Chapter 8. Equilibrium Solutions of the Vlasov Equation

Topics or concepts to learn in Chapter 8:

1. What is wave equation?
2. How to find the general solution form of a wave-equation-like partial differential equation?
3. Determine the field-free equilibrium solution of the Vlasov equation.
4. Determine the equilibrium solution of the Vlasov equation with a given electrostatic potential profile.
5. Determine the equilibrium solution of the Vlasov equation with a uniform background magnetic field.

Suggested Readings:

(1) Section 6.2 in Nicholson (1983)

(2) Section 7.7 in Krall and Trivelpiece (1973)

Linear wave analysis is a powerful tool in studying general wave dispersion relations in hydrodynamics and plasma physics. However, linear wave analysis will be meaningless if the background state was not an equilibrium state. In this chapter, we demonstrate how to obtain equilibrium solutions of Vlasov equation to pave the road for studying linear waves in Vlasov plasma in later Chapters 9 and 11.

8.1. Characteristic Curves of a Partial Differential Equation

Consider a function $A = A(x,t)$, which satisfies the following partial differential equation

$$\frac{\partial A}{\partial x} + \frac{1}{c}\frac{\partial A}{\partial t} = 0 \tag{8.1}$$

Eq. (8.1) is a wave equation. Eq. (8.1) implies that one can find two functions $\xi(x,t)$, and $\eta(x,t)$, so that

$$\left.\frac{\partial A}{\partial \eta}\right|_{\xi = const.} = 0 \tag{8.2}$$

If Jacobian determinant of $\xi = \xi(x,t)$ and $\eta = \eta(x,t)$ is non-zero, then we can find the inverse function $x(\xi,\eta)$, and $t(\xi,\eta)$, such that

$A = \overline{A}(\xi(x,t),\eta(x,t)) = \overline{A}(\xi(x,t)) = A(x(\xi,\eta),t(\xi,\eta))$.

Thus Eq. (8.2) can be written as,

$$\left.\frac{\partial A}{\partial \eta}\right|_{\xi=const.} = \left.\frac{\partial A}{\partial x}\frac{\partial x}{\partial \eta}\right|_{\xi=const.} + \left.\frac{\partial A}{\partial t}\frac{\partial t}{\partial \eta}\right|_{\xi=const.} = 0 \tag{8.3}$$

Comparing Eqs. (8.1) and (8.3), yields

$$\left.\frac{\partial x}{\partial \eta}\right|_{\xi=const.} = 1 \tag{8.4}$$

$$\left.\frac{\partial t}{\partial \eta}\right|_{\xi=const.} = \frac{1}{c} \tag{8.5}$$

Multiplying Eq. (8.5) by the wave speed c and then deducting the resulting equation from Eq.(8.4), it yields,

$$\left.\frac{\partial (x-ct)}{\partial \eta}\right|_{\xi=const.} = 0$$

If we choose $\xi = x - ct$, then $A = \overline{A}(\xi(x,t)) = \overline{A}(x-ct)$ will be the solution of Eq. (8.1). Characteristic curves of Eq. (8.1) are $x - ct = \xi = const.$ contours. Each characteristic curve corresponds to a specific value of ξ and a specific value of A. To obtain solution of A, over entire $x-t$ domain, one needs provide just enough information on each characteristic curve. Namely, one needs give initial or boundary conditions of A at *one and only one point* on each characteristic curve.

8.2. Equilibrium Solutions of Time-Independent Vlasov-Maxwell Equations

A set of equilibrium solutions of Vlasov-Maxwell system includes equilibrium distribution functions $f_{i0} = f_{i0}(\mathbf{x},\mathbf{v})$, $f_{e0} = f_{e0}(\mathbf{x},\mathbf{v})$, equilibrium electric field $\mathbf{E}_0(\mathbf{x})$, and equilibrium magnetic field $\mathbf{B}_0(\mathbf{x})$. This set of equilibrium solutions should satisfy the following equations.

The steady-state Vlasov equation of the αth species:

$$\mathbf{v}\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0 \tag{8.6}$$

The steady-state Maxwell's equations:

$$\varepsilon_0 \nabla\cdot\mathbf{E}_0 = \sum_\alpha \iiint e_\alpha f_{\alpha 0} d^3 v \tag{8.7}$$

$$\nabla \cdot \mathbf{B}_0 = 0 \tag{8.8}$$

$$\nabla \times \mathbf{E}_0 = 0 \tag{8.9}$$

$$(\nabla \times \mathbf{B}_0) / \mu_0 = \mathbf{J}_0 = \sum_{\alpha} \iiint e_\alpha f_{\alpha 0} \mathbf{v} d^3 v \tag{8.10}$$

where $\alpha = i, e$, $e_i = e$, and $e_e = -e$.

Eqs. (8.8) and (8.9) yield, respectively,

$$\mathbf{B}_0 = \nabla \times \mathbf{A}_0 \tag{8.11}$$

$$\mathbf{E}_0 = -\nabla \Phi_0 \tag{8.12}$$

Table 8.1. lists examples of equilibrium solutions of given $\mathbf{E}_0$ and $\mathbf{B}_0$.

Table 8.1. Examples of equilibrium solutions of given $\mathbf{E}_0$ and $\mathbf{B}_0$.

Case 1	Case 2	Case 3
$\mathbf{E}_0 = \mathbf{B}_0 = 0$	$\mathbf{E}_0 = -\hat{x}\dfrac{d\Phi}{dx}$, $\mathbf{B}_0 = 0$	$\mathbf{E}_0 = 0$, $\mathbf{B}_0 = \hat{z}B_0$
$f_{i0} = f_{i0}(v_x, v_y, v_z)$ $f_{e0} = f_{e0}(v_x, v_y, v_z)$ and f_{i0}, f_{e0} satisfy $\iiint (f_{i0} - f_{e0}) d^3 v = 0$ $\iiint \mathbf{v}(f_{i0} - f_{e0}) d^3 v = 0$	$f_{i0} = f_{i0}(\frac{1}{2} m_i v_x^2 + e\Phi(x), v_y, v_z)$ $f_{e0} = f_{e0}(\frac{1}{2} m_e v_x^2 - e\Phi(x), v_y, v_z)$ and f_{i0}, f_{e0} satisfy $\dfrac{d^2\Phi(x)}{dx^2} = \dfrac{-e}{\varepsilon_0} \iiint (f_{i0} - f_{e0}) d^3 v$ $\iiint \mathbf{v}(f_{i0} - f_{e0}) d^3 v = 0$	$f_{i0} = f_{i0}(v_x^2 + v_y^2, v_z)$ $f_{e0} = f_{e0}(v_x^2 + v_y^2, v_z)$ and f_{i0}, f_{e0} satisfy $\iiint (f_{i0} - f_{e0}) d^3 v = 0$ $\iiint \mathbf{v}(f_{i0} - f_{e0}) d^3 v = 0$

Exercise 8.1.

Derive equilibrium solutions shown in Table 8.1 based on procedures described in Section 8.1.

Exercise 8.2.

If we ignore Maxwell's equations, we can find another type of solutions for the steady-state Vlasov equation in Case 3. Show that for $\mathbf{E}_0 = 0, \mathbf{B}_0 = \hat{z}B_0$,

$$f_{i0} = f_{i0}\left(\frac{eB_0}{m_i} x + v_y, \frac{eB_0}{m_i} y - v_x, v_z\right)$$

and

$$f_{e0} = f_{e0}(\frac{-eB_0}{m_e}x + v_y, \frac{-eB_0}{m_e}y - v_x, v_z)$$

are equilibrium solutions of the steady-state Vlasov equations but cannot be the equilibrium solutions of the steady-state Vlasov-Maxwell equations.

References

Krall, N. A., and A. W. Trivelpiece (1973), *Principles of Plasma Physics*, McGraw-Hill Book Company, New York.

Nicholson, D. R. (1983), *Introduction to Plasma Theory*, John Wiley & Sons, New York.