Chapter 7. Particle Motions With Multiple Time Scales

Topics or concepts to learn in Chapter 7:

1. Periodic motions in different time scales
2. Gyro motion and magnetic moment
3. Bounce motion and the mirror point. What is pitch angle? What is loss-cone distribution?
4. Drift motions and their applications to the space plasma phenomena

(a) How to separate motions in different time scale?
(b) $\mathbf{E}\times\mathbf{B}$ drift and the moving frame dependent electric field
(c) Gravitational drift
(d) Curvature drift
(e) Gradient-B drift
(f) The diamagnetic effect: the diamagnetic drift, the diamagnetic current, and the magnetization current.
(g) The polarization drift, the polarization current and the Alfvén waves
(h) The ponderomotive force

Suggested Readings:

(1) Chapter 2 in Nicholson (1983)
(2) Appendix I in Krall and Trivelpiece (1973)
(3) Chapter 2 in F. F. Chen (1984)

7.1. Periodic Motions and Drift Motions of a Charged Particle

Action variable ($J = \oint p\, dq$) is adiabatic invariant under slow change of parameters (Goldstein, 1980). Action of a quasi-periodic motion is conserved if parameters, which affect the periodic motion, are nearly steady and nearly uniform. Three periodic motions may be found in magnetized plasma. They are (1) gyro motion around the magnetic field, (2) bounce motion in a magnetic mirror machine and (3) periodic drift motion around a magnetic mirror machine, where magnetic mirror machine is characterized by non-uniform magnetic field strength along magnetic field line.

Exercise 7.1.

Consider a charge particle moving in a nearly steady and nearly uniform magnetic field. Show that if variation of magnetic field $\delta\mathbf{B}(\mathbf{x},t)$ is small compare with the background magnetic field $\mathbf{B}$ in one gyro period and in one gyro radius (i.e., $|\delta\mathbf{B}| << |\mathbf{B}|$), then the particle's magnetic moment is conserved. That is

$$\mu = \frac{\frac{1}{2}mv_\perp^2}{B} \approx \text{constant}$$

Exercise 7.2.

Consider a charge particle moving in a steady magnetic mirror machine, in which magnitude of magnetic field is non-uniform along the magnetic field line. Discuss changes of particle's velocity along its bounce trajectory for different pitch angles at minimum B along a field line. Discuss the formation of the loss-cone distribution.

Before introducing the third type of periodic motion (i.e., a periodic drift motion), we need first introduce different types of drift motion in a magnetized plasma. Let us consider a charged particle moving in a nearly steady and nearly uniform magnetic field. If this particle's magnetic moment is conserved, its perpendicular velocity $\mathbf{v}_\perp$ can be decomposed into two components. One is a high frequency gyro velocity $\mathbf{v}_{gyro}$. The other is a low frequency or nearly time independent drift motion $\mathbf{v}_{drift}$. Namely,

$$\mathbf{v}_\perp = \mathbf{v}_{gyro} + \mathbf{v}_{drift}$$

In general, a *low frequency equation of motion* can be obtained by averaging the *original equation of motion* over a gyro period. We can obtain the guiding center drift velocity $\mathbf{v}_{drift}$ from the *low frequency equation of motion.*

7.1.1. $\mathbf{E}\times\mathbf{B}$ Drift

Let us consider a charge particle moving in a system with a uniform magnetic field $\mathbf{B}$ and a uniform electric field $\mathbf{E}$, which is in the direction perpendicular to the local magnetic field $\mathbf{B}$. If this particle has no velocity component parallel to the local magnetic field and magnetic moment of this particle is conserved, then we can decompose velocity of this particle into

$\mathbf{v} = \mathbf{v}_{gyro} + \mathbf{v}_{drift}$

where $\mathbf{v}_{gyro}$ is the high frequency gyro motion velocity and $\mathbf{v}_{drift}$ is a time independent guiding center drift velocity. Equation of motion of this charge particle is

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{7.1}$$

Averaging Eq. (7.1) over one gyro period ($\tau = 2\pi/\Omega_c$, where $\Omega_c = |q|B/m$), we can obtain equation for low frequency guiding-center motion,

$$\mathbf{E} + \mathbf{v}_{drift} \times \mathbf{B} = 0 \tag{7.2}$$

Solution of $\mathbf{v}_{drift}$ in Eq. (7.2) is the $\mathbf{E} \times \mathbf{B}$ *drift velocity*

$$\boxed{\mathbf{v}_{drift} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}} \tag{7.3}$$

Note that if both ions and electrons follow $\mathbf{E} \times \mathbf{B}$ drift, then there will be no low frequency electric current generated by ions' and electrons' $\mathbf{E} \times \mathbf{B}$-drift. In the Earth ionosphere E-region, electrons follow $\mathbf{E} \times \mathbf{B}$ drift, but ions do not. As a result, electrons' $\mathbf{E} \times \mathbf{B}$ drift can lead to Hall current in the E-region ionosphere. Hall current is in $-\mathbf{E} \times \mathbf{B}$ direction. Large-scale plasma flow in magnetosphere and interplanetary space are mainly governed by $\mathbf{E} \times \mathbf{B}$ drift, whereas, electric field information is mainly carried by Alfvén wave along the magnetic field line. Thus, Alfvén wave and $\mathbf{E} \times \mathbf{B}$ drift together play important roles on determining large-scale plasma flow in space.

