}\mathbf{J}}{en})/(1-\frac{\rho_c}{en})\}$$

contain second order or third order time and/or spatial derivatives of the electric field or the magnetic field. The last three terms in Eq. (B.6), i.e.,

$$\frac{\rho_c}{ne}\mathbf{E},\ \frac{\mathbf{J}}{ne} \times \mathbf{B}, \text{ and } -\frac{1}{ne}\nabla \cdot \mathbf{P}_e$$

contain first order derivative terms of electric field or magnetic field in space. Only the term $-(\mathbf{E} + \mathbf{V} \times \mathbf{B})$ in Eq. (B.6) contains no time derivative or spatial derivative of the electric field and magnetic field. It can be shown that the first two terms and the third last term $\rho_c \mathbf{E}/ne$ in Eq. (B.6) can be ignored when the spatial scale length is much greater than the electron inertial length (c/ω_{pe}). The last two terms in Eq. (B.6) can be ignored when the scale length is much greater than the ions' inertial length (c/ω_{pi}).

For magnetohydrodynamic (MHD) plasma, the spatial scale length is much greater than the ions' inertial length. Thus, the MHD Ohm's law becomes,

$$\boxed{\mathbf{E} = -\mathbf{V} \times \mathbf{B}} \tag{B.7}$$

For ion-scale phenomena, the spatial scale length is much greater than the electrons' inertial length but equal or slightly larger than the ions' inertial length. Thus, the ion-scale Ohm's law becomes,

$$\boxed{\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \frac{\mathbf{J}}{ne}\times\mathbf{B} - \frac{1}{ne}\nabla\cdot\mathbf{P}_e} \tag{B.8}$$

An easy way to obtain Eq. (B.8):

Equation (B.8) can also be obtained directly from the electrons' momentum equation, i.e.,

$$m_e n_e(\frac{\partial}{\partial t} + \mathbf{V}_e\cdot\nabla)\mathbf{V}_e = -\nabla\cdot\mathbf{P}_e - en_e(\mathbf{E} + \mathbf{V}_e\times\mathbf{B}) \tag{B.9}$$

For ion-scale phenomena, we can ignore the electron-inertial term on the left-hand side of Eq. (B.9), it yields

$$-\frac{1}{en_e}\nabla\cdot\mathbf{P}_e - (\mathbf{E} + \mathbf{V}_e\times\mathbf{B}) = 0 \tag{B.9a}$$

It can be shown that, for $|n_i - n_e| << n_e \approx n$ and for $m_i << m_e$, the flow velocity of the electrons is approximately equal to,

$$\mathbf{V}_e = \mathbf{V} - \frac{\mathbf{J}}{ne} \tag{B.10}$$

Substituting Eq. (B.10) into Eq. (B.9a), it yields equation (B.8).

For isotropic electron pressure ($\mathbf{P}_e = \mathbf{1}p_e$) and for $|n_i - n_e| << n$, Eq. (B.8) is reduced to

$$\boxed{\mathbf{E}_\perp = -\mathbf{V}\times\mathbf{B} + \frac{\mathbf{J}}{ne}\times\mathbf{B} - \frac{1}{ne}\nabla_\perp p_e} \tag{B.11}$$

or

$$\boxed{\mathbf{E}_\perp = -\mathbf{V}_e\times\mathbf{B} - \frac{1}{ne}\nabla_\perp p_e} \tag{B.11a}$$

and

$$\boxed{\mathbf{E}_\| = -\frac{1}{ne}\nabla_\| p_e} \tag{B.12}$$

where subscripts $\|$ and $\perp$ denote directions perpendicular to and parallel to the local magnetic field, respectively.

For $\nabla_\perp p_e \to 0$, Eqs. (B.11) and (B.11a) are reduced to

$$\boxed{\mathbf{E}_\perp = -\mathbf{V}\times\mathbf{B} + \frac{\mathbf{J}}{ne}\times\mathbf{B}} \tag{B.13}$$

or

$$\boxed{\mathbf{E}_{\perp} = -\mathbf{V}_e \times \mathbf{B}} \qquad \text{(B.13a)}$$

where Eqs. (B.13) and (B.13a) are commonly called *the Hall-MHD Ohm's law*.

For the electron-scale phenomena or for the ion-electron cross-scale phenomena, the Ohm's law is given by Eq. (B.3) or Eq. (B.6). To study the electron-scale fluid phenomena, or the ion-electron cross-scale fluid phenomena, it would be better to use the electron-ion two-fluid equations, instead of the one-fluid equations with complicated generalized Ohm's law.