Chapter 3. Deriving the Fluid Equations From the Vlasov Equation

Topics or concepts to learn in Chapter 3:

1. The basic equations for study kinetic plasma physics: The Vlasov-Maxwell equations
2. Definition of fluid variables: number density, mass density, average velocity, thermal pressure (including scalar pressure and pressure tensor), heat flux, and entropy function
3. Derivation of plasma fluid equations from Vlasov-Maxwell equations:

(a) The ion-electron two-fluid equations

(b) The one-fluid equations and the MHD (magnetohydrodynamic) equations

(c) The continuity equations of the number density, the mass density, and the charge density

(d) The momentum equation

(e) The momentum of the plasma and the E-, B- fields

(f) The momentum flux (the pressure tensor) of the plasma and the E-, B- fields

(g) The energy of the plasma and the E-, B- fields

(h) The energy flux of the plasma and the E-, B- fields

(i) The energy equations and "the equations of state"

(j) The MHD Ohm's law and the generalized Ohm's law

Suggested Readings:

(1) Section 7.1 in Nicholson (1983)

(2) Chapter 3 in Krall and Trivelpiece (1973)

(3) Chapter 3 in F. F. Chen (1984)

3.1. The Vlasov-Maxwell System

Vlasov equation of the αth species, shown in Eq. (2.7), can be rewritten as

$$\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha(\mathbf{x},\mathbf{v},t) + \frac{e_\alpha}{m_\alpha}[\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)]\cdot\nabla_{\mathrm{v}} f_\alpha(\mathbf{x},\mathbf{v},t) = 0 \tag{3.1}$$

or

$$\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} + \nabla\cdot\{\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t)\} + \frac{e_\alpha}{m_\alpha}\nabla_{\mathrm{v}}\cdot\{[\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t)\} = 0 \tag{3.1'}$$

where $\nabla = \partial/\partial\mathbf{x}$ and $\nabla_{\mathrm{v}} = \partial/\partial\mathbf{v}$.

Maxwell's equations in Table 1.1. can be rewritten as

$$\boxed{\begin{aligned}\nabla\cdot\mathbf{E}(\mathbf{x},t) &= \frac{\rho_c(\mathbf{x},t)}{\varepsilon_0} \\ &= \frac{1}{\varepsilon_0}\sum_\alpha e_\alpha n_\alpha(\mathbf{x},t) \\ &= \frac{1}{\varepsilon_0}\sum_\alpha e_\alpha \iiint f_\alpha(\mathbf{x},\mathbf{v},t)d^3v\end{aligned}} \tag{3.2}$$

$$\nabla\cdot\mathbf{B}(\mathbf{x},t) = 0 \tag{3.3}$$

$$\nabla\times\mathbf{E}(\mathbf{x},t) = -\frac{\partial\mathbf{B}(\mathbf{x},t)}{\partial t} \tag{3.4}$$

$$\boxed{\begin{aligned}\nabla\times\mathbf{B}(\mathbf{x},t) &= \mu_0\mathbf{J}(\mathbf{x},t) + \mu_0\varepsilon_0\frac{\partial\mathbf{E}(\mathbf{x},t)}{\partial t} \\ &= \mu_0[\sum_\alpha e_\alpha n_\alpha(\mathbf{x},t)\mathbf{V}_\alpha(\mathbf{x},t)] + \mu_0\varepsilon_0\frac{\partial\mathbf{E}(\mathbf{x},t)}{\partial t} \\ &= \mu_0[\sum_\alpha e_\alpha \iiint \mathbf{v}f_\alpha(\mathbf{x},\mathbf{v},t)d^3v] + \mu_0\varepsilon_0\frac{\partial\mathbf{E}(\mathbf{x},t)}{\partial t}\end{aligned}} \tag{3.5}$$

The Vlasov equations of ions and electrons and the Maxwell's equations are the governing equations of the Vlasov-Maxwell system, which includes eight unknowns ($f_i, f_e, \mathbf{E}, \mathbf{B}$) and eight independent equations in a six-dimensional phase space.

