Chapter 9. Electrostatic Linear Waves in the Vlasov Plasma

Topics or concepts to learn in Chapter 9:

1. What is Landau contour? Why do we need to use the Landau contour when we determine the electrostatic dispersion relation in the kinetic plasma?
2. Landau damping?
3. What is the kinetic electrostatic dispersion relation of a plasma with field-free background equilibrium?
4. Using Nyquist method to determine whether a uniform plasma with a given velocity distribution is stable or not.
5. Gardner's Theorem
6. Penrose Criterion

Suggested Readings:

(1) Sections 6.3~6.9 in Nicholson (1983)

(2) Sections 8.1~8.7 and Section 9.6 in Krall and Trivelpiece (1973)

(3) Sections 7.1~7.9 in F. F. Chen (1984)

(4) Penrose (1960)

9.1. Landau Contour

In this chapter, we use the electrostatic waves as an example to show the importance of the Landau contour in studying linear waves in the Vlasov plasma. Basic equations to be used in this study include Vlasov equations of the αth species:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} = 0 \tag{9.1}$$

and Poisson equation

$$\nabla\cdot\mathbf{E} = -\nabla^2\Phi = \frac{e}{\varepsilon_0}(n_i - n_e) = \frac{e}{\varepsilon_0}\iiint (f_i - f_e)d\mathbf{v} \tag{9.2}$$

Let us consider a field-free ($\mathbf{E}_0 = 0,\ \ \mathbf{B}_0 = 0$) plasma. Equilibrium distributions of the field-free plasma satisfy the following conditions

$$\iiint f_{i0}\,d\mathbf{v} = \iiint f_{e0}\,d\mathbf{v} = n_0$$

and

$f_{\alpha 0} = f_{\alpha 0}(\mathbf{v})$.

For linear electrostatic waves, we can assume that $\mathbf{E} = \mathbf{E}_1 = -\nabla\Phi_1$, $\mathbf{B} = \mathbf{B}_1 = 0$, and $f_\alpha = f_{\alpha 0} + f_{\alpha 1}$, where $f_{\alpha 1} << f_{\alpha 0}$. In general, the linearized Vlasov equation (9.1) can be written as

$$L_{\alpha 0} f_{\alpha 1} = -L_{\alpha 1} f_{\alpha 0} \tag{9.3}$$

where

$$L_{\alpha 0} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial}{\partial \mathbf{v}} \tag{9.4}$$

$$L_{\alpha 1} = \frac{e_\alpha}{m_\alpha}(\mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1)\cdot\frac{\partial}{\partial \mathbf{v}} \tag{9.5}$$

are the zeroth order and the first order differential operators, respectively. For linear electrostatic waves in field-free background plasma, $L_{\alpha 0}$ and $L_{\alpha 1}$ are reduced to the following forms,

$$L_{\alpha 0} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} \tag{9.6}$$

$$L_{\alpha 1} = \frac{e_\alpha}{m_\alpha}\left(-\frac{\partial \Phi_1}{\partial \mathbf{x}}\right)\cdot\frac{\partial}{\partial \mathbf{v}} \tag{9.7}$$

Substituting Eqs. (9.6) and (9.7) into Eq. (9.3) yields

$$\frac{\partial f_{\alpha 1}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{\alpha 1}}{\partial \mathbf{x}} = \frac{e_\alpha}{m_\alpha}\frac{\partial \Phi_1}{\partial \mathbf{x}}\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} \tag{9.8}$$

Linearizing Poisson equation (9.2) yields

$$\frac{\partial^2 \Phi_1}{\partial \mathbf{x}^2} = \frac{e}{\varepsilon_0}\iiint (f_{e1} - f_{i1})\,d\mathbf{v} \tag{9.9}$$

Integration transform methods, such as Fourier transform and Laplace transform can reduce linear differential equations to algebraic equations. Thus, they are commonly used in linear wave analysis.

Fourier transform is defined by

$$\mathcal{F}[A(\mathbf{x},t)] = \overline{A}(\mathbf{k},t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}} A(\mathbf{x},t)\,d\mathbf{x} \tag{9.10}$$

Inverse-Fourier transform is defined by

$$\mathcal{F}^{-1}[\overline{A}(\mathbf{k},t)] = A(\mathbf{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\overline{A}(\mathbf{k},t)\,d\mathbf{k} \tag{9.11}$$

Laplace transform is defined by

$$\mathcal{L}\,[\bar{A}(\mathbf{k},t)]=\tilde{A}(\mathbf{k},p)=\int_0^\infty e^{-pt}\bar{A}(\mathbf{k},t)dt \quad \text{with } \operatorname{Re}(p)\ge p_0 \tag{9.12}$$

where p_0 is chosen such that $e^{-p_0 t}\bar{A}(\mathbf{k},t)\to 0$ as $t\to\infty$.

Inverse-Laplace transform is defined by

$$\mathcal{L}^{-1}[\tilde{A}(\mathbf{k},p)]=\bar{A}(\mathbf{k},t)=\frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}e^{pt}\tilde{A}(\mathbf{k},p)dp \tag{9.13}$$

It can be easily shown that

$$\mathcal{F}[\frac{\partial}{\partial x}A(x,t)]=ik\bar{A}(k,t) \tag{9.14}$$

and

$$\mathcal{L}\,[\frac{\partial}{\partial t}\bar{A}(k,t)]=p\tilde{A}(k,p)-\bar{A}(k,t=0) \tag{9.15}$$

Or, in more general forms

$$\mathcal{F}[\frac{\partial^n A(x,t)}{\partial x^n}]=(ik)^n\bar{A}(k,t) \tag{9.16}$$

and

$$\mathcal{L}\,[\frac{\partial^n}{\partial t^n}\bar{A}(k,t)]=p^n\tilde{A}(k,p)-[p^{n-1}\bar{A}(k,t=0)+p^{n-2}\left.\frac{\partial\bar{A}(k,t)}{\partial t}\right|_{t=0}+\cdots+\left.\frac{\partial^{n-1}\bar{A}(k,t)}{\partial t^{n-1}}\right|_{t=0}] \tag{9.17}$$

As a result, Fourier transform and Laplace transform convert the $(\mathbf{x},t)$ domain linear differential equation to an algebraic equation in $(\mathbf{k},p)$ domain.

Fourier transform and Laplace transform of Eqs. (9.8) and (9.9), yields

$$[p\tilde{f}_{\alpha 1}(\mathbf{k},\mathbf{v},p)-\bar{f}_{\alpha 1}(\mathbf{k},\mathbf{v},t=0)]+i\mathbf{v}\cdot\mathbf{k}\tilde{f}_{\alpha 1}(\mathbf{k},\mathbf{v},p)=\frac{e_\alpha}{m_\alpha}i\mathbf{k}\tilde{\mathbf{\Phi}}_1(\mathbf{k},p)\cdot\frac{\partial f_{\alpha 0}(\mathbf{v})}{\partial\mathbf{v}} \tag{9.18}$$

$$k^2\tilde{\mathbf{\Phi}}_1(\mathbf{k},p)=\frac{e}{\varepsilon_0}\iiint[\tilde{f}_{i1}(\mathbf{k},\mathbf{v},p)-\tilde{f}_{e1}(\mathbf{k},\mathbf{v},p)]d\mathbf{v} \tag{9.19}$$

where $\operatorname{Re}(p)\ge p_0$.

Let $\mathbf{k}=\hat{k}k$, $\mathbf{k}\cdot\mathbf{v}=ku$ and $\iiint f(\mathbf{v})d\mathbf{v}=\int F(u)du$. Integrating equation (9.18) over a velocity space with components perpendicular to the wave propagation direction $\hat{k}$, it yields

$$[p\tilde{F}_{\alpha 1}(\mathbf{k},u,p)-\bar{F}_{\alpha 1}(\mathbf{k},u,t=0)]+iku\tilde{F}_{\alpha 1}(\mathbf{k},u,p)=\frac{e_\alpha}{m_\alpha}ik\tilde{\mathbf{\Phi}}_1(\mathbf{k},p)\frac{dF_{\alpha 0}(u)}{du} \tag{9.20}$$

or

$$\tilde{F}_{\alpha 1}(\mathbf{k},u,p)=\frac{\frac{e_\alpha}{m_\alpha}ik\tilde{\Phi}_1(\mathbf{k},p)\frac{dF_{\alpha 0}(u)}{du}+\overline{F}_{\alpha 1}(\mathbf{k},u,t=0)}{p+iku} \tag{9.21}$$

Poisson equation (9.19) can be rewritten as

$$k^2\tilde{\Phi}_1(\mathbf{k},p)=\frac{e}{\varepsilon_0}\int[\tilde{F}_{i1}(\mathbf{k},u,p)-\tilde{F}_{e1}(\mathbf{k},u,p)]du \tag{9.22}$$

Substituting Eq. (9.21) into Eq. (9.22) yields

$$\begin{aligned}k^2\tilde{\Phi}_1(\mathbf{k},p)&=\frac{e}{\varepsilon_0}\int[\tilde{F}_{i1}(\mathbf{k},u,p)-\tilde{F}_{e1}(\mathbf{k},u,p)]du\\&=\sum_\alpha\frac{e_\alpha}{\varepsilon_0}[\frac{e_\alpha}{m_\alpha}ik\tilde{\Phi}_1(\mathbf{k},p)\int\frac{dF_{\alpha 0}(u)/du}{p+iku}du+\int\frac{\overline{F}_{\alpha 1}(\mathbf{k},u,t=0)}{p+iku}du]\\&=ik\tilde{\Phi}_1(\mathbf{k},p)\sum_\alpha\frac{e_\alpha}{\varepsilon_0}[\frac{e_\alpha}{m_\alpha}\int\frac{dF_{\alpha 0}(u)/du}{p+iku}du]+\sum_\alpha\frac{e_\alpha}{\varepsilon_0}[\int\frac{\overline{F}_{\alpha 1}(\mathbf{k},u,t=0)}{p+iku}du]\end{aligned}$$

or

$$k^2\tilde{\Phi}_1(\mathbf{k},p)=\frac{\sum_\alpha\frac{e_\alpha}{\varepsilon_0}[\int\frac{\overline{F}_{\alpha 1}(\mathbf{k},u,t=0)}{p+iku}du]}{1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha 0}^2}{n_0}\int\frac{dF_{\alpha 0}(u)/du}{u-(ip/k)}du]} \quad \text{where } \operatorname{Re}(p)\geq p_0. \tag{9.23}$$

where $\omega_{p\alpha 0}^2=n_0e^2/\varepsilon_0 m_\alpha$.

The Poisson equation can be written as $\nabla\cdot\mathbf{E}=\rho_c/\varepsilon_0=[\rho_{cb}(\mathbf{E},\mathbf{B})+\rho_{cf}]/\varepsilon_0$ or $\nabla\cdot\mathbf{D}=\rho_{cf}$. We consider the plasma as a dielectric medium without free charge density (i.e., the charge density free from the inference of the electric field and magnetic field), which yields $\rho_{cf}=0$.

Let $\mathbf{D}_1(\mathbf{x},t)=\varepsilon_0(\overset{\Rightarrow}{\varepsilon}_r\cdot\mathbf{E})_1=\iiint\mathsf{D}(\mathbf{x}-\mathbf{x}',t)\cdot\mathbf{E}_1(\mathbf{x}',t)d\mathbf{x}'+\frac{1}{k^2}\nabla N(\mathbf{x},t)$. It yields

$$\mathcal{L}\{\mathcal{F}[\nabla\cdot\mathbf{D}_1(\mathbf{x},t)]\}=i\mathbf{k}\cdot[\tilde{\mathsf{D}}(\mathbf{k},p)\cdot\tilde{\mathbf{E}}_1(\mathbf{k},p)]+i\mathbf{k}\cdot\frac{i\mathbf{k}\tilde{N}(\mathbf{k},p)}{k^2}=k^2\tilde{D}_{kk}(\mathbf{k},p)\tilde{\Phi}_1(\mathbf{k},p)-\tilde{N}(\mathbf{k},p)$$

For $\nabla\cdot\mathbf{D}_1(\mathbf{x},t)=0$, it yields $k^2\tilde{\Phi}_1(\mathbf{k},p)=\tilde{N}(\mathbf{k},p)/\tilde{D}_{kk}(\mathbf{k},p)$. Since $\tilde{\mathsf{D}}(\mathbf{k},p)$ is closely related to the dielectric function $\overset{\Rightarrow}{\varepsilon}_r$, it is commonly called $\tilde{\mathsf{D}}(\mathbf{k},p)$ the dielectric tensor. For electrostatic wave, let $D(\mathbf{k},p)=\tilde{D}_{kk}(\mathbf{k},p)$, which is given by the denominator in Eq. (9.23). Namely,

$$D(\mathbf{k},p)=1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha 0}^2}{n_0}\int\frac{dF_{\alpha 0}(u)/du}{u-(ip/k)}du] \quad \text{where } \operatorname{Re}(p)\geq p_0. \tag{9.24}$$

Thus, inverse-Laplace transform of Eq. (9.23) becomes

$$\mathcal{L}^{-1}[k^2\tilde{\Phi}_1(\mathbf{k},p)]=k^2\bar{\Phi}_1(\mathbf{k},t)=\frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}e^{pt}\frac{\sum_\alpha\frac{e_\alpha}{\varepsilon_0}[\int\frac{\bar{F}_{\alpha 1}(\mathbf{k},u,t=0)}{p+iku}du]}{D(\mathbf{k},p)}dp \tag{9.25}$$

Since $e^{pt}\to 0$ as $\operatorname{Re}(p)\to-\infty$, we usually deform the integration contour of the inverse-Laplace transform to

$$\mathcal{L}^{-1}[\tilde{A}(p)]=\frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}e^{pt}\tilde{A}(p)dp=\frac{1}{2\pi i}\int_{-\infty-i\infty}^{-\infty+i\infty}e^{pt}\tilde{A}(p)dp+\sum_j R_j e^{p_j t}=\sum_j R_j e^{p_j t} \tag{9.26}$$

where p_j is the pole of the function $\tilde{A}(p)$ and R_j is the residue of $\tilde{A}(p)$ at pole p_j.

However, care must be taken when we deform integration contour of the inverse-Laplace transform. As we can see, both the denominator and the numerator in Eq. (9.25) consist of another integration in the velocity space with a pole at $u=ip/k$. When we change the integration path of p in the inverse-Laplace transform, the location of pole in the velocity-space integration will also move to another location. Before changing the integration path in the inverse-Laplace transform, the pole is always located in the upper half plane of the complex- u space (i.e., $\operatorname{Im}(u)>0$), because we choose the path $\operatorname{Re}(p)\geq p_0>0$ in Eqs. (9.24) and (9.25). Therefore, the integration along the $\operatorname{Re}(u)$ axis (i.e., the path along $\operatorname{Im}(u)=0$) is the same as the integration along the path of $\operatorname{Im}(u)\to-\infty$. However, after deforming the integration contour of the inverse-Laplace transform from Eq. (9.25) to Eq. (9.26), the integration along the $\operatorname{Re}(u)$ axis will be different from the integration along the path of $\operatorname{Im}(u)\to-\infty$, if the pole $\operatorname{Re}(p_j)\leq 0$. Since the function $\tilde{A}(p)$ in Eq. (9.26) must be an analytic function except at poles p_j (i.e., the *analytic continuation* of $\tilde{A}(p)$), we need to choose a different integration path when $\operatorname{Re}(p_j)\leq 0$ such that the integration in the complex-u space is still the same as the integration along the path of $\operatorname{Im}(u)\to-\infty$. Landau (1946) pointed out that, if $\operatorname{Re}(p_j)\leq 0$, we must choose an integration path of u, in which the pole in the velocity-space integration always locate on the left-hand side of the integration path, so that there is no singularity between the integration path and the path of $\operatorname{Im}(u)\to-\infty$ in the complex-u space. The new integration path in the velocity space is called *Landau contour,* which is defined below:

