Chapter 4. Deriving the Vlasov Equation From the Liouville Equation

Topics or concepts to learn in Chapter 4:

1. Liouville equation of the system probability density function f_{N_0} in $6N_0$-dimensional space.
2. The probability density function f_{N_0} is a point, thus it is incompressible in the in the $6N_0$-dimensional space.
3. How to reduce the probability density function from the $6N_0$-dimensional space to the 6-dimensional space.
4. What assumptions have been made to obtain the Vlasov equation from the Liouville equation.

Suggested Readings:

(1) Chapter 4 in Nicholson (1983)

(2) Chapter 2 in Krall and Trivelpiece (1973)

Exercise 4.1.

Read Chapters 3 and 4 of Nicholson (1983) carefully. Understand the philosophical differences between the Klimontovich equation and the Liouville equation.

4.1. Liouville Equation

Let us consider a system with N_0 particles. Let $N(\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_{N_0},\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_{N_0},t)$ be the density function of the system, i.e.,

$$N(\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_{N_0},\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_{N_0},t)=\prod_{k=1}^{N_0}\delta[\mathbf{x}_k-\mathbf{X}_k(t)]\delta[\mathbf{v}_k-\mathbf{V}_k(t)] \tag{4.1}$$

It can be shown that

$$\frac{\partial}{\partial t}N=\prod_{k=1}^{N_0}\{\sum_{i=1}^{N_0}[-\dot{\mathbf{X}}_i\cdot\frac{\partial}{\partial\mathbf{x}_i}]+\sum_{i=1}^{N_0}[-\dot{\mathbf{V}}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}]\{\delta[\mathbf{x}_k-\mathbf{X}_k(t)]\delta[\mathbf{v}_k-\mathbf{V}_k(t)]\}\}$$

$$=\prod_{k=1}^{N_0}\{\sum_{i=1}^{N_0}[-\mathbf{V}_i\cdot\frac{\partial}{\partial\mathbf{x}_i}]+\sum_{i=1}^{N_0}[-\sum_{j=1}^{N_0}\mathbf{a}_{ji}(\mathbf{X}_i,\mathbf{X}_j,\mathbf{V}_i,\mathbf{V}_j;m_i,m_j,e_i,e_j)\cdot\frac{\partial}{\partial\mathbf{v}_i}]\{\delta[\mathbf{x}_k-\mathbf{X}_k(t)]\delta[\mathbf{v}_k-\mathbf{V}_k(t)]\}\}$$

$$=-\sum_{i=1}^{N_0}[\mathbf{v}_i\cdot\frac{\partial N}{\partial\mathbf{x}_i}]-\sum_{i=1}^{N_0}[\sum_{j=1}^{N_0}\mathbf{a}_{ji}(\mathbf{x}_i,\mathbf{x}_j,\mathbf{v}_i,\mathbf{v}_j;m_i,m_j,e_i,e_j)\cdot\frac{\partial N}{\partial\mathbf{v}_i}]$$

or

$$\frac{\partial}{\partial t}N+\sum_{i=1}^{N_0}[\mathbf{v}_i\cdot\frac{\partial N}{\partial\mathbf{x}_i}]+\sum_{i=1}^{N_0}[\sum_{j=1}^{N_0}\mathbf{a}_{ji}\cdot\frac{\partial N}{\partial\mathbf{v}_i}]=\frac{DN}{Dt}=0 \tag{4.2}$$

where $m_i\mathbf{a}_{ji}$ is the force that the jth particle acting on the ith particle. Thus, we have $\mathbf{a}_{jj}=0$. Eq. (4.2) is the Liouville equation of the system density function N.

Let us now consider an ensemble of such N_0-particle systems. From these systems, we can obtain a probability density function $f_{N_0}(\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_{N_0},\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_{N_0},t)$ of a N_0-particle system. The joint probability density function f_{N_0} of the system satisfies

$$\frac{Df_{N_0}}{Dt}=\frac{\partial}{\partial t}f_{N_0}+\sum_{i=1}^{N_0}[\mathbf{v}_i\cdot\frac{\partial f_{N_0}}{\partial\mathbf{x}_i}]+\sum_{i=1}^{N_0}[\sum_{j=1}^{N_0}\mathbf{a}_{ji}\cdot\frac{\partial f_{N_0}}{\partial\mathbf{v}_i}]=0 \tag{4.3}$$

Eq. (4.3) is the Liouville equation of the system probability density function f_{N_0}. The probability density function f_{N_0} is incompressible in the in $6N_0$-dimensional space.