3.2. The Fluid Variables

Before introducing the fluid equations, we need to define fluid variables of plasma. The number density of the αth species, in Eq. (3.2), is defined by

$$n_\alpha(\mathbf{x},t) \equiv \iiint f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.6}$$

The average velocity of the αth species, in Eq. (3.5), is defined by

$$\mathbf{V}_\alpha(\mathbf{x},t) \equiv \frac{\iiint \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v}{n_\alpha(\mathbf{x},t)} \tag{3.7}$$

The particle flux of the αth species is

$$n_\alpha \mathbf{V}_\alpha = \iiint \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.8}$$

The mass flux of the αth species is

$$m_\alpha n_\alpha \mathbf{V}_\alpha = \iiint m_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.9}$$

The charge flux of the αth species is

$$e_\alpha n_\alpha \mathbf{V}_\alpha = \iiint e_\alpha \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.10}$$

The momentum flux, or the kinetic pressure, of the αth species is

$$m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + \mathsf{P}_\alpha(\mathbf{x},t) = \iiint m_\alpha \mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.11}$$

where $m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha$ is the dynamic pressure, and $\mathsf{P}_\alpha(\mathbf{x},t)$ is the thermal pressure tensor. The thermal pressure tensor $\mathsf{P}_\alpha(\mathbf{x},t)$ in Eq. (3.11) is defined by

$$\mathsf{P}_\alpha(\mathbf{x},t) \equiv \iiint m_\alpha [\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)][\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.12}$$

Since $\mathsf{P}_\alpha(\mathbf{x},t)$ is a second rank symmetric tensor, trace of $\mathsf{P}_\alpha(\mathbf{x},t)$ is invariant after an orthonormal coordinate transformation. For an isotropic pressure, we have

$$\mathsf{P}_\alpha(\mathbf{x},t) = \mathbf{1}\, p_\alpha(\mathbf{x},t) = \begin{pmatrix} p_\alpha(\mathbf{x},t) & 0 & 0 \\ 0 & p_\alpha(\mathbf{x},t) & 0 \\ 0 & 0 & p_\alpha(\mathbf{x},t) \end{pmatrix}$$

Thus, in general we can define a scalar thermal pressure $p_\alpha(\mathbf{x},t)$

$$p_\alpha(\mathbf{x},t) \equiv \frac{1}{3} \mathrm{trace}[\mathsf{P}_\alpha(\mathbf{x},t)] \tag{3.13}$$

The flux of the total kinetic pressure of the αth species is

$$m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + (\mathsf{P}_\alpha \mathbf{V}_\alpha)^S + \mathsf{Q}_\alpha(\mathbf{x},t) = \iiint m_\alpha \mathbf{v}\mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.14}$$

where the heat-flux tensor $\mathsf{Q}_\alpha(\mathbf{x},t)$ is a third rank tensor, which is defined by

$$\mathsf{Q}_\alpha(\mathbf{x},t) \equiv \iiint m_\alpha [\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)][\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)][\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.15}$$

$(\mathsf{P}_\alpha \mathbf{V}_\alpha)^S$ in Eq. (3.14) is a symmetric third rank tensor, which is defined by

$$(\mathsf{P}_\alpha \mathbf{V}_\alpha)^S \equiv \mathsf{P}_\alpha \mathbf{V}_\alpha + \mathbf{V}_\alpha \mathsf{P}_\alpha + \iiint m_\alpha (\mathbf{v} - \mathbf{V}_\alpha)\mathbf{V}_\alpha(\mathbf{v} - \mathbf{V}_\alpha) f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.16}$$

