Appendix A. Static Electric Field and Magnetic Field

A.1. General Solutions

For static electric field and magnetic field, we have

$$\frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \frac{\partial \mathbf{E}}{\partial t} = 0$$

$$\nabla \times \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\mathbf{E} = -\nabla \Phi \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$-\nabla^2 \Phi = \frac{\rho_c(\mathbf{r})}{\varepsilon_0} \qquad -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}(\mathbf{r})$$

where $\nabla \cdot \mathbf{A} = 0$ (Coulomb gauge) has been assumed.

Let us consider a Green's function, which satisfies Poisson equation with a source term of three-dimensional delta function, i.e.,

$$\nabla^2 G = \delta(\mathbf{r}) = \frac{\delta(r)}{4\pi r^2}$$

where $\int \delta(\mathbf{r}) d\mathbf{r} = 1 = \iiint \delta(\mathbf{r}) r^2 \sin\theta \, dr \, d\theta \, d\phi$ and $\int \delta(r) dr = 1$

One can easily show that solution of this Green's function, is

$$G(\mathbf{r}) = -\frac{1}{4\pi r} = -\frac{1}{4\pi |\mathbf{r}|}$$

Proof:

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} G = \frac{\delta(r)}{4\pi r^2}$$

$$\Rightarrow \frac{d}{dr} r^2 \frac{d}{dr} G = \frac{\delta(r)}{4\pi}$$

$$\Rightarrow r^2 \frac{d}{dr} G = \frac{1}{4\pi}$$

$$\Rightarrow \frac{d}{dr} G = \frac{1}{4\pi r^2}$$

$$\Rightarrow G = -\frac{1}{4\pi r}$$

Thus,

$$
\begin{aligned}
-\nabla^2\Phi &= \frac{\rho_c(\mathbf{r})}{\varepsilon_0} \\
&= \int \frac{\rho_c(\mathbf{r}')}{\varepsilon_0}\delta(\mathbf{r}-\mathbf{r}')d\mathbf{r}' \\
&= \int \frac{\rho_c(\mathbf{r}')}{\varepsilon_0}\nabla^2 G(\mathbf{r}-\mathbf{r}')d\mathbf{r}' \\
&= \nabla^2 \int \frac{\rho_c(\mathbf{r}')}{\varepsilon_0} G(\mathbf{r}-\mathbf{r}')d\mathbf{r}' \\
&= \nabla^2 \int \frac{\rho_c(\mathbf{r}')(-1)}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'
\end{aligned}
$$

Namely,

$$\boxed{\Phi(\mathbf{r}) = \int \frac{\rho_c(\mathbf{r}')}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'} \tag{A.1}$$

Likewise

$$\boxed{\mathbf{A}(\mathbf{r}) = \int \frac{\mu_0\mathbf{J}(\mathbf{r}')}{4\pi |\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' + \nabla\psi} \tag{A.2}$$

where we choose integration constant $\nabla\psi$ such that $\nabla\cdot\mathbf{A}=0$. Namely,

$$\nabla^2\psi = \nabla\cdot\mathbf{A}(\mathbf{r}) - \nabla\cdot\left(\int \frac{\mu_0\mathbf{J}(\mathbf{r}')}{4\pi |\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right) = -\nabla\cdot\left(\int \frac{\mu_0\mathbf{J}(\mathbf{r}')}{4\pi |\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right)$$

A.2. Solutions of Special Cases

If source terms, $\rho_c(\mathbf{r}')$ and $\mathbf{J}(\mathbf{r}')$, in Eqs. (A.1) and (A.2) are confined in a small volume, then we can assume that

$$|\mathbf{r}-\mathbf{r}'| \approx |\mathbf{r}| = r,$$

Let the total charge in the source volume be

$$Q = \int \rho_c(\mathbf{r}')d\mathbf{r}'$$

Then, the scalar potential in Eq. (A.1) can be rewritten as

$$\boxed{\Phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r}} \tag{A.3}$$

Electrostatic electric field, in spherical coordinate system (r,θ,ϕ), becomes

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}} \tag{A.4}$$

Likewise, we can assume the source term in Eq. (A.2) is a local electric current density along

z direction, i.e.,

$$\mathbf{J} = \hat{z} J_z \ .$$

If we also assume that the integration volume is a cylindrical region with cross-section Δa_z and length Δl_z, so that

$$\int d\mathbf{r}' = \iiint d^3 r' = (\iint da_z)\Delta l_z \approx \Delta a_z \Delta l_z$$

By definition, the local electric current becomes

$$I_z = \iint J_z \, da_z \ ,$$

As a result, we can rewrite the vector potential $\mathbf{A} = \hat{z} A_z$ in cylindrical coordinate system (R, ϕ, z), as

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0 (I_z \Delta l_z)}{4\pi \sqrt{R^2 + z^2}} \hat{z} + \nabla \psi} \tag{A.5}$$

where

$$\nabla^2 \psi = -\nabla \cdot \left(\frac{\mu_0 (I_z \Delta l_z)}{4\pi \sqrt{R^2 + z^2}} \hat{z} \right)$$

Thus, magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, in cylindrical coordinate system, becomes

$$\boxed{\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 (I_z \Delta l_z)}{4\pi} \frac{R}{(R^2 + z^2)^{3/2}}} \tag{A.6}$$

Let us consider the following special cases:

Case 1: magnetic field on $z = 0$ plane

$$\boxed{\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 (I_z \Delta l_z)}{4\pi} \frac{1}{R^2}} \tag{A.7}$$

Case 2: magnetic field nearly along the z-axis with $z >> R \to 0$,

$$\boxed{\mathbf{B}(\mathbf{r}) \approx \hat{\phi} \frac{\mu_0 (I_z \Delta l_z)}{4\pi} \frac{R}{z^3}} \tag{A.8}$$

Case 3: magnetic field at center of a circle current loop with circle radius R

$$\boxed{\mathbf{B}(\mathbf{r}) = \hat{z} \frac{\mu_0 (I_\phi 2\pi R)}{4\pi} \frac{1}{R^2} = \hat{z} \frac{\mu_0 I_\phi}{2R}} \tag{A.9}$$

Case 4: magnetic field at distance r from an infinite long line current

$$\boxed{\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 I_z}{2\pi r}} \tag{A.10}$$