4.2. BBGKY Hierarchy

We define the reduced probability distributions f_k

$$f_k(\mathbf{x}_1,\cdots,\mathbf{x}_k,\mathbf{v}_1,\cdots,\mathbf{v}_k,t)\equiv V^k\int d\mathbf{x}_{k+1}\,d\mathbf{v}_{k+1}\cdots d\mathbf{x}_{N_0}\,d\mathbf{v}_{N_0}f_{N_0}(\mathbf{x}_1,\cdots,\mathbf{x}_{N_0},\mathbf{v}_1,\cdots,\mathbf{v}_{N_0},t) \tag{4.4}$$

Integration Eq. (4.3) over $\mathbf{x}_{N_0}$ and $\mathbf{v}_{N_0}$ space and making use of the definition in Eq. (4.4), it yields an equation of reduced probability distributions f_{N_0-1}, that is

$$\frac{\partial f_{N_0-1}}{\partial t}+\sum_{i=1}^{N_0-1}[\mathbf{v}_i\cdot\frac{\partial f_{N_0-1}}{\partial\mathbf{x}_i}]+\sum_{i=1}^{N_0-1}[\sum_{j=1}^{N_0-1}\mathbf{a}_{ji}\cdot\frac{\partial f_{N_0-1}}{\partial\mathbf{v}_i}]+V^{N_0-1}\sum_{i=1}^{N_0-1}[\int d\mathbf{x}_{N_0-1}d\mathbf{v}_{N_0-1}\mathbf{a}_{iN_0}\cdot\frac{\partial f_{N_0}}{\partial\mathbf{v}_i}]=0 \tag{4.5}$$

Thus, the desired equation for f_{N_0-1} does not only depend on f_{N_0-1} itself, but also depend on f_{N_0} as shown in the last term in Eq. (4.5).

Likewise, we can obtain the equation for the probability density function f_k

$$\frac{\partial}{\partial t} f_k + \sum_{i=1}^{k}[\mathbf{v}_i \cdot \frac{\partial f_k}{\partial \mathbf{x}_i}] + \sum_{i=1}^{k}[\sum_{j=1}^{k}\mathbf{a}_{ji} \cdot \frac{\partial f_k}{\partial \mathbf{v}_i}] + \frac{N_0 - k}{V}\sum_{i=1}^{k}[\int d\mathbf{x}_{k+1} d\mathbf{v}_{k+1}\mathbf{a}_{i,k+1} \cdot \frac{\partial f_{k+1}}{\partial \mathbf{v}_i}] = 0 \qquad (4.6)$$

Again, the desired equation for f_k does not only depend on f_k itself, but also depend on f_{k+1}. This is the BBGKY hierarchy (Bogoliubov, 1946; Born and Green 1949; Kirkwood 1946; 1947; and Yvon 1935)

For $k = 1$, Eq. (4.6) yields,

$$\frac{\partial}{\partial t} f_1 + \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{x}_1} + \frac{N_0 - 1}{V}\int d\mathbf{x}_2 d\mathbf{v}_2 \mathbf{a}_{12} \cdot \frac{\partial f_2}{\partial \mathbf{v}_1} = 0 \qquad (4.7)$$

Since the joint probability function f_2 can be rewritten as

$$f_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2, t) = f_1(\mathbf{x}_1, \mathbf{v}_1, t) f_1(\mathbf{x}_2, \mathbf{v}_2, t) + g(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2, t) \qquad (4.8)$$

where $g(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2, t)$ is the correlation function. If we ignore two-particle interactions, we can assume that $g(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2, t)$ vanishes. Then, substituting Eq. (4.8) into Eq. (4.7), it yields

$$\frac{\partial}{\partial t} f_1 + \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{x}_1} + n_0 \int d\mathbf{x}_2 d\mathbf{v}_2 \mathbf{a}_{12} f_1(\mathbf{x}_2, \mathbf{v}_2, t) \cdot \frac{\partial f_1(\mathbf{x}_1, \mathbf{v}_1, t)}{\partial \mathbf{v}_1} = 0 \qquad (4.9)$$

where $n_0 = N_0 / V \approx (N_0 - 1)/V$.

Let $\mathbf{a}(\mathbf{x}_1, \mathbf{v}_1, t) \equiv n_0 \int d\mathbf{x}_2\, d\mathbf{v}_2 \mathbf{a}_{12} f_1(\mathbf{x}_2, \mathbf{v}_2, t)$. Eq. (4.9) can be rewritten as

$$\frac{\partial}{\partial t} f_1 + \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{x}_1} + \mathbf{a} \cdot \frac{\partial f_1(\mathbf{x}_1, \mathbf{v}_1, t)}{\partial \mathbf{v}_1} = 0 \qquad (4.10)$$

Eq. (4.10) is the Vlasov equation in the six-dimensional space.

References

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