The kinetic energy flux of the αth species is

$$\frac{1}{2} m_\alpha n_\alpha \mathbf{V}_\alpha \cdot \mathbf{V}_\alpha \mathbf{V}_\alpha + \frac{3}{2} p_\alpha \mathbf{V}_\alpha + \mathsf{P}_\alpha \cdot \mathbf{V}_\alpha + \mathbf{q}_\alpha(\mathbf{x},t) = \iiint \frac{1}{2} m_\alpha \mathbf{v} \cdot \mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v$$

or

$$(\frac{1}{2} m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2} p_\alpha)\mathbf{V}_\alpha + \mathsf{P}_\alpha \cdot \mathbf{V}_\alpha + \mathbf{q}_\alpha(\mathbf{x},t) = \iiint \frac{1}{2} m_\alpha v^2 \mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.17}$$

where the heat-flux vector $\mathbf{q}_\alpha(\mathbf{x},t)$ is defined by

$$\mathbf{q}_\alpha(\mathbf{x},t) \equiv \iiint \frac{1}{2} m_\alpha [\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)] \cdot [\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)][\mathbf{v} - \mathbf{V}_\alpha(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.18}$$

For advanced study:

If it is needed, we can define a fourth rank tensor $\mathsf{R}_\alpha(\mathbf{x},t)$

$$\mathsf{R}_\alpha(\mathbf{x},t) \equiv \iiint m_\alpha (\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha) f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.19}$$

and a fifth rank tensor $\mathsf{S}_\alpha(\mathbf{x},t)$

$$\mathsf{S}_\alpha(\mathbf{x},t) \equiv \iiint m_\alpha (\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha) f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.20}$$

so that

$$m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + (\mathsf{P}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha)^S + (\mathsf{Q}_\alpha \mathbf{V}_\alpha)^S + \mathsf{R}_\alpha = \iiint m_\alpha \mathbf{v}\mathbf{v}\mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \tag{3.21}$$

and

$$\begin{aligned} &m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + (\mathsf{P}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha)^S + (\mathsf{Q}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha)^S + (\mathsf{R}_\alpha \mathbf{V}_\alpha)^S + \mathsf{S}_\alpha \\ &= \iiint m_\alpha \mathbf{v}\mathbf{v}\mathbf{v}\mathbf{v}\mathbf{v} f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \end{aligned} \tag{3.22}$$

where the superscript s denotes a higher-rank symmetric tensor. An example of the 2nd-rank symmetric tensor is shown in (3.16).

Exercise 3.0.1

Let us define a generalized local entropy function S_α of the αth species:

$$S_\alpha = -\int \frac{f_\alpha}{n_\alpha} \ln f_\alpha d^3v + \text{constant}$$

Let f_α be a normal distribution function with number density n_α, temperature T_α, and zero average velocity. Determine how the entropy S_α varies with varying of the number density n_α and the thermal pressure p_α.

In addition to the general definitions of the fluid variables, we shall encounter the following integrations in deriving the fluid equations in the next section.

$$\boxed{\iiint \frac{\partial}{\partial \mathbf{v}} \cdot \{[\mathbf{E}(\mathbf{x},t) + \mathbf{v} \times \mathbf{B}(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t)\} d^3v = 0} \tag{3.23}$$

$$\boxed{\iiint \mathbf{v} \frac{\partial}{\partial \mathbf{v}} \cdot \{[\mathbf{E}(\mathbf{x},t) + \mathbf{v} \times \mathbf{B}(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t)\} d^3v = -n_\alpha(\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B})} \tag{3.24}$$

$$\boxed{\begin{aligned} &\iiint \mathbf{vv} \frac{\partial}{\partial \mathbf{v}} \cdot \{[\mathbf{E}(\mathbf{x},t) + \mathbf{v} \times \mathbf{B}(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t)\} d^3v \\ &= -n_\alpha(\mathbf{E}\mathbf{V}_\alpha)^S - n_\alpha[\mathbf{V}_\alpha(\mathbf{V}_\alpha \times \mathbf{B})]^S - \iiint \{(\mathbf{v} - \mathbf{V}_\alpha)[(\mathbf{v} - \mathbf{V}_\alpha) \times \mathbf{B}]\}^S f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \end{aligned}} \tag{3.25}$$

The following two integrations are useful for the higher moment integrations of the Vlasov equation.