$$\boxed{h(p)=\int_L \frac{g(u)}{u-(ip/k)}du=\begin{cases}\int_{-\infty}^{+\infty}\frac{g(u)}{u-(ip/k)}du & \text{if } \mathrm{Re}(p)>0\\ \wp\int_{-\infty}^{+\infty}\frac{g(u)}{u-(ip/k)}du+\pi i g(ip/k) & \text{if } \mathrm{Re}(p)=0\\ \int_{-\infty}^{+\infty}\frac{g(u)}{u-(ip/k)}du+2\pi i g(ip/k) & \text{if } \mathrm{Re}(p)<0\end{cases}} \tag{9.27}$$

where the principle value of the integration at $\mathrm{Re}(p)=0$ is defined by

$$\wp\int_{-\infty}^{+\infty}\frac{g(u)}{u-(ip/k)}du=\lim_{\delta\to 0^+}[\int_{-\infty}^{(ip/k)-\delta}\frac{g(u)}{u-(ip/k)}du+\int_{(ip/k)+\delta}^{+\infty}\frac{g(u)}{u-(ip/k)}du] \tag{9.28}$$

After choosing Landau contour, we can now remove the condition $\mathrm{Re}(p)\geq p_0$ in Eqs. (9.23), (9.24), and rewrite them as

$$\boxed{\tilde{\Phi}_1(\mathbf{k},p)=\frac{\frac{1}{ik^3}\sum_\alpha \frac{e_\alpha}{\varepsilon_0}[\int_L \frac{\overline{F}_{\alpha 1}(\mathbf{k},u,t=0)}{u-(ip/k)}du]}{D(\mathbf{k},p)}} \tag{9.29}$$

$$\boxed{D(\mathbf{k},p)=1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha 0}^2}{n_0}\int_L \frac{dF_{\alpha 0}(u)/du}{u-(ip/k)}du]} \tag{9.30}$$

Inverse-Laplace transform of $\tilde{\Phi}_1(\mathbf{k},p)$ can be simplified as

$$\mathcal{L}^{-1}[\tilde{\Phi}_1(\mathbf{k},p)]=\overline{\Phi}_1(\mathbf{k},t)=\frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}e^{pt}\tilde{\Phi}_1(\mathbf{k},p)dp=\sum_j R_j(\mathbf{k},p)e^{p_j(\mathbf{k})t} \tag{9.31}$$

where $p_j(\mathbf{k})$ is a root of $D(\mathbf{k},p)=0$, $R_j(\mathbf{k},p)=\lim_{p\to p_j}\{[p-p_j(\mathbf{k})]\tilde{\Phi}_1(\mathbf{k},p)\}$ is the residue of $\tilde{\Phi}_1(\mathbf{k},p)$ at $p=p_j(\mathbf{k})$. $\tilde{\Phi}_1(\mathbf{k},p)$ and $D(\mathbf{k},p)$ are given in Eqs. (9.29) and (9.30), respectively. For a given wave number $\mathbf{k}$, deformation of integration contour of inverse-Laplace transform in p-domain and corresponding Landau contour due to displacement of poles in u-domain are illustrated in Figure 9.1.

We can write above results in terms of frequency by defining $\omega=ip$, or $p=-i\omega$. The inverse-Laplace transform becomes

$$\mathcal{L}^{-1}[\tilde{\Phi}_1(\mathbf{k},\omega)]=\overline{\Phi}_1(\mathbf{k},t)$$
$$=\frac{-i}{2\pi i}\int_{(p_0-i\infty)/(-i)}^{(p_0+i\infty)/(-i)}e^{-i\omega t}\tilde{\Phi}_1(\mathbf{k},\omega)d\omega \qquad \mathrm{Im}(\omega)>p_0$$
$$=\frac{1}{2\pi}\int_{ip_0-\infty}^{ip_0+\infty}e^{-i\omega t}\tilde{\Phi}_1(\mathbf{k},\omega)d\omega \qquad \mathrm{Im}(\omega)>p_0$$

To remove the condition $\mathrm{Im}(\omega) > p_0$, we need define Landau contour

$$\boxed{h(\omega) = \int_L \frac{g(u)}{u-(\omega/k)}du = \begin{cases} \int_{-\infty}^{+\infty} \frac{g(u)}{u-(\omega/k)}du & \text{if } \mathrm{Im}(\omega) > 0 \\ \wp\int_{-\infty}^{+\infty} \frac{g(u)}{u-(\omega/k)}du + \pi i g(\omega/k) & \text{if } \mathrm{Im}(\omega) = 0 \\ \int_{-\infty}^{+\infty} \frac{g(u)}{u-(\omega/k)}du + 2\pi i g(\omega/k) & \text{if } \mathrm{Im}(\omega) < 0 \end{cases}} \tag{9.27a}$$

where

$$\wp\int_{-\infty}^{+\infty} \frac{g(u)}{u-(\omega/k)}du = \lim_{\delta\to 0^+}\left[\int_{-\infty}^{(\omega/k)-\delta} \frac{g(u)}{u-(\omega/k)}du + \int_{(\omega/k)+\delta}^{+\infty} \frac{g(u)}{u-(\omega/k)}du\right] \tag{9.28a}$$

Thus, $\tilde{\Phi}_1(\mathbf{k},\omega)$ becomes

$$\boxed{\tilde{\Phi}_1(\mathbf{k},\omega) = \frac{\frac{1}{ik^3}\sum_\alpha \frac{e_\alpha}{\varepsilon_0}\left[\int_L \frac{\bar{F}_{\alpha 1}(\mathbf{k},u,t=0)}{u-(\omega/k)}du\right]}{D(\mathbf{k},\omega)}} \tag{9.29a}$$

where

$$\boxed{D(\mathbf{k},\omega) = 1 - \frac{1}{k^2}\sum_\alpha \left[\frac{\omega_{p\alpha 0}^2}{n_0}\int_L \frac{dF_{\alpha 0}(u)/du}{u-(\omega/k)}du\right]} \tag{9.30a}$$

The inverse-Laplace transform of $\tilde{\Phi}_1(\mathbf{k},\omega)$ becomes

$$\mathcal{L}^{-1}[\tilde{\Phi}_1(\mathbf{k},\omega)] = \bar{\Phi}_1(\mathbf{k},t) = \sum_j R_j(\mathbf{k},\omega) e^{-i\omega_j(\mathbf{k})t} \tag{9.31a}$$

where $\omega_j(\mathbf{k})$ is a root of $D(\mathbf{k},\omega) = 0$, and $R_j(\mathbf{k},\omega) = \lim_{\omega\to\omega_j}\{[\omega-\omega_j(\mathbf{k})]\tilde{\Phi}_1(\mathbf{k},\omega)\}$ is the residue of $\tilde{\Phi}_1(\mathbf{k},\omega)$ at $\omega = \omega_j(\mathbf{k})$. $\tilde{\Phi}_1(\mathbf{k},\omega)$ and $D(\mathbf{k},\omega)$ are given in Eq. (9.29a) and (9.30a), respectively. For a given wave number $\mathbf{k}$, deformation of integration contour of inverse-Laplace transform in ω-domain and corresponding Landau contour due to displacement of poles in u-domain are illustrated in Figure 9.2.

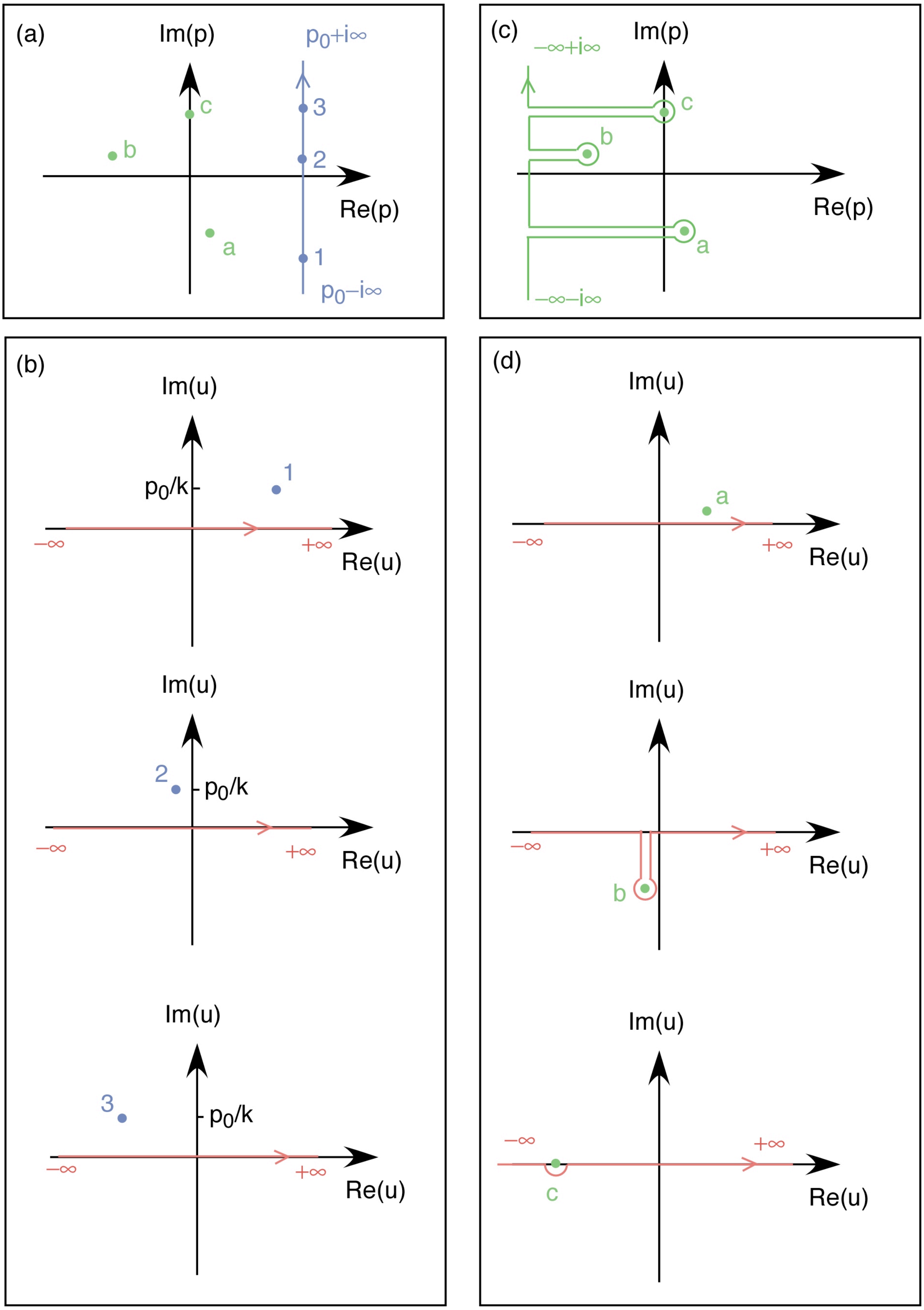

Figure 9.1. Integration contours of the inverse-Laplace transform and Landau contours, where panel (a) shows original integration contour of the inverse-Laplace transform in the p-domain defined in (9.25), panel (b) shows original integration path and poles ($u = ip/k$) in the u-domain, panel (c) shows deformation of the integration contours of the inverse-Laplace transform in the p-domain described in Eq. (9.26), and panel (d) shows displacement of poles and integration path of the Landau contours in the u-domain.

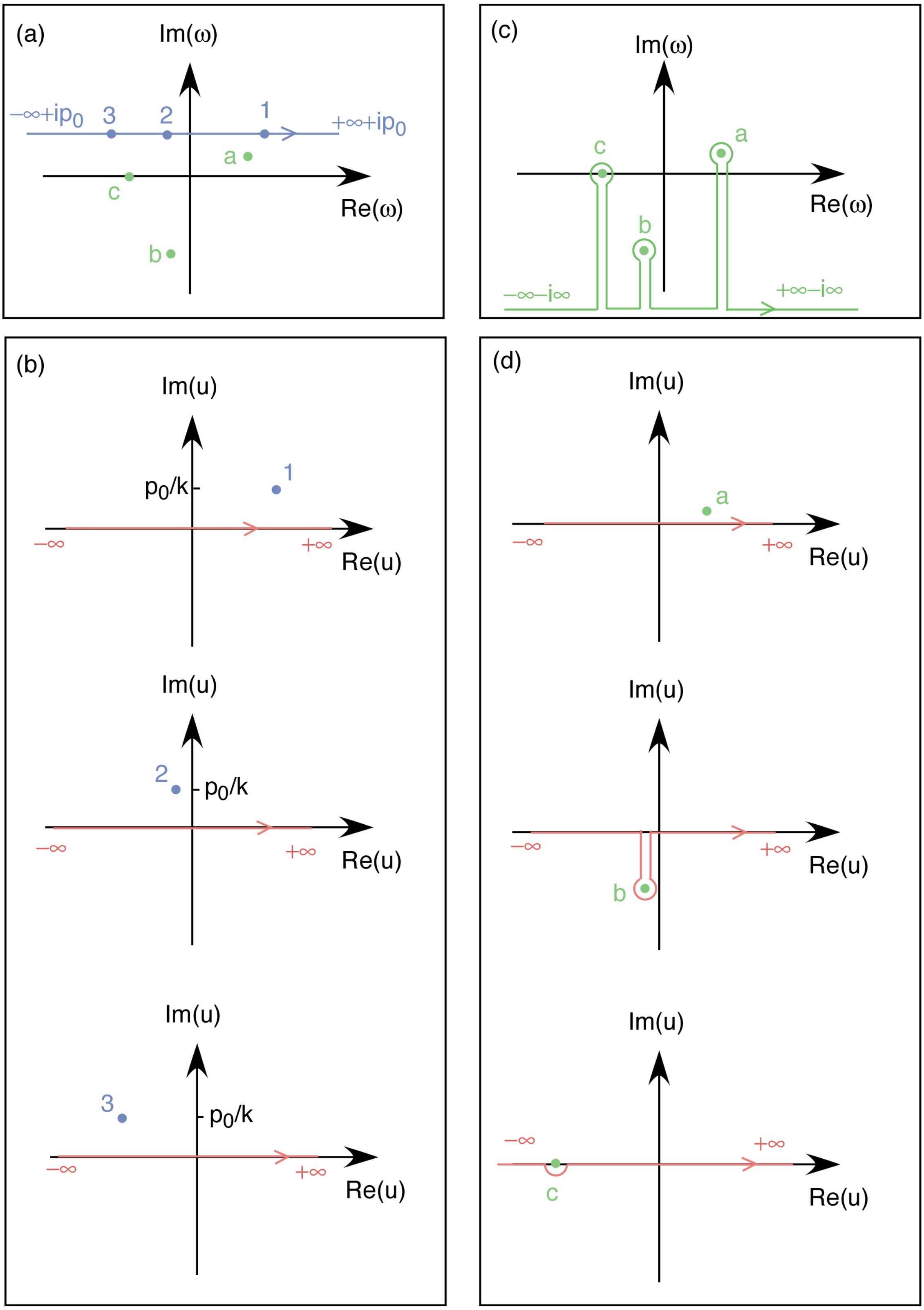

Figure 9.2. Integration contours of the inverse-Laplace transform and Landau contours, where panel (a) shows original integration contour of the inverse-Laplace transform in the ω-domain defined in (9.25), panel (b) shows original integration path and poles ($u = \omega / k$) in the u-domain, panel (c) shows deformation of the integration contours of the inverse-Laplace transform in the ω-domain described in Eq. (9.26), and panel (d) shows displacement of poles and integration path of the Landau contours in the u-domain.

9.2. Linear Dispersion Relations of Electrostatic Waves

Let $\omega = \omega_r + i\omega_i$ and $D(\mathbf{k},\omega) = \mathrm{Re}[D(\mathbf{k},\omega)] + i\,\mathrm{Im}[D(\mathbf{k},\omega)]$. The dispersion relation $D(\mathbf{k},\omega)=0$ implies $\mathrm{Re}[D(\mathbf{k},\omega)]=0$ and $\mathrm{Im}[D(\mathbf{k},\omega)]=0$. Substitution Eq. (9.27a) into Eq. (9.30a) yields,

$$\boxed{D(\mathbf{k},\omega) = \begin{cases} 1-\dfrac{1}{k^2}\displaystyle\sum_\alpha\{\dfrac{\omega_{p\alpha 0}^2}{n_0}\int_{-\infty}^{+\infty}\dfrac{dF_{\alpha 0}(u)/du}{u-(\omega/k)}du\} & \text{if } \omega_i>0 \\ 1-\dfrac{1}{k^2}\displaystyle\sum_\alpha\{\dfrac{\omega_{p\alpha 0}^2}{n_0}[\wp\int_{-\infty}^{+\infty}\dfrac{dF_{\alpha 0}(u)/du}{u-(\omega_r/k)}du+\pi i\left.\dfrac{dF_{\alpha 0}(u)}{du}\right|_{u=\frac{\omega_r}{k}}]\} & \text{if } \omega_i=0 \\ 1-\dfrac{1}{k^2}\displaystyle\sum_\alpha\{\dfrac{\omega_{p\alpha 0}^2}{n_0}[\int_{-\infty}^{+\infty}\dfrac{dF_{\alpha 0}(u)/du}{u-(\omega/k)}du+2\pi i\left.\dfrac{dF_{\alpha 0}(u)}{du}\right|_{u=\frac{\omega}{k}}]\} & \text{if } \omega_i<0 \end{cases}} \qquad (9.32)$$

The integration in Eq. (9.32) can be rewritten as

$$\boxed{\begin{aligned}\int_{-\infty}^{+\infty}\frac{dF_{\alpha 0}(u)/du}{u-\dfrac{\omega_r+i\omega_i}{k}}du &= \int_{-\infty}^{+\infty}(u-\frac{\omega_r}{k}+i\frac{\omega_i}{k})\frac{dF_{\alpha 0}(u)/du}{(u-\frac{\omega_r}{k})^2+(\frac{\omega_i}{k})^2}du \\ &= \int_{-\infty}^{+\infty}(u-\frac{\omega_r}{k})\frac{dF_{\alpha 0}(u)/du}{(u-\frac{\omega_r}{k})^2+(\frac{\omega_i}{k})^2}du + i\frac{\omega_i}{k}\int_{-\infty}^{+\infty}\frac{dF_{\alpha 0}(u)/du}{(u-\frac{\omega_r}{k})^2+(\frac{\omega_i}{k})^2}du\end{aligned}} \qquad (9.33)$$

Substituting Eq. (9.33) into Eq. (9.32), we can obtain $D(\mathbf{k},\omega)$ under different ω_i conditions.

Case 1 $\omega_i > 0$

$$\boxed{\begin{aligned}\mathrm{Re}[D(\mathbf{k},\omega)] &= 1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha 0}^2}{n_0}\int_{-\infty}^{\infty}(u-\frac{\omega_r}{k})\frac{dF_{\alpha 0}(u)/du}{(u-\frac{\omega_r}{k})^2+(\frac{\omega_i}{k})^2}du] \\ &= 1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha 0}^2}{n_0}\int_{-\infty}^{\infty}\frac{[(u-\frac{\omega_r}{k})^2-(\frac{\omega_i}{k})^2]F_{\alpha 0}(u)}{[(u-\frac{\omega_r}{k})^2+(\frac{\omega_i}{k})^2]^2}du]\end{aligned}} \qquad (9.34)$$

$$\boxed{\mathrm{Im}[D(\mathbf{k},\omega)] = -\frac{\omega_i}{k}\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha 0}^2}{n_0}\int_{-\infty}^{\infty}\frac{dF_{\alpha 0}(u)/du}{(u-\frac{\omega_r}{k})^2+(\frac{\omega_i}{k})^2}du]} \qquad (9.35)$$

Case 2 $\omega_i = 0$

(Note that the solutions of Case 2 are included in the solutions of Case 3. The expression of $\mathrm{Re}[D(\mathbf{k},\omega_r)]$ and $\mathrm{Im}[D(\mathbf{k},\omega_r)]$ presented in Case 2 is only good for the Nyquist method to be discussed in Section 9.4)

$$\boxed{\begin{aligned}\mathrm{Re}[D(\mathbf{k},\omega_r)] &= 1-\frac{1}{k^2}\sum_\alpha\Big[\frac{\omega_{p\alpha0}^2}{n_0}\wp\int_{-\infty}^{\infty}\frac{dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})}du\Big] \\ &= 1-\frac{1}{k^2}\sum_\alpha\Big[\frac{\omega_{p\alpha0}^2}{n_0}\wp\int_{-\infty}^{+\infty}\frac{F_{\alpha0}(u)-F_{\alpha0}(u=\frac{\omega_r}{k})}{(u-\frac{\omega_r}{k})^2}du\Big]\end{aligned}} \tag{9.36}$$

$$\boxed{\mathrm{Im}[D(\mathbf{k},\omega_r)] = -\pi\frac{1}{k^2}\sum_\alpha\Big[\frac{\omega_{p\alpha0}^2}{n_0}\left.\frac{dF_{\alpha0}(u)}{du}\right|_{u=\frac{\omega_r}{k}}\Big]} \tag{9.37}$$

Note that the result of integration by part of the principle-value integration in equation (9.36) can be found in Section 9.4 in the proof of equation (9.45).

Case 3 $\omega_i < 0$, but $\omega_i \to 0$ and $|\omega_i| << \omega_r$,

$$\boxed{\begin{aligned}\mathrm{Re}[D(\mathbf{k},\omega_r)] &= 1-\frac{1}{k^2}\sum_\alpha\Big[\frac{\omega_{p\alpha0}^2}{n_0}\wp\int_{-\infty}^{\infty}\frac{dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})}du\Big] \\ &= 1-\frac{1}{k^2}\sum_\alpha\Big[\frac{\omega_{p\alpha0}^2}{n_0}\wp\int_{-\infty}^{+\infty}\frac{F_{\alpha0}(u)-F_{\alpha0}(u=\frac{\omega_r}{k})}{(u-\frac{\omega_r}{k})^2}du\Big]\end{aligned}} \tag{9.38}$$

$$\boxed{\mathrm{Im}[D(\mathbf{k},\omega)] = -\frac{\omega_i}{k}\frac{1}{k^2}\sum_\alpha\Big[\frac{\omega_{p\alpha0}^2}{n_0}\wp\int_{-\infty}^{\infty}\frac{dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})^2}du\Big]-\pi\frac{1}{k^2}\sum_\alpha\Big[\frac{\omega_{p\alpha0}^2}{n_0}\left.\frac{dF_{\alpha0}(u)}{du}\right|_{u=\frac{\omega_r}{k}}\Big]} \tag{9.39}$$

Note that the result of integration by part of the principle-value integration in equation (9.38) can be found in Section 9.4 in the proof of equation (9.45).

Case 4 $\omega_i < 0$,

$$\boxed{\begin{aligned}\mathrm{Re}[D(\mathbf{k},\omega)] &= 1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha0}^2}{n_0}\int_{-\infty}^{\infty}(u-\frac{\omega_r}{k})\frac{dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})^2+(\frac{\omega_i}{k})^2}du] \\ &\quad + 2\pi\frac{1}{k^2}\sum_\alpha\{\frac{\omega_{p\alpha0}^2}{n_0}\mathrm{Im}[\frac{dF_{\alpha0}(u)}{du}\bigg|_{u=\frac{\omega_r+i\omega_i}{k}}]\} \\ &= 1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha0}^2}{n_0}\int_{-\infty}^{\infty}\frac{[(u-\frac{\omega_r}{k})^2-(\frac{\omega_i}{k})^2]F_{\alpha0}(u)}{[(u-\frac{\omega_r}{k})^2+(\frac{\omega_i}{k})^2]^2}du] \\ &\quad + 2\pi\frac{1}{k^2}\sum_\alpha\{\frac{\omega_{p\alpha0}^2}{n_0}\mathrm{Im}[F'_{\alpha0}(u=\frac{\omega_r+i\omega_i}{k})]\}\end{aligned}} \qquad (9.40)$$

$$\boxed{\begin{aligned}\mathrm{Im}[D(\mathbf{k},\omega)] &= -\frac{\omega_i}{k}\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha0}^2}{n_0}\int_{-\infty}^{\infty}\frac{dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})^2+(\frac{\omega_i}{k})^2}du] \\ &\quad - 2\pi\frac{1}{k^2}\sum_\alpha\{\frac{\omega_{p\alpha0}^2}{n_0}\mathrm{Re}[\frac{dF_{\alpha0}(u)}{du}\bigg|_{u=\frac{\omega_r+i\omega_i}{k}}]\}\end{aligned}} \qquad (9.41)$$

9.3. Landau Damping

Since wave amplitude is proportional to $e^{\omega_i t}$, wave amplitude decreases with time if $\omega_i < 0$. Decreasing on wave amplitude is called wave damping. Wave amplitude in Case 4 may damp too fast that can hardly observed. Waves in Case 3 with $\omega_i \to 0$ are waves with slow damping rate. By solving Eq. (9.38) one can obtain solution of ω_r for given wave number k and distribution functions $F_{\alpha0}(u)$. Then we can substitute this ω_r into Eq. (9.39) to obtain the damping rate ω_i, i.e.,

$$\boxed{\omega_i = k\frac{\pi\sum_\alpha[\frac{\omega_{p\alpha0}^2}{n_0}\frac{dF_{\alpha0}(u)}{du}\bigg|_{u=\frac{\omega_r}{k}}]}{\sum_\alpha[\frac{\omega_{p\alpha0}^2}{n_0}\wp\int_{-\infty}^{\infty}\frac{-dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})^2}du]}} \qquad (9.42)$$

Since $\omega_{pe} >> \omega_{pi}$, if $[dF_{e0}(u)/du]_{u=\omega_r/k} < 0$, the denominator in Eq. (9.42) should be positive and numerator in Eq. (9.42) is negative. Thus, an electrostatic wave with frequency ω_r and phase speed ω_r/k, will under go Landau damping if $[dF_{e0}(u)/du]_{u=\omega_r/k} < 0$. Landau damping can only be found in kinetic plasma linear dispersion relation.

Physical picture of Landau damping process can be understood by phase-space trajectories of charge particles in wave moving frame. It can also be shown that the damping process does not occur uniformly in space, and the damping rate in Eq. (9.42) is only applicable to the initial phase of the Landau damping at a time interval $\Delta t < \pi/\omega_b$, where $2\pi/\omega_b$ is the average bounce period of trapped particles (e.g., Nicholson, 1983, page 96).

9.4. Nyquist Method

For the case with slow damping rate as discussed in Case 3, the normal mode wave frequency ω_r can be obtained by solving Eq. (9.38). Solution obtained from Eq. (9.38) is similar to the one obtained from fluid dispersion relation if the background plasma is of normal distribution in velocity space. We can estimate wave frequency ω_r from fluid equations and then substitute it into Eq. (9.42) to obtain Landau damping rate ω_i.

For $\omega_i > 0$, wave amplitude increases with time. The system is unstable to electrostatic perturbations. One needs to solve Eqs. (9.34) and (9.35) simultaneously, to obtain solutions of ω_r and ω_i. However, it is not an easy task to solve the integration equations (9.34) and (9.35), simultaneously. Using Nyquist method, one can determine stability of a system under electrostatic perturbation qualitatively, without actually solving Eqs. (9.34) and (9.35).

Let us consider the following integrations in D-domain and in ω-domain

$$i2\pi N = \oint_{C_D} d\ln D = \oint_{C_D} \frac{dD}{D} = \oint_{C_\omega} \frac{\partial D(\mathbf{k},\omega)/\partial\omega}{D(\mathbf{k},\omega)} d\omega \tag{9.43}$$

where N denotes number of zeros of dielectric function D in the closed loop C_D in the D domain or the closed loop C_ω in the ω domain. If we choose C_ω to be a closed loop over the upper-half of the complex ω plane, then N will be the number of unstable modes in the system. Namely, if $N > 0$, the system is unstable to electrostatic disturbance.

What is C_D if C_ω is a closed loop over the upper-half of the complex ω plane?

It can be shown that

$$\oint_{C_\omega}[\frac{\partial D(\mathbf{k},\omega)/\partial\omega}{D(\mathbf{k},\omega)}]d\omega=\{\lim_{R_\omega\to\infty}\int_0^\pi[\frac{\partial D(\mathbf{k},R_\omega e^{i\theta})/\partial\theta}{D(\mathbf{k},R_\omega e^{i\theta})}]d\theta\}+\int_{-\infty}^{\infty}[\frac{\partial D(\mathbf{k},\omega_r)/\partial\omega_r}{D(\mathbf{k},\omega_r)}]d\omega_r \tag{9.44}$$

where $\omega=\omega_r+i\omega_i=R_\omega e^{i\theta}$. Eq. (9.34) and Eq. (9.35) yield

$\lim_{R_\omega\to\infty}\mathrm{Re}[D(\mathbf{k},\omega)]=1$

and

$\lim_{R_\omega\to\infty}\mathrm{Im}[D(\mathbf{k},\omega)]=0$.

Thus the first term on the right-hand side of the integration path in Eq. (9.44) is reduced to a point at $D(\mathbf{k},\omega)=1$ in the complex D-domain.

The integration path of the second term on the right-hand side of Eq. (9.44) is along the real axis, where $\omega_i=0$. Thus, we have to use Eqs. (9.36) and (9.37) to determine the corresponding integration path in the complex D-domain. Namely, for $\omega_i=0$

$$\mathrm{Re}[D(\mathbf{k},\omega_r)]=1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha0}^2}{n_0}\wp\int_{-\infty}^{\infty}\frac{dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})}du] \tag{9.36a}$$

$$\mathrm{Im}[D(\mathbf{k},\omega_r)]=-\pi\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha0}^2}{n_0}\left.\frac{dF_{\alpha0}(u)}{du}\right|_{u=\frac{\omega_r}{k}}] \tag{9.37a}$$

Since $dF_{\alpha0}(u)/du|_{u\to\pm\infty}\to0^{\mp}$, Eq. (9.37a) yields $\mathrm{Im}[D(\mathbf{k},\omega_r\to\pm\infty)]\to0^{\pm}$.

Figure 9.3 illustrates (a) a distribution function $F_{e0}(u)$, and corresponding integration path $1\to2\to3\to4$ in (b) ω-domain and in (c) D-domain. The possible corresponding integration path $4\to5\to6\to7\to1$ in D-domain will be discussed later in Figure 9.4.

For $N>0$, the integration path C_D must circle around the branch point $(\mathrm{Re}(D),\mathrm{Im}(D))=(0,0)$ N times. It is possible to estimate N by examining the intersections of the loop C_D and the real axis in D-domain.

Let $\mathrm{Im}[D(\mathbf{k},\omega_{r,n-1})]=\mathrm{Im}[D(\mathbf{k},\omega_{r,n})]=\mathrm{Im}[D(\mathbf{k},\omega_{r,n+1})]=0$, where $\omega_{r,n-1}<\omega_{r,n}<\omega_{r,n+1}$ are three consecutive roots of $\mathrm{Im}(D)=0$ along the real axis in the ω-domain. Then, loop C_D will intersect with the real axis in the D-domain at points $D_{r,n-1}=\mathrm{Re}[D(\mathbf{k},\omega_{r,n-1})]$, $D_{r,n}=\mathrm{Re}[D(\mathbf{k},\omega_{r,n})]$, and $D_{r,n+1}=\mathrm{Re}[D(\mathbf{k},\omega_{r,n+1})]$. If $D_{r,n-1}\cdot D_{r,n}<0$ and $D_{r,n}\cdot D_{r,n+1}<0$, then loop C_D will circle around the branch point $(\mathrm{Re}(D),\mathrm{Im}(D))=(0,0)$ at least once, which implies $N>0$ and the system is unstable to electrostatic perturbations.

One can determine $D_{r,n}=\mathrm{Re}[D(\mathbf{k},\omega_{r,n})]$ from Eq. (9.36a), where $\omega_{r,n}$ is the root of $\mathrm{Im}[D(\mathbf{k},\omega_{r,n})]=0$ in Eq. (9.37a). For convenience, we use integrating by parts to rewrite Eq. (9.36a) as

$$\mathrm{Re}[D(\mathbf{k},\omega_r)]=1-\frac{1}{k^2}\sum_\alpha\{\frac{\omega_{p\alpha0}^2}{n_0}[\wp\int_{-\infty}^{\infty}\frac{F_{\alpha0}(u)-F_{\alpha0}(u=\frac{\omega_r}{k})}{(u-\frac{\omega_r}{k})^2}du]\} \tag{9.45}$$

For $\omega_{pe}>>\omega_{pi}$, $\mathrm{Im}[D(\mathbf{k},\omega_{r,n})]=0$ implies $dF_{e0}(u)/du=0$. If electron distribution function, F_{e0}, has local maximum or local minimum at $u=u_n$, then $[dF_{e0}/du]_{u=u_n}=0$. We can choose $\omega_{r,n}=ku_n$ so that $\mathrm{Im}[D(\mathbf{k},\omega_{r,n})]=0$. The corresponding real part D at $u=u_n$, i.e., $D_{r,n}=\mathrm{Re}[D(\mathbf{k},\omega_{r,n})]$ is given in Eq. (9.45).

Using the example shown in Figure 9.3, one can estimate real part D based on Eq. (9.45) at points, where $u=\omega_r/k$. Figure 9.4 illustrates possible integration path C_D in D-domain for different wave number k, where (a) is for large wave number k, (b) is for medium wave number k, and (c) is for small wave number k. No unstable mode can be found when the wave number is too small or too large. One unstable wave mode can be found in the case with medium wave number k.

Exercise 9.1.

If Re(D) at point 5 is greater than 1, sketches possible integration path C_D for the three cases shown in Figure 9.4.

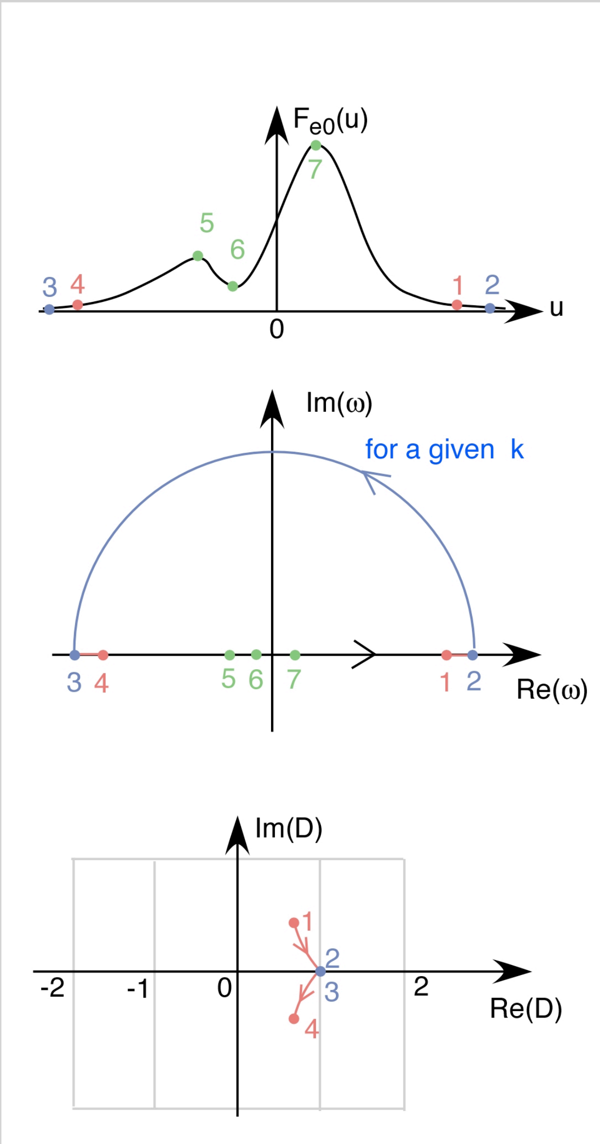

Figure 9.3. Integration path in the ω-domain and the D-domain of the Nyquist method for a given distribution function $F_{e0}(u)$. See text for discussion in details.

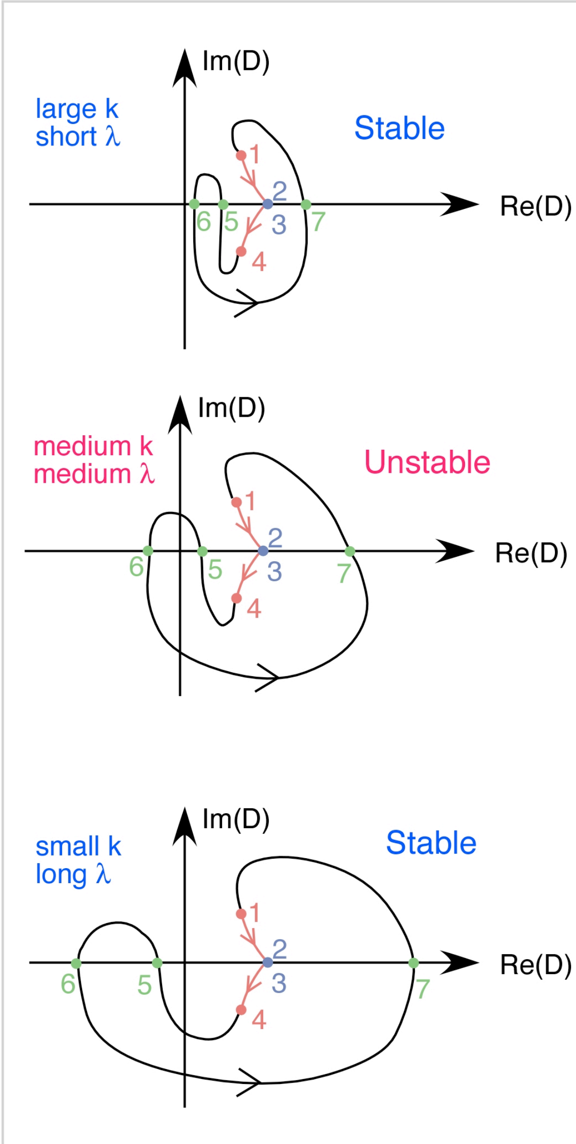

Figure 9.4. Possible integration paths C_D in the D-domain for different wave number k and for the given distribution function $F_{e0}(u)$ shown in Figure 9.3. See text for discussion in details.

Discussion:

Some textbooks (e.g., Nicholson, 1983; Sturrock, 1994) used incorrect statements to prove the above equation (9.45). The authors in those textbooks claimed that because $dF_{\alpha 0}(u)/du$ at $u=\omega_r/k$ is equal to zero, therefore one can add a dummy constant $-dF_{\alpha 0}(u)/du\big|_{u=\omega_r/k}$ in the numerator of Eq. (9.36a).

Namely,

$$\mathrm{Re}[D(\mathbf{k},\omega_r)] = 1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha 0}^2}{n_0}\wp\int_{-\infty}^{\infty}\frac{dF_{\alpha 0}(u)/du-[dF_{\alpha 0}(u)/du]_{u=\omega_r/k}}{(u-\frac{\omega_r}{k})}du] \tag{9.36b}$$

They believe that "integration by parts of Eq. (9.36b) can yield Eq. (9.45)." However, the authors in those textbooks cannot explain the following paradox. Since one can also add a dummy constant $dF_{\alpha 0}(u)/du\big|_{u=\omega_r/k}$ in the numerator of Eq. (9.36a), which yields

$$\mathrm{Re}[D(\mathbf{k},\omega_r)] = 1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha 0}^2}{n_0}\wp\int_{-\infty}^{\infty}\frac{dF_{\alpha 0}(u)/du+[dF_{\alpha 0}(u)/du]_{u=\omega_r/k}}{(u-\frac{\omega_r}{k})}du] \tag{9.36c}$$

Following "their" integration by parts procedure, Eq. (9.36c) can result in an equation, which is totally different from Eq. (9.45). This paradox is a result of incorrect "integration by parts" of principle value of a discontinuous integration function.

A correct proof of equation (9.45) is given below. According to the following statements, Eq. (9.45) is correct even if $dF_{\alpha 0}(u=\omega_r/k)/du$ is not equal to zero.

Proof of Eq.(9.45):

$$\mathrm{Re}[D(\mathbf{k},\omega_r)]=1-\frac{1}{k^2}\sum_\alpha[\frac{\omega_{p\alpha0}^2}{n_0}\wp\int_{-\infty}^{\infty}\frac{dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})}du]$$

$$=1-\frac{1}{k^2}\sum_\alpha\{\frac{\omega_{p\alpha0}^2}{n_0}\lim_{\delta\to0}[\int_{-\infty}^{\frac{\omega_r}{k}-\delta}\frac{dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})}du+\int_{\frac{\omega_r}{k}+\delta}^{\infty}\frac{dF_{\alpha0}(u)/du}{(u-\frac{\omega_r}{k})}du]\}$$

$$=1-\frac{1}{k^2}\sum_\alpha\{\frac{\omega_{p\alpha0}^2}{n_0}\lim_{\delta\to0}[\frac{F_{\alpha0}(u)}{(u-\frac{\omega_r}{k})}\Bigg|_{-\infty}^{\frac{\omega_r}{k}-\delta}+\int_{-\infty}^{\frac{\omega_r}{k}-\delta}\frac{F_{\alpha0}(u)}{(u-\frac{\omega_r}{k})^2}du+\frac{F_{\alpha0}(u)}{(u-\frac{\omega_r}{k})}\Bigg|_{\frac{\omega_r}{k}+\delta}^{\infty}+\int_{\frac{\omega_r}{k}+\delta}^{\infty}\frac{F_{\alpha0}(u)}{(u-\frac{\omega_r}{k})^2}du]\}$$

$$=1-\frac{1}{k^2}\sum_\alpha\{\frac{\omega_{p\alpha0}^2}{n_0}\lim_{\delta\to0}[\wp\int_{-\infty}^{\infty}\frac{F_{\alpha0}(u)}{(u-\frac{\omega_r}{k})^2}du+F_{\alpha0}(u=\frac{\omega_r}{k})(\frac{1}{-\delta}-\frac{1}{\delta})]\}$$

$$=1-\frac{1}{k^2}\sum_\alpha\{\frac{\omega_{p\alpha0}^2}{n_0}[\wp\int_{-\infty}^{\infty}\frac{F_{\alpha0}(u)}{(u-\frac{\omega_r}{k})^2}du+F_{\alpha0}(u=\frac{\omega_r}{k})\wp\int_{-\infty}^{\infty}\frac{-1}{(u-\frac{\omega_r}{k})^2}du]\}$$

$$=1-\frac{1}{k^2}\sum_\alpha\{\frac{\omega_{p\alpha0}^2}{n_0}[\wp\int_{-\infty}^{\infty}\frac{F_{\alpha0}(u)-F_{\alpha0}(u=\frac{\omega_r}{k})}{(u-\frac{\omega_r}{k})^2}du]\}$$

Gardner's Theorem (Gardner, 1963)

For a distribution function with only one single maximum at $u=u_1$, we have $F_{\alpha0}(u)-F_{\alpha0}(u_1)<0$ for all u. Thus $\mathrm{Re}[D(\mathbf{k},\omega_r)]$ in Eq. (9.45) is always greater than 1. Namely, loop C_D and the real axis $\mathrm{Im}(D)=0$ have only two intersections and both intersections are for $\mathrm{Re}(D)>0$. Thus, the system is stable to all electrostatic perturbation. A more general proof based on nonlinear stability analysis can be found in literature (Gardner, 1963). On the other hand, if the distribution function has more then one local maximum, then there must be at least one local minimum on the distribution. If there is a local minimum at $u=u_n$ ($[dF_{e0}/du]_{u=u_n}=0$), and

$$\wp\int_{-\infty}^{\infty}\frac{F_{e0}(u_0)-F_{e0}(u)}{(u-u_0)^2}du<0$$

then one can always find unstable waves with wavelength long enough such that

$$\wp\int_{-\infty}^{\infty}\frac{F_{e0}(u_n)-F_{e0}(u)}{(u-u_n)^2}du<-\frac{k^2n_0}{\omega_{pe0}^2}<0$$

It can be shown that the system is unstable to electrostatic perturbations at certain range of wave numbers as discussed in Figure 9.4 and Exercise 9.2. (e.g., Nicholson, 1983, page 104)

Penrose Criterion (Penrose, 1960):

If a system is unstable to electrostatic perturbation, then there is at least one local minimum in the electron distribution at $u = u_0$, such that $[dF_{e0} / du]_{u=u_0} = 0$ and

$$\wp\int_{-\infty}^{\infty} \frac{F_{e0}(u_0) - F_{e0}(u)}{(u - u_0)^2} du < 0$$

References

Chen, F. F. (1984), *Introduction to Plasma Physics and Controlled Fusion, Volume 1: Plasma Physics*, 2nd edition, Plenum Press, New York.

Gardner, C. S. (1963), Bound on the energy available from a plasma, *Phys. Fluid, 6*, 839.

Krall, N. A., and A. W. Trivelpiece (1973), *Principles of Plasma Physics,* McGraw-Hill, New York, p.452.

Landau, L. D. (1946), *J. Phys. (U.S.S.R.) 10,* 25.

Nicholson, D. R. (1983), *Introduction to Plasma Theory*, John Wiley & Sons, New York.

Penrose, R. (1960), Electrostatic instabilities of a uniform non-Maxwellian plasma, *Phys. Fluids, 3*, 258.

Sturrock, P. A. (1994), *Plasma Physics,* Cambridge University Press, New York.

The answer to Exercise 9.1 is given below.