Chapter 4. Deriving the Vlasov Equation From the Liouville Equation

Topics or concepts to learn in Chapter 4:
1. Liouville equation of the system probability density function \( f_{N_0} \) in \( 6N_0 \)-dimensional space.
2. The probability density function \( f_{N_0} \) is a point, thus it is incompressible in the \( 6N_0 \)-dimensional space.
3. How to reduce the probability density function from the \( 6N_0 \)-dimensional space to the \( 6 \)-dimensional space.
4. What assumptions have been made to obtain the Vlasov equation from the Liouville equation.

Suggested Readings:
(1) Chapter 4 in Nicholson (1983)
(2) Chapter 2 in Krall and Trivelpiece (1973)

Exercise 4.1.
Read Chapters 3 and 4 of Nicholson (1983) carefully. Understand the philosophical differences between the Klimontovich equation and the Liouville equation.

4.1. Liouville Equation

Let us consider a system with \( N_0 \) particles. Let \( N(x_1, x_2, \ldots, x_{N_0}, v_1, v_2, \ldots, v_{N_0}, t) \) be the density function of the system, i.e.,

\[
N(x_1, x_2, \ldots, x_{N_0}, v_1, v_2, \ldots, v_{N_0}, t) = \prod_{k=1}^{N_0} \delta[x_k - X_k(t)] \delta[v_k - V_k(t)]
\]  

(4.1)

It can be shown that
\[
\frac{\partial}{\partial t} N = \prod_{k=1}^{N_0} \left\{ \sum_{i=1}^{N_0} \frac{\partial}{\partial x_i} \right\} \left\{ \sum_{i=1}^{N_0} \left[ -\mathbf{v}_j \cdot \frac{\partial}{\partial \mathbf{v}_j} \right] \{ \delta(x_k - X_k(t)) \delta(v_k - V_k(t)) \} \right\}
\]

\[
= \prod_{k=1}^{N_0} \left\{ \sum_{i=1}^{N_0} \frac{\partial}{\partial x_i} \right\} \left\{ \sum_{j=1}^{N_0} \left[ -\sum_{i=1}^{N_0} a_{ji} (x_i, x_j, v_i, v_j; m_i, m_j, e_i, e_j) \right] \frac{\partial}{\partial v_j} \{ \delta(x_k - X_k(t)) \delta(v_k - V_k(t)) \} \right\}
\]

\[
= -\sum_{i=1}^{N_0} \left[ \mathbf{v}_i \cdot \frac{\partial N}{\partial x_i} \right] - \sum_{i=1}^{N_0} \left[ \sum_{j=1}^{N_0} a_{ji} (x_i, x_j, v_i, v_j; m_i, m_j, e_i, e_j) \right] \frac{\partial N}{\partial v_j}
\]

or

\[
\frac{\partial}{\partial t} N + \sum_{i=1}^{N_0} \left[ \mathbf{v}_i \cdot \frac{\partial N}{\partial x_i} \right] + \sum_{i=1}^{N_0} \left[ \sum_{j=1}^{N_0} a_{ji} \right] \frac{\partial N}{\partial v_j} = \frac{DN}{Dt} = 0 \tag{4.2}
\]

where \( m_i a_{ji} \) is the force that the \( j \)th particle acting on the \( i \)th particle. Thus, we have \( a_{jj} = 0 \). Eq. (4.2) is the Liouville equation of the system density function \( N \).

Let us now consider an ensemble of such \( N_0 \)-particle systems. From these systems, we can obtain a probability density function \( f_{N_0} (x_1, x_2, \ldots, x_{N_0}, v_1, v_2, \ldots, v_{N_0}, t) \) of a \( N_0 \)-particle system. The joint probability density function \( f_{N_0} \) of the system satisfies

\[
\frac{DF_{N_0}}{Dt} = \frac{\partial}{\partial t} f_{N_0} + \sum_{i=1}^{N_0} \left[ \mathbf{v}_i \cdot \frac{\partial f_{N_0}}{\partial x_i} \right] + \sum_{i=1}^{N_0} \left[ \sum_{j=1}^{N_0} a_{ji} \right] \frac{\partial f_{N_0}}{\partial v_j} = 0 \tag{4.3}
\]

Eq. (4.3) is the Liouville equation of the system probability density function \( f_{N_0} \). The probability density function \( f_{N_0} \) is incompressible in the in \( 6 N_0 \)-dimensional space.

### 4.2. BBGKY Hierarchy

We define the reduced probability distributions \( f_k \)

\[
f_k (x_1, \ldots, x_k, v_1, \ldots, v_k, t) \equiv V^k \int dx_{k+1} \cdots dx_{N_0} d\mathbf{v}_{N_0} f_{N_0} (x_1, \ldots, x_{N_0}, v_1, \ldots, v_{N_0}, t) \tag{4.4}
\]

Integration Eq. (4.3) over \( x_{N_0} \) and \( \mathbf{v}_{N_0} \) space and making use of the definition in Eq. (4.4), it yields an equation of reduced probability distributions \( f_{N_0-1} \), that is

\[
\frac{df_{N_0-1}}{dt} + \sum_{i=1}^{N_0-1} \left[ \mathbf{v}_i \cdot \frac{\partial f_{N_0-1}}{\partial x_i} \right] + \sum_{i=1}^{N_0-1} \left[ \sum_{j=1}^{N_0-1} a_{ji} \right] \frac{\partial f_{N_0-1}}{\partial v_j} + V^{N_0-1} \sum_{i=1}^{N_0-1} \int dx_{N_0} d\mathbf{v}_{N_0} a_{ib_{N_0}} \frac{\partial f_{N_0}}{\partial v_i} = 0 \tag{4.5}
\]

Thus, the desired equation for \( f_{N_0-1} \) does not only depend on \( f_{N_0-1} \) itself, but also depend on \( f_{N_0} \) as shown in the last term in Eq. (4.5).
Likewise, we can obtain the equation for the probability density function $f_k$

$$\frac{\partial}{\partial t} f_k + \sum_{i=1}^{k} [v_i \cdot \frac{\partial f_k}{\partial x_i}] + \sum_{j=1}^{k} [a_j \cdot \frac{\partial f_k}{\partial v_j}] + \frac{N_0}{V} - \frac{k}{V} \sum_{i=1}^{k} \left[ \int dx_{k+1} dv_{k+1} a_{i,k+1} \cdot \frac{\partial f_{k+1}}{\partial v_i} \right] = 0 \quad (4.6)$$

Again, the desired equation for $f_k$ does not only depend on $f_k$ itself, but also depend on $f_{k+1}$. This is the BBGKY hierarchy (Bogoliubov, 1946; Born and Green 1949; Kirkwood 1946; 1947; and Yvon 1935).

For $k=1$, Eq. (4.6) yields,

$$\frac{\partial}{\partial t} f_1 + v_1 \cdot \frac{\partial f_1}{\partial x_1} + \frac{N_0}{V} - \frac{1}{V} \int dx_2 dv_2 a_{12} \cdot \frac{\partial f_2}{\partial v_1} = 0 \quad (4.7)$$

Since the joint probability function $f_2$ can be rewritten as

$$f_2(x_1, x_2, v_1, v_2, t) = f_1(x_1, v_1, t) f_2(x_2, v_2, t) + g(x_1, x_2, v_1, v_2, t) \quad (4.8)$$

where $g(x_1, x_2, v_1, v_2, t)$ is the correlation function. If we ignore two-particle interactions, we can assume that $g(x_1, x_2, v_1, v_2, t)$ vanishes. Then, substituting Eq. (4.8) into Eq. (4.7), it yields

$$\frac{\partial}{\partial t} f_1 + v_1 \cdot \frac{\partial f_1}{\partial x_1} + n_0 \int dx_2 dv_2 a_{12} f_1(x_2, v_2, t) \cdot \frac{\partial f_1(x_1, v_1, t)}{\partial v_1} = 0 \quad (4.9)$$

where $n_0 = N_0 / V = (N_0 - 1) / V$.

Let $a(x_1, v_1, t) \equiv n_0 \int dx_2 dv_2 a_{12} f_1(x_2, v_2, t)$. Eq. (4.9) can be rewritten as

$$\frac{\partial}{\partial t} f_1 + v_1 \cdot \frac{\partial f_1}{\partial x_1} + a \cdot \frac{\partial f_1(x_1, v_1, t)}{\partial v_1} = 0 \quad (4.10)$$

Eq. (4.10) is the Vlasov equation in the six-dimensional space.

**References**


Company, New York.