$$\boxed{\begin{aligned} &\iiint \mathbf{vvv} \frac{\partial}{\partial \mathbf{v}} \cdot \{[\mathbf{E}(\mathbf{x},t) + \mathbf{v} \times \mathbf{B}(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t)\} d^3v \\ &= -n_\alpha(\mathbf{E}\mathbf{V}_\alpha\mathbf{V}_\alpha)^S - \frac{n_\alpha}{m_\alpha}(\mathbf{E}\mathsf{P}_\alpha)^S - n_\alpha[\mathbf{V}_\alpha\mathbf{V}_\alpha(\mathbf{V}_\alpha \times \mathbf{B})]^S - \frac{n_\alpha}{m_\alpha}[\mathsf{P}_\alpha(\mathbf{V}_\alpha \times \mathbf{B})]^S \\ &\quad - \iiint \{(\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)[(\mathbf{v} - \mathbf{V}_\alpha) \times \mathbf{B}]\}^S f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \end{aligned}} \tag{3.26}$$

$$\boxed{\begin{aligned} &\iiint \mathbf{vvvv} \frac{\partial}{\partial \mathbf{v}} \cdot \{[\mathbf{E}(\mathbf{x},t) + \mathbf{v} \times \mathbf{B}(\mathbf{x},t)] f_\alpha(\mathbf{x},\mathbf{v},t)\} d^3v \\ &= -n_\alpha(\mathbf{E}\mathbf{V}_\alpha\mathbf{V}_\alpha\mathbf{V}_\alpha)^S - \frac{n_\alpha}{m_\alpha}(\mathbf{E}\mathsf{P}_\alpha\mathbf{V}_\alpha)^S - \frac{n_\alpha}{m_\alpha}(\mathbf{E}\mathsf{Q}_\alpha)^S \\ &\quad - n_\alpha[\mathbf{V}_\alpha\mathbf{V}_\alpha\mathbf{V}_\alpha(\mathbf{V}_\alpha \times \mathbf{B})]^S - \frac{n_\alpha}{m_\alpha}[\mathsf{P}_\alpha\mathbf{V}_\alpha(\mathbf{V}_\alpha \times \mathbf{B})]^S - \frac{n_\alpha}{m_\alpha}[\mathsf{Q}_\alpha(\mathbf{V}_\alpha \times \mathbf{B})]^S \\ &\quad - \iiint \{(\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)[(\mathbf{v} - \mathbf{V}_\alpha) \times \mathbf{B}]\}^S f_\alpha(\mathbf{x},\mathbf{v},t) d^3v \end{aligned}} \tag{3.27}$$

and so forth.

Exercise 3.1.

Verify Eqs. (3.23), (3.24), (3.25), and (3.26).

Hint: There are three terms in the integration of Eq. (3.23), they are all in the following form:

$$\iint [\int \frac{\partial}{\partial v_x} E_x f_\alpha(\mathbf{x},\mathbf{v},t) dv_x] d^2v = \iint E_x [f_\alpha(\mathbf{x},\mathbf{v},t)]_{v_x=-\infty}^{v_x=+\infty} d^2v = 0$$

There are nine terms in the integration of Eq. (3.24), in which six of them are in the following form:

$$\iint [v_y \int \frac{\partial}{\partial v_x} E_x f_\alpha(\mathbf{x},\mathbf{v},t) dv_x] d^2v = \iint v_y E_x [f_\alpha(\mathbf{x},\mathbf{v},t)]_{v_x=-\infty}^{v_x=+\infty} d^2v = 0$$

and the rest three of them are in the following form:

$$\begin{aligned} &\iint [\int v_x \frac{\partial}{\partial v_x} E_x f_\alpha(\mathbf{x},\mathbf{v},t) dv_x] d^2v \\ &= \iint E_x \{[v_x f_\alpha(\mathbf{x},\mathbf{v},t)]_{v_x=-\infty}^{v_x=+\infty} - \int f_\alpha(\mathbf{x},\mathbf{v},t) dv_x\} d^2v = 0 - E_x n_\alpha \end{aligned}$$