Exercise 7.3.

Let us consider an electron moving in a system with $\mathbf{E} = \hat{y}60mV/m$, $\mathbf{B} = \hat{z}200nT$. Please determine gyro speed, sketch trajectory of the electron, and describe the physical meaning of the trajectory, if at $t = 0$, the initial velocity of the electron is

(1) $\mathbf{v} = +\hat{x}800km/s$

(2) $\mathbf{v} = +\hat{x}600km/s$

(3) $\mathbf{v} = +\hat{x}400km/s$

(4) $\mathbf{v} = +\hat{x}300km/s$

(5) $\mathbf{v} = +\hat{x}200km/s$

Exercise 7.4.

Explain formation of (1) the plasma tail (or ion tail) of a comet, (2) the plasmasphere of Earth, and (3) the plasma sheet in the Earth magnetotail based on $\mathbf{E} \times \mathbf{B}$ drift of plasmas. Discuss the formation of cross-field electric field ($\mathbf{E}_{\perp\mathbf{B}}$) in these three cases.

7.1.2. Gravitational Drift

Let us consider a charge particle moving in a system with uniform magnetic field $\mathbf{B}$ and uniform gravitational field $\mathbf{g}$, which is in the direction perpendicular to the local magnetic field $\mathbf{B}$. If this particle has no velocity component parallel to the local magnetic field and magnetic moment of this particle is conserved, then we can decompose velocity of this particle into

$$\mathbf{v} = \mathbf{v}_{gyro} + \mathbf{v}_{drift}$$

where $\mathbf{v}_{gyro}$ is the high frequency gyro motion velocity and $\mathbf{v}_{drift}$ is a time independent guiding center drift velocity. Equation of motion of this charge particle is

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} + q\mathbf{v}\times\mathbf{B} \tag{7.4}$$

Averaging Eq. (7.4) over one gyro period ($\tau = 2\pi/\Omega_c$, where $\Omega_c = |q|B/m$), we can obtain equation for low frequency guiding-center motion,

$$m\mathbf{g} + q\mathbf{v}_{drift}\times\mathbf{B} = 0 \tag{7.5}$$

Solution of $\mathbf{v}_{drift}$ in Eq. (7.5) is the *gravitational drift velocity*

$$\boxed{\mathbf{v}_{drift} = \frac{m\mathbf{g}\times\mathbf{B}}{qB^2}} \tag{7.6}$$

Drift speed of gravitational drift increases with increasing particle's mass. Gravitational drift provides an important electric current source in the low-latitude ionosphere and in the solar atmosphere.

Exercise 7.5.

Show that the gravitational drift in the low-latitude ionosphere is unstable to a surface perturbation at bottom-side of the nighttime ionosphere. This is called gravitational Rayleigh-Taylor (GRT) instability. The GRT instability can produce plasma cavities in the ionosphere and initiate the observed equatorial spread F (ESF) irregularities (e.g., Kelley, 1989, pp.121-122).

7.1.3. Curvature Drift

Consider a charge particle with constant magnetic moment and non-zero velocity component parallel to the local magnetic field. If curvature of the magnetic field line is non-zero, then the particle's field-aligned moving frame will become a non-inertial frame. Let us consider a time scale in which the particle's parallel speed $v_{\parallel}$ is nearly constant. Equation of motion in this non-inertial moving frame can be approximately written as

$$m\frac{d\mathbf{v}}{dt} = \frac{\hat{R}_B m v_{\parallel}^2}{R_B} + q\mathbf{v}\times\mathbf{B} \tag{7.7}$$

We can decompose velocity of this particle into

$$\mathbf{v} = \mathbf{v}_{gyro} + \mathbf{v}_{drift}$$

where $\mathbf{v}_{gyro}$ is the high frequency gyro motion velocity and $\mathbf{v}_{drift}$ is a low frequency (or nearly time independent) drift velocity. Averaging Eq. (7.7) over one gyro period ($\tau = 2\pi/\Omega_c$, where $\Omega_c = |q|B/m$), we can obtain equation for low frequency guiding-center motion in the $v_{\parallel}$ non-inertial moving frame

$$\frac{\hat{R}_B m v_{\parallel}^2}{R_B} + q\mathbf{v}_{drift}\times\mathbf{B} = 0 \tag{7.8}$$

Solution of $\mathbf{v}_{drift}$ in Eq. (7.8) is the *curvature drift velocity*, which can be written as

$$\boxed{\mathbf{v}_{drift} = \frac{m v_{\parallel}^2}{qB^2}\left(\frac{\hat{R}_B}{R_B}\times\mathbf{B}\right)} \tag{7.9}$$

It is shown in **Appendix D** that curvature drift velocity in Eq. (7.9) can be rewritten as

$$\boxed{\mathbf{v}_{drift} = \frac{m v_{\parallel}^2}{qB^2}\left[(\nabla\times\mathbf{B})_{\perp} - \frac{\nabla_{\perp}B}{B}\times\mathbf{B}\right]} \tag{7.10}$$

Drift speed of curvature drift increases with increasing $m v_{\parallel}^2$ (which is proportion to particle's kinetic energy in the direction parallel to local magnetic field). Curvature drift carried by energetic ions during magnetic storm and substorm periods can enhance partial ring current in the pre-midnight and midnight region.

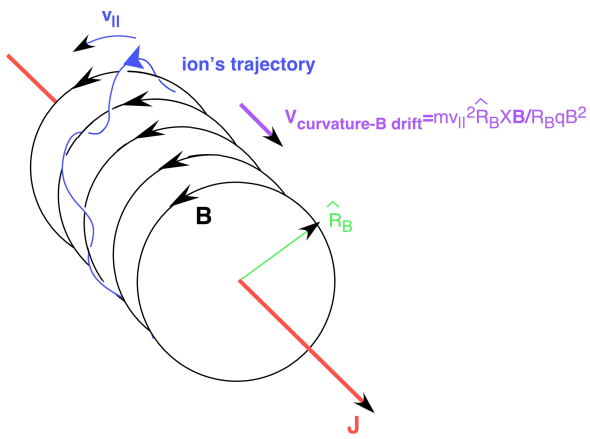

Figure 7.1. A sketch of the curvature drift of an ion moving in a non-uniform magnetic field.

7.1.4. Gradient B Drift

Let us consider a charge particle moving in a system with non-uniform magnetic field $\mathbf{B}(\mathbf{r})$. If the non-uniformity of the magnetic field is small enough such that we can use the first two terms in Taylor expansion to estimate magnetic field based on magnetic field information at guiding center of the charge particle. Namely,

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}(\boldsymbol{r}_{g.c.}) + (\boldsymbol{r} - \boldsymbol{r}_{g.c.}) \cdot (\nabla \boldsymbol{B})\big|_{\boldsymbol{r}_{g.c.}} + \cdots \tag{7.11}$$

where $\boldsymbol{r} - \boldsymbol{r}_{g.c.} = \boldsymbol{r}_{gyro}$.

If this particle has no velocity component parallel to the local magnetic field and magnetic moment of this particle is conserved then we can decompose velocity of this particle into

$$\mathbf{v} = \mathbf{v}_{gyro} + \mathbf{v}_{drift}$$

where $\mathbf{v}_{gyro}$ is the high frequency gyro motion velocity and $\mathbf{v}_{drift}$ is a time independent guiding center drift velocity. Equation of motion of this charge particle can be approximately written as

$$m\frac{d\boldsymbol{v}}{dt} = q\boldsymbol{v} \times \boldsymbol{B} \approx q(\boldsymbol{v}_{gyro} + \boldsymbol{v}_{drift}) \times [\boldsymbol{B}(\boldsymbol{r}_{g.c.}) + \boldsymbol{r}_{gyro} \cdot \nabla \boldsymbol{B}] \tag{7.12}$$

Averaging Eq. (7.12) over one gyro period ($\tau = 2\pi/\Omega_c$, where $\Omega_c = |q|B/m$), we can obtain equation for low frequency guiding-center motion

$$\boldsymbol{v}_{drift} \times \boldsymbol{B}(\boldsymbol{r}_{g.c.}) + \left\langle \boldsymbol{v}_{gyro} \times (\boldsymbol{r}_{gyro} \cdot \nabla \boldsymbol{B}) \right\rangle = 0 \tag{7.13}$$

where the notation $\langle f \rangle$ denotes time average value of f. It is shown in **Appendix E** that the average value in Eq. (7.13) can be rewritten as

$$\left\langle \boldsymbol{v}_{gyro} \times \boldsymbol{r}_{gyro} \cdot \nabla \boldsymbol{B} \right\rangle = \frac{mv_{gyro}^2}{2qB}(-\nabla_\perp B)$$

Thus, Eq. (7.13) becomes

$$\mathbf{v}_{drift} \times \mathbf{B}(\mathbf{r}_{g.c.}) + \frac{mv_{gyro}^2}{2qB}(-\nabla_\perp B) = 0 \tag{7.14}$$

Solution of $\mathbf{v}_{drift}$ in Eq. (7.14) is the *gradient-B drift velocity (or grad-B drift velocity)*

$$\boxed{\boldsymbol{v}_{drift} = \frac{mv_{gyro}^2}{2qB}\frac{(-\nabla_\perp B) \times \boldsymbol{B}}{B^2}} \tag{7.15}$$

The gradient-B drift speed increases with increasing $mv_{gyro}^2/2$.

For $v_{drift} << v_{gyro}$, the perpendicular speed, $v_\perp$, of the charge particle is approximately equal to v_{gyro}. Thus, it is commonly using the following expression to denote *gradient-B drift*

$$\boxed{\mathbf{v}_{drift} = \frac{mv_\perp^2}{2qB}\frac{(-\nabla_\perp B)\times\mathbf{B}}{B^2}} \tag{7.16}$$

In this case, the *gradient-B drift speed* increases with increasing perpendicular kinetic energy. Gradient-B drift cancels magnetic gradient effect in magnetization current to be discussed in section 7.2. As a result, the net current (diamagnetic current, to be discussed in section 7.2) has little dependence on the magnetic gradient. Both gradient-B drift and curvature drift of the energetic particles in the ring current region can reduce time scale of the third periodic motion (periodically drifting around the Earth) from 24-hour co-rotating period to only a few hours. Thus, the third adiabatic invariant condition may be applicable to these energetic particles in the ring current region.

7.2. Fluid Drift

Let us consider a non-uniform plasma system with a sharp density or pressure gradient in the direction perpendicular to the ambient magnetic field. Since gyro motion of a charge particle can reduce/enhance magnetic field magnitude inside/outside its orbit. The net effects of gyro motions in high-density (or high-pressure) region can result in an effective electric current located at the density-gradient (or pressure-gradient) region. In this section, we shall use ions' and electrons' momentum equations to determine drift velocity of ions and electrons at the pressure-gradient region. Similarly, one-fluid momentum equation is used to determine effective electric current (so-called *diamagnetic current*) at the pressure-gradient region.

7.2.1. Ions' Diamagnetic Drift

Momentum equation of ion fluid

$$n_i m_i(\frac{\partial \mathbf{V}_i}{\partial t} + \mathbf{V}_i \cdot \nabla\mathbf{V}_i) = -\nabla p_i + n_i e(\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) \tag{7.17}$$

where n_i, $\mathbf{V}_i$, and p_i are ions' number density, flow velocity, and thermal pressure, respectively. For steady state ($\partial / \partial t = 0$) and for $\mathbf{V}_i \cdot \nabla\mathbf{V}_i = 0$, $\mathbf{E} = 0$, Eq. (7.17) yields

$$-\nabla p_i + n_i e\mathbf{V}_i \times \mathbf{B} = 0 \tag{7.18}$$

Thus, we obtain ions' diamagnetic drift velocity

$$\boxed{\mathbf{V}_i = \frac{-\nabla p_i \times \mathbf{B}}{n_i e B^2}} \tag{7.19}$$

7.2.2. Electrons' Diamagnetic Drift

Momentum equation of electron fluid

$$n_e m_e (\frac{\partial \mathbf{V}_e}{\partial t} + \mathbf{V}_e \cdot \nabla \mathbf{V}_e) = -\nabla p_e - n_e e(\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) \tag{7.20}$$

where $n_e, \mathbf{V}_e$, and p_e are electrons' number density, flow velocity, and thermal pressure, respectively. For steady state ($\partial/\partial t = 0$) and for $\mathbf{V}_e \cdot \nabla \mathbf{V}_e = 0$, $\mathbf{E} = 0$, Eq. (7.20) yields

$$-\nabla p_e - n_e e \mathbf{V}_e \times \mathbf{B} = 0 \tag{7.21}$$

Thus, we obtain electrons' diamagnetic drift velocity

$$\boxed{\mathbf{V}_e = \frac{-\nabla p_e \times \mathbf{B}}{n_e(-e)B^2} = \frac{\nabla p_e \times \mathbf{B}}{n_e e B^2}} \tag{7.22}$$

7.2.3. Diamagnetic Current Density

We define one-fluid mass density ρ to be

$$\rho = n_i m_i + n_e m_e \tag{7.23}$$

and flow velocity $\mathbf{V}$ to be ions and electrons center of mass flow velocity

$$\mathbf{V} = \frac{n_i m_i \mathbf{V}_i + n_e m_e \mathbf{V}_e}{n_i m_i + n_e m_e} \tag{7.24}$$

We can also define one-fluid thermal pressure satisfies

$$\left[n_i m_i (\frac{\partial \mathbf{V}_i}{\partial t} + \mathbf{V}_i \cdot \nabla \mathbf{V}_i) + \nabla p_i \right] + \left[n_e m_e (\frac{\partial \mathbf{V}_e}{\partial t} + \mathbf{V}_e \cdot \nabla \mathbf{V}_e) + \nabla p_e \right]$$

$$= \rho(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}) + \nabla p \tag{7.25}$$

Then, Eq. (7.17) + Eq. (7.18) yields one-fluid momentum equation

$$\rho(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}) = -\nabla p + \rho_c \mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{7.26}$$

For steady state ($\partial/\partial t = 0$) and for $\mathbf{V} \cdot \nabla \mathbf{V} = 0$, $\mathbf{E} = 0$, Eq. (7.26) becomes

$$-\nabla p + \mathbf{J} \times \mathbf{B} = 0 \tag{7.27}$$

Thus, we obtain diamagnetic current density

$$\boxed{\mathbf{J} = \frac{-\nabla p \times \mathbf{B}}{B^2}} \tag{7.28}$$

Most current sheets in the space plasma are maintained by a density or pressure gradient. One can obtain electric current direction at magnetopause, plasmapause, and plasma sheet boundary layer (PSBL) based on Eq. (7.28).

Exercise 7.6.

Determine electric current direction at:

(1) dayside magnetopause

(2) nightside magnetopause

(3) plasmapause

(4) plasma sheet boundary layer

For convenience, we shall use **V** to denote flow velocity and use **v** to denote a single particle velocity. Fluid drift motion plays an important role on generating electric currents in our magnetosphere. These current systems can generate new magnetic field components to make our magnetosphere different from a dipole field structure.

7.2.4. Magnetization Current

The diamagnetic current obtained in last subsection is indeed a net current of (1) current due to diamagnetic motion of charge particles, which is called *magnetization current* (Longmire, 1963), (2) current due to particles' curvature drift, and (3) current due to particles' gradient-B drift.

By definition, *magnetization current* is

$$\mathbf{J} = \nabla \times \mathbf{M} = \nabla \times \sum_i (-\mu_i \hat{B}) \tag{7.29}$$

where $-\mu_i \hat{B}$ is the magnetic moment of the ith particle.

Exercise 7.7.

Show that for low temperature plasma with isotropic pressure the net current due to curvature drift and gradient-B drift discussed in sections 7.1.3 and 7.1.4 and the magnetization current in Eq. (7.29) is equal to the diamagnetic current in Eq. (7.28).

For high temperature plasma, we have to use kinetic approach to determine the net current. The net current obtained from kinetic approach is not identical to the diamagnetic current in Eq. (7.28). Kinetic approach is an advanced subject of plasma physics, which will be discussed later in Chapters 8-11.

7.3. Drift Motion in Time-Dependent Fields

7.3.1. Polarization Drift

The low frequency wave, such as the Alfvén-mode/Fast-mode wave in the MHD plasma, can carry electric field along the B-field line/stream line. The time variation of the electric field at the wave front of the low frequency wave can lead to polarization drift of particles and result in polarization current.

Let us consider a uniform magnetic field, $\mathbf{B} = \hat{z} B$, and a time-dependent electric field, which increases linearly with time. Let $\mathbf{E} = \hat{y}\, E(t) = \hat{y}\, \dot{E}t$.

The equation of motion becomes

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\hat{y}\, \dot{E}t + \mathbf{v} \times \hat{z} B) \tag{7.30}$$

Let

$$\mathbf{v} = \mathbf{v}_{gyro} + \mathbf{v}_{E\times B\ drift} + \mathbf{v}_{polarization\ drift} \tag{7.31}$$

where

$$\mathbf{v}_{E\times B\ drift} = \frac{\hat{y}\, \dot{E}t \times \hat{z} B}{B^2} = \hat{x}\frac{\dot{E}t}{B} \tag{7.32}$$

or

$$\hat{y}\, \dot{E}t + \mathbf{v}_{E\times B\ drift} \times \hat{z} B = 0 \tag{7.33}$$

and

$$\frac{d\mathbf{v}_{E\times B\ drift}}{dt} = \hat{x}\frac{\dot{E}}{B} \tag{7.34}$$

Substituting Eqs. (7.31) and (7.32) into Eq. (7.30), and then averaging the resulting equation over the gyro period, $2\pi/(qB/m)$, and then making use of the Eqs (7.33) and (7.34), it yields

$$\frac{d\mathbf{v}_{E\times B\ drift}}{dt} = \hat{x}\frac{\dot{E}}{B} = \frac{q}{m}(\mathbf{v}_{polarization\ drift} \times \hat{z} B) \tag{7.35}$$

The solution of Eq. (7.35) is

$$\mathbf{v}_{polarization\ drift} = \frac{-(m\dfrac{d\mathbf{v}_{E\times B\ drift}}{dt})\times\hat{z}B}{qB^2} = \frac{-(\hat{x}m\dfrac{\dot{E}}{B})\times\hat{z}B}{qB^2} = \hat{y}\frac{m}{q}\frac{\dot{E}}{B^2} \tag{7.36}$$

7.3.2. Ponderomotive Force

The motion of a charge particle under the influence of a high-frequency non-uniform longitudinal wave or transverse wave shows a drift motion with nearly constant acceleration. The acceleration of the drift motion is due to the presence of ponderomotive force of the non-uniform wave field.

7.3.2.1 Ponderomotive Force in a High-Frequency Non-uniform Longitudinal E-Field

Let us consider a high frequency non-uniform longitudinal electric field $\mathbf{E} = \hat{x}E_0(x)\sin\omega t$. The equation of motion becomes

$$\frac{dv_x}{dt} = \frac{q}{m}E_0(x)\sin\omega t \tag{7.37}$$

For $t=0$, $v_x \approx 0$, and $x \approx x_0$, integrating Eq. (7.37) once, it yields

$$v_x = \frac{q}{m\omega}E_0(x)(1-\cos\omega t) \tag{7.38a}$$

Integrating Eq. (7.38a) once, it yields

$$x - x_0 = \frac{q}{m\omega^2}E_0(x)(\omega t - \sin\omega t) \tag{7.38b}$$

The non-uniform wave amplitude can be written as

$$E_0(x) = E_0(x_0) + (x-x_0)\left.\frac{dE_0}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2}\left.\frac{d^2E_0}{dx^2}\right|_{x=x_0} + \cdots \tag{7.39}$$

Substituting Eq. (7.38b) into Eq. (7.39), and then substituting the resulting equation into Eq. (7.37) it yields

$$\begin{aligned}\frac{dv_x}{dt} &= \frac{q}{m}\{E_0(x_0) + [\frac{q}{m\omega^2}E_0(x)(\omega t - \sin\omega t)]\left.\frac{dE_0}{dx}\right|_{x=x_0} + \cdots\}\sin\omega t \\ &= \frac{q}{m}\{E_0(x_0)\sin\omega t + \frac{q}{2m\omega^2}\left.\frac{dE_0^2}{dx}\right|_{x=x_0}\omega t\sin\omega t - \frac{q}{2m\omega^2}\left.\frac{dE_0^2}{dx}\right|_{x=x_0}\sin^2\omega t\}\end{aligned} \tag{7.40}$$

Averaging Eq. (7.40) over the wave period, $2\pi/\omega$, it yields

$$\left\langle \frac{dv_x}{dt} \right\rangle_{2\pi/\omega} \approx \frac{-q^2}{m^2\omega^2}(\frac{1}{2}+\frac{1}{4})\frac{dE_0^2}{dx}\bigg|_{x=x_0} = -\frac{3}{4}\frac{q^2}{m^2\omega^2}\frac{dE_0^2}{dx}\bigg|_{x=x_0} \tag{7.41}$$

where $\langle t \sin\omega t \rangle_{2\pi/\omega} = -1/\omega$ and $\langle \sin^2 \omega t \rangle_{2\pi/\omega} = 1/2$

The ponderomotive force of the non-uniform high frequency longitudinal electric field is

$$\mathbf{F}_p = m\left\langle \frac{d\mathbf{v}}{dt} \right\rangle_{2\pi/\omega} \approx \hat{x}\{-\frac{3}{4}\frac{q^2}{m\omega^2}\frac{dE_0^2}{dx}\} \tag{7.42}$$

7.3.2.2 Ponderomotive Force in a High-Frequency Non-uniform EM Wave Field

Let us consider a high-frequency non-uniform transverse electric field and magnetic field

$\mathbf{E} = \hat{x} E_0(z)\cos\omega t$

$\mathbf{B} = \hat{y} B_0(z)\cos\omega t$

The equation of motion of a relative charge particle is

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \hat{x}qE_0(z)\cos\omega t + q\frac{\mathbf{p}}{\gamma m}\times\hat{y}B_0(z)\cos\omega t \tag{7.43}$$

or

$$\frac{dp_x}{dt} = qE_0(z)\cos\omega t - q\frac{p_z}{\gamma m}B_0(z)\cos\omega t \tag{7.43a}$$

$$\frac{dp_z}{dt} = q\frac{p_x}{\gamma m}B_0(z)\cos\omega t \tag{7.43b}$$

For $t=0$, $\mathbf{p}\approx 0$, and $z \approx z_0$, integrating Eq. (7.43a) once, it yields

$$p_x \approx \frac{q}{\omega}E_0(z)\sin\omega t \tag{7.44}$$

Substituting Eq. (7.44) into Eq. (7.43b). it yields

$$\frac{dp_z}{dt} \approx \frac{q}{\gamma m}[\frac{q}{\omega}E_0(z)\sin\omega t]B_0(z)\cos\omega t = \frac{q^2}{2\gamma m\omega}E_0(z)B_0(z)\sin 2\omega t \tag{7.45}$$

Integrating Eq. (7.45) once, it yields

$$p_z \approx \frac{q^2}{2\gamma m\omega}E_0(z)B_0(z)\frac{1-\cos 2\omega t}{2\omega} \tag{7.46}$$

Since $p_z = \gamma m v_z$, we have

$$v_z \approx \frac{q^2}{2(\gamma m)^2 \omega} E_0(z) B_0(z) \frac{1-\cos 2\omega t}{2\omega} \tag{7.46a}$$

Integrating Eq. (7.46a) once, it yields

$$z - z_0 \approx \frac{q^2}{(2\omega)^2} \frac{1}{(\gamma m)^2} E_0(z) B_0(z) (t - \frac{\sin 2\omega t}{2\omega}) \tag{7.47}$$

Define

$$E_{00}(z) = E_0(z) B_0(z) / \gamma \tag{7.48}$$

Substituting Eq. (7.48) into Eq. (7.45), it yields

$$\frac{dp_z}{dt} \approx \frac{q^2}{(2\omega)m} E_{00}(z) \sin 2\omega t \tag{7.45a}$$

Substituting Eq. (7.48) into Eq. (7.47), it yields

$$z - z_0 \approx \frac{q^2}{(2\omega)^2 m^2} \frac{1}{\gamma} E_{00}(z)(t - \frac{\sin 2\omega t}{2\omega}) \tag{7.47a}$$

The non-uniform $E_{00}(z)$ can be written as

$$E_{00}(z) = E_{00}(z_0) + (z - z_0) \left.\frac{dE_{00}}{dz}\right|_{z=z_0} + \frac{(z-z_0)^2}{2} \left.\frac{d^2 E_{00}}{dz^2}\right|_{z=z_0} + \cdots \tag{7.49}$$

Substituting Eq. (7.47a) into (7.49), then substituting the resulting equation into Eq. (7.45a), it yields

$$\begin{aligned}
\frac{dp_z}{dt} &\approx \frac{q^2}{(2\omega)m} E_{00}(z) \sin 2\omega t \\
&\approx \frac{q^2}{(2\omega)m} \{E_{00}(z_0) + (z - z_0) \left.\frac{dE_{00}}{dz}\right|_{z=z_0} + ...\} \sin 2\omega t \\
&\approx \frac{q^2}{(2\omega)m} \{E_{00}(z_0) + [\frac{q^2}{(2\omega)^2 m^2} \frac{1}{\gamma} E_{00}(z)(t - \frac{\sin 2\omega t}{2\omega})] \left.\frac{dE_{00}}{dz}\right|_{z=z_0} + ...\} \sin 2\omega t \\
&\approx \frac{q^2}{(2\omega)m} \{E_{00}(z_0) \sin 2\omega t + \frac{q^2}{(2\omega)^2 m^2} \frac{1}{\gamma} (t \sin 2\omega t - \frac{\sin^2 2\omega t}{2\omega}) \frac{1}{2} \left.\frac{dE_{00}^2}{dz}\right|_{z=z_0} \}
\end{aligned} \tag{7.50}$$

Averaging the Eq. (7.50) over the period, $2\pi/(2\omega)$, it yields

$$
\begin{aligned}
\left\langle \frac{dp_z}{dt} \right\rangle_{\frac{2\pi}{2\omega}} &\approx \frac{q^2}{(2\omega)m}\{E_{00}(z_0)\langle \sin 2\omega t\rangle_{\frac{2\pi}{2\omega}} + \frac{q^2}{(2\omega)^2 m^2}\frac{1}{\gamma}\left\langle t\sin 2\omega t - \frac{\sin^2 2\omega t}{2\omega}\right\rangle_{\frac{2\pi}{2\omega}} \frac{1}{2}\frac{dE_{00}^2}{dz}\bigg|_{z=z_0} \} \\
&= \frac{q^2}{(2\omega)m}\{E_{00}(z_0)\cdot 0 + \frac{q^2}{(2\omega)^2 m^2}\frac{1}{\gamma}(-\frac{1}{2\omega} - \frac{1}{2}\frac{1}{2\omega})\frac{1}{2}\frac{dE_{00}^2}{dz}\bigg|_{z=z_0} \} \\
&= \frac{-q^4}{(2\omega)^4 m^3}\frac{1}{\gamma}\frac{3}{4}\frac{dE_{00}^2}{dz}\bigg|_{z=z_0} \\
&= \frac{-q^4}{(2\omega)^4 m^3}\frac{1}{\gamma}\frac{3}{4}\frac{d(E_0(z)B_0(z)/\gamma)^2}{dz}\bigg|_{z=z_0} = \mathbf{F}_p
\end{aligned}
\tag{7.51a}
$$

where $\mathbf{F}_p$ is the ponderomotive force at $z = z_0$

The following statements may not be correct if the wave reflection takes place.

From Faraday's law, we have $E_0(z) = (\omega/k)B_0(z)$. For high frequency EM wave, we have $cB_0(z) = E_0(z)$. Since the momentum per unit mass $u \sim O(p/m) \sim O(qA/m) \sim O(qE/m\omega)$, we can define an EM wave induced transverse momentum per unit mass to be $u_T(z) = qE_0(z)/m(2\omega)$.

For high frequency EM wave, Eq. (7.51a) can be written as

$$
\begin{aligned}
\left\langle \frac{dp_z}{dt} \right\rangle_{\frac{2\pi}{2\omega}} &= \frac{-q^4}{(2\omega)^4 m^3}\frac{1}{\gamma}\frac{3}{4}\frac{d(E_0(z)B_0(z)/\gamma)^2}{dz}\bigg|_{z=z_0} \\
&= \frac{-q^4}{(2\omega)^4 m^3}\frac{1}{\gamma}\frac{3}{4}\frac{d(E_0^2(z)/c\gamma)^2}{dz}\bigg|_{z=z_0} \\
&= \frac{-q^4}{(2\omega)^4 m^3}\frac{1}{\gamma}\frac{3}{2}(\frac{E_0^2(z)}{c\gamma})\frac{d}{dz}(\frac{E_0^2(z)}{c\gamma})\bigg|_{z=z_0} \\
&= -\frac{m}{\gamma}\frac{3}{2}(\frac{q^2E_0^2(z)}{m^2(2\omega)^2 c\gamma})\frac{d}{dz}(\frac{q^2E_0^2(z)}{m^2(2\omega)^2 c\gamma})\bigg|_{z=z_0} \\
&= -\frac{m}{\gamma}\frac{3}{2}(\frac{u_T^2(z)}{c\gamma})\frac{d}{dz}(\frac{u_T^2(z)}{c\gamma})\bigg|_{z=z_0}
\end{aligned}
\tag{7.51b}
$$

Let $v_T(z) = u_T(z)/\gamma$, it yields

$$
\mathbf{F}_p = \left\langle \frac{dp_z}{dt} \right\rangle_{\frac{2\pi}{2\omega}} = -\frac{3}{2}\frac{\gamma m}{c^2}u_T(z)v_T(z)\frac{d}{dz}[u_T(z)v_T(z)]\bigg|_{z=z_0}
\tag{7.51c}
$$

Likewise, the longitudinal momentum can be rewritten as

$$
\begin{aligned}
p_z &= \frac{q^2}{(2\omega)^2 m}\frac{E_0(z)B_0(z)}{\gamma}(1-\cos 2\omega t) \\
&= \frac{q^2}{(2\omega)^2 m}E_{00}(z)(1-\cos 2\omega t) \\
&= \frac{q^2}{(2\omega)^2 m}\{E_{00}(z_0)+(z-z_0)\left.\frac{dE_{00}}{dz}\right|_{z=z_0}+...\}(1-\cos 2\omega t) \\
&= \frac{q^2}{(2\omega)^2 m}\{E_{00}(z_0)+[\frac{q^2}{(2\omega)^2 m^2}\frac{1}{\gamma}E_{00}(z)(t-\frac{\sin 2\omega t}{2\omega})]\left.\frac{dE_{00}}{dz}\right|_{z=z_0}+...\}(1-\cos 2\omega t) \\
&= \frac{q^2}{(2\omega)^2 m}\{E_{00}(z_0)(1-\cos 2\omega t)+[\frac{q^2}{(2\omega)^2 m^2}\frac{1}{\gamma}(t-\frac{\sin 2\omega t}{2\omega})(1-\cos 2\omega t)]\frac{1}{2}\left.\frac{dE_{00}^2}{dz}\right|_{z=z_0}+...\}
\end{aligned}
\tag{7.52}
$$

The average of the longitudinal momentum over a period of $2\pi/(2\omega)$ is

$$
\begin{aligned}
&\langle p_z\rangle_{\frac{2\pi}{2\omega}} \\
&= \frac{q^2}{(2\omega)^2 m}\{E_{00}(z_0)\langle 1-\cos 2\omega t\rangle_{\frac{2\pi}{2\omega}}+[\frac{q^2}{(2\omega)^2 m^2}\frac{1}{\gamma}\left\langle(t-\frac{\sin 2\omega t}{2\omega})(1-\cos 2\omega t)\right\rangle_{\frac{2\pi}{2\omega}}]\frac{1}{2}\left.\frac{dE_{00}^2}{dz}\right|_{z=z_0}+...\} \\
&\approx \frac{q^2}{(2\omega)^2 m}E_{00}(z_0)(1-0) \\
&\quad+\frac{q^2}{(2\omega)^2 m}[\frac{q^2}{(2\omega)^2 m^2}\frac{1}{\gamma}\left\langle(t-\frac{\sin 2\omega t}{2\omega}-t\cos 2\omega t+\frac{\sin 2\omega t}{2\omega}\cos 2\omega t)\right\rangle_{\frac{2\pi}{2\omega}}]\frac{1}{2}\left.\frac{dE_{00}^2}{dz}\right|_{z=z_0} \\
&= \frac{q^2}{(2\omega)^2 m}E_{00}(z_0)+[\frac{q^4}{(2\omega)^4 m^3}\frac{1}{\gamma}(\frac{\pi}{2\omega}-0-\frac{1}{2\omega}+0)]\frac{1}{2}\left.\frac{dE_{00}^2}{dz}\right|_{z=z_0} \\
&= \frac{q^2}{(2\omega)^2 m}E_{00}(z_0)+\frac{q^4}{(2\omega)^4 m^3}\frac{1}{\gamma}\frac{(\pi-1)}{2\omega}\frac{1}{2}\left.\frac{dE_{00}^2}{dz}\right|_{z=z_0}
\end{aligned}
\tag{7.53}
$$

The following statements may not be correct if the wave reflection takes place.

For high frequency EM wave, Eq. (7.53) can be written as

$$
\begin{aligned}
\langle p_z \rangle_{\frac{2\pi}{2\omega}} &= \frac{q^2}{(2\omega)^2 m} E_{00}(z_0) + \frac{q^4}{(2\omega)^4 m^3} \frac{1}{\gamma} \frac{(\pi-1)}{2\omega} \frac{1}{2} \frac{dE_{00}^2}{dz}\bigg|_{z=z_0} \\
&= \frac{q^2}{(2\omega)^2 m} \frac{E_0(z_0)B_0(z_0)}{\gamma} + \frac{q^4}{(2\omega)^4 m^3} \frac{1}{\gamma} \frac{(\pi-1)}{2\omega} \left(\frac{E_0(z)B_0(z)}{\gamma}\right) \frac{d}{dz} \left(\frac{E_0(z)B_0(z)}{\gamma}\right)\bigg|_{z=z_0} \\
&= \frac{q^2}{(2\omega)^2 m} \frac{E_0^2(z_0)}{\gamma c} + \frac{q^4}{(2\omega)^4 m^3} \frac{1}{\gamma} \frac{(\pi-1)}{2\omega} \left(\frac{E_0^2(z)}{\gamma c}\right) \frac{d}{dz} \left(\frac{E_0^2(z)}{\gamma c}\right)\bigg|_{z=z_0} \\
&= m \frac{u_T^2(z_0)}{\gamma c} + m \frac{1}{\gamma} \frac{(\pi-1)}{2\omega} \frac{u_T^2(z)}{\gamma c} \frac{d}{dz} \left(\frac{u_T^2(z)}{\gamma c}\right)\bigg|_{z=z_0}
\end{aligned}
\tag{7.54}
$$

Exercise 7.8

Determine the ponderomotive force of a high-frequency non-uniform EM wave field with $\mathbf{p} = (p_{x0}, p_{y0}, p_{z0})$ at $t = 0$

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