There are 27 terms in the integration of Eq. (3.25), in which six of them are in the following form:

$$\iint [v_y v_z \int \frac{\partial}{\partial v_x} E_x f_\alpha(\mathbf{x},\mathbf{v},t) dv_x] d^2v = \iint v_y v_z E_x [f_\alpha(\mathbf{x},\mathbf{v},t)]_{v_x=-\infty}^{v_x=+\infty} d^2v = 0$$

another six of them are in the following type:

$$\iint [v_y v_y \int \frac{\partial}{\partial v_x} E_x f_\alpha(\mathbf{x},\mathbf{v},t) dv_x] d^2v = \iint v_y v_y E_x [f_\alpha(\mathbf{x},\mathbf{v},t)]_{v_x=-\infty}^{v_x=+\infty} d^2v = 0$$

another twelve of them are in the following form:

$$\begin{aligned} &\iint [\int v_y v_x \frac{\partial}{\partial v_x} E_x f_\alpha(\mathbf{x},\mathbf{v},t) dv_x] d^2v \\ &= \iint v_y E_x \{[v_x f_\alpha(\mathbf{x},\mathbf{v},t)]_{v_x=-\infty}^{v_x=+\infty} - \int f_\alpha(\mathbf{x},\mathbf{v},t) dv_x\} d^2v = 0 - V_{\alpha y} E_x n_\alpha \end{aligned}$$

and the rest three of them are in the following form:

$$\begin{aligned} &\iint [\int v_x v_x \frac{\partial}{\partial v_x} E_x f_\alpha(\mathbf{x},\mathbf{v},t) dv_x] d^2v \\ &= \iint E_x \{[v_x v_x f_\alpha(\mathbf{x},\mathbf{v},t)]_{v_x=-\infty}^{v_x=+\infty} - \int 2v_x f_\alpha(\mathbf{x},\mathbf{v},t) dv_x\} d^2v \\ &= 0 - 2E_x V_{\alpha x} n_\alpha = -(E_x V_{\alpha x} + V_{\alpha x} E_x) n_\alpha \end{aligned}$$

3.3. The Fluid Equations

Fluid equations can be obtained from integration of the Vlasov equation in the velocity space. For instance (e.g., Rossi and Olbert, 1970; Chao, 1970):

We can obtain the continuity equation of the αth species from $\iiint (3.1)_\alpha d^3v$.

We can obtain the momentum equation of the αth species from $\iiint m_\alpha \mathbf{v}(3.1)_\alpha d^3v$.

We can obtain the pressure equation of the αth species from $\iiint m_\alpha \mathbf{vv}(3.1)_\alpha d^3v$.

We can obtain the energy equation of the αth species from $\iiint \frac{1}{2} m_\alpha v^2 (3.1)_\alpha d^3v$.

If we consider entire plasma system as a single fluid medium, the following integrations are useful in obtaining one-fluid plasma equations.

$\sum_\alpha \iiint m_\alpha (3.1)_\alpha d^3v$ yields the one-fluid mass continuity equation.

$\sum_\alpha \iiint m_\alpha \mathbf{v}(3.1)_\alpha d^3v$ yields the one-fluid momentum equation.

$\sum_\alpha \iiint m_\alpha \mathbf{vv}(3.1)_\alpha d^3v$ yields the one-fluid pressure equation.

$\sum_\alpha \iiint \frac{1}{2} m_\alpha v^2 (3.1)_\alpha d^3v$ yields the one-fluid energy equation.

$\sum_\alpha \iiint e_\alpha (3.1)_\alpha d^3v$ yields the one-fluid charge continuity equation.

$\sum_\alpha \iiint e_\alpha \mathbf{v}(3.1)_\alpha d^3v$ yields the one-fluid Ohm's law.

3.3.1. The Fluid Equations of the αth Species

The continuity equation of the αth species, obtained from $\iiint (3.1')d^3v$, is

$$\boxed{\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{V}_\alpha) = 0} \tag{3.28}$$

The momentum equation of the αth species, obtained from $\iiint m_\alpha \mathbf{v}(3.1')d^3v$, is

$$\boxed{\frac{\partial}{\partial t}(m_\alpha n_\alpha \mathbf{V}_\alpha) + \nabla \cdot (m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + \mathbf{P}_\alpha) - e_\alpha n_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}) = 0} \tag{3.29}$$

The pressure equation of the αth species, obtained from $\iiint m_\alpha \mathbf{vv}(3.1')d^3v$, is

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}(m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + \mathsf{P}_\alpha) + \nabla\cdot[m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + (\mathsf{P}_\alpha \mathbf{V}_\alpha)^S + \mathsf{Q}_\alpha]\\ &-e_\alpha n_\alpha(\mathbf{E}\mathbf{V}_\alpha + \mathbf{V}_\alpha \mathbf{E}) - e_\alpha n_\alpha[\mathbf{V}_\alpha(\mathbf{V}_\alpha \times \mathbf{B}) + (\mathbf{V}_\alpha \times \mathbf{B})\mathbf{V}_\alpha]\\ &-e_\alpha \iiint \{(\mathbf{v}-\mathbf{V}_\alpha)[(\mathbf{v}-\mathbf{V}_\alpha)\times\mathbf{B}] + [(\mathbf{v}-\mathbf{V}_\alpha)\times\mathbf{B}](\mathbf{v}-\mathbf{V}_\alpha)\} f_\alpha(\mathbf{x},\mathbf{v},t)\} d^3v = 0\end{aligned}} \tag{3.30}$$

The energy equation of the αth species, obtained from $\iiint \frac{1}{2} m_\alpha v^2 (3.1')d^3v$, is

$$\boxed{\frac{\partial}{\partial t}(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha) + \nabla\cdot[(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha)\mathbf{V}_\alpha + \mathsf{P}_\alpha\cdot\mathbf{V}_\alpha + \mathbf{q}_\alpha] - e_\alpha n_\alpha \mathbf{E}\cdot\mathbf{V}_\alpha = 0} \tag{3.31}$$

As we can see that the continuity equation is a scalar equation, the momentum equation is a vector equation, the pressure equation is a second-rank-tensor equation, and the energy equation is a scalar equation. Likewise, we can obtain a third-rank-tensor heat-flux equation from $\iiint m_\alpha \mathbf{vvv}(3.1')d^3v$, and a fourth-rank-tensor equation from $\iiint m_\alpha \mathbf{vvvv}(3.1')d^3v$, and so forth.

If the αth species has reached to a thermal-dynamic-equilibrium state, then we have $\mathsf{P}_\alpha = \mathbf{1}\, p_\alpha$, and all the higher order rank tensors are vanished, i.e., $\mathsf{Q}_\alpha = 0$, $\mathsf{R}_\alpha = 0$, $\mathsf{S}_\alpha = 0$, … and so forth. In this case, the six-dimensional Vlasov equation can be replaced by the three-dimensional equations (3.28), (3.29), and (3.31). Otherwise, we have to use infinite number of three-dimensional equations to replace the six-dimensional Vlasov equation. Thus, the fluid variables are meaningful and fluid equations are useful only when the α th species is in a thermal-dynamic-equilibrium state and can remain in a quasi-thermal-dynamic-equilibrium state after they interaction with the waves.

3.3.2. The Two-Fluid Equations in the Convective-Time-Derivative Form

In fluid mechanics, it is commonly use dA/dt to denote the time derivatives of $A(\mathbf{x},t)$ along the trajectory of a fluid element in a velocity field $\mathbf{V}(\mathbf{x},t)$. Namely, $dA/dt \equiv (\partial/\partial t + \mathbf{V}\cdot\nabla)A$, where $\mathbf{V}\cdot\nabla A$ is the convective time derivative of $A(\mathbf{x},t)$. Equations obtained in the last section 3.3.1 can be rewritten in a convective-time-derivative form. The convective-time-derivative term is a second-order small term in the linear wave analysis of waves in a uniform background plasma. Thus, equations obtained in this section are particularly useful in linear wave analysis.

The continuity equation (3.28) can be rewritten as

$$(\frac{\partial}{\partial t} + \mathbf{V}_\alpha \cdot \nabla) n_\alpha = -n_\alpha \nabla \cdot \mathbf{V}_\alpha \tag{3.32}$$

The momentum equation (3.29) can be rewritten as

$$m_\alpha n_\alpha (\frac{\partial}{\partial t} + \mathbf{V}_\alpha \cdot \nabla) \mathbf{V}_\alpha = -\nabla \cdot \mathsf{P}_\alpha + e_\alpha n_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}) \tag{3.33}$$

The energy equation (3.31) can be rewritten as

$$\frac{3}{2}[(\frac{\partial}{\partial t} + \mathbf{V}_\alpha \cdot \nabla) p_\alpha] + \frac{3}{2} p_\alpha (\nabla \cdot \mathbf{V}_\alpha) + (\mathsf{P}_\alpha \cdot \nabla) \cdot \mathbf{V}_\alpha + \nabla \cdot \mathbf{q}_\alpha = 0 \tag{3.34}$$

For $\nabla \cdot \mathbf{q}_\alpha = 0$ and isotropic pressure $\mathsf{P}_\alpha = \mathbf{1} p_\alpha$, the energy equation (3.34) is reduced to the the well-known adiabatic equation of state

$$\frac{3}{2}[\frac{d}{dt} \ln(p_\alpha n_\alpha^{-5/3})] = \frac{3}{2}[(\frac{\partial}{\partial t} + \mathbf{V}_\alpha \cdot \nabla) \ln(p_\alpha n_\alpha^{-5/3})] = 0 \tag{3.35}$$

Exercise 3.2.

Verify Eqs. (3.33) and (3.34).

Answer to Exercise 3.2.

Proof of Eq. (3.33):

Substituting Eq. (3.28) into Eq. (3.29) yields

$$\frac{\partial}{\partial t}(m_\alpha n_\alpha \mathbf{V}_\alpha) + \nabla \cdot (m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + \mathsf{P}_\alpha) - e_\alpha n_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B})$$

$$= m_\alpha \mathbf{V}_\alpha [\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{V}_\alpha)] + m_\alpha n_\alpha (\frac{\partial}{\partial t} + \mathbf{V}_\alpha \cdot \nabla) \mathbf{V}_\alpha + \nabla \cdot \mathsf{P}_\alpha - e_\alpha n_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B})$$

$$= 0 + m_\alpha n_\alpha (\frac{\partial}{\partial t} + \mathbf{V}_\alpha \cdot \nabla) \mathbf{V}_\alpha + \nabla \cdot \mathsf{P}_\alpha - e_\alpha n_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}) = 0$$

Proof of Eq. (3.34):

Substituting Eqs. (3.28) and (3.33) into Eq. (3.31), it yields

$$\frac{\partial}{\partial t}(\frac{1}{2} m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2} p_\alpha) + \nabla \cdot [(\frac{1}{2} m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2} p_\alpha) \mathbf{V}_\alpha + \mathsf{P}_\alpha \cdot \mathbf{V}_\alpha + \mathbf{q}_\alpha] - e_\alpha n_\alpha \mathbf{E} \cdot \mathbf{V}_\alpha$$

$$= \frac{1}{2} m_\alpha V_\alpha^2 [\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{V}_\alpha)] + \mathbf{V}_\alpha \cdot [m_\alpha n_\alpha (\frac{\partial}{\partial t} + \mathbf{V}_\alpha \cdot \nabla) \mathbf{V}_\alpha + \nabla \cdot \mathsf{P}_\alpha - e_\alpha n_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B})]$$

$$+ \frac{3}{2}[(\frac{\partial}{\partial t} + \mathbf{V}_\alpha \cdot \nabla) p_\alpha] + \frac{3}{2} p_\alpha (\nabla \cdot \mathbf{V}_\alpha) + (\mathsf{P}_\alpha \cdot \nabla) \cdot \mathbf{V}_\alpha + \nabla \cdot \mathbf{q}_\alpha$$

$$= 0 + 0 + \frac{3}{2}[(\frac{\partial}{\partial t} + \mathbf{V}_\alpha \cdot \nabla) p_\alpha] + \frac{3}{2} p_\alpha (\nabla \cdot \mathbf{V}_\alpha) + (\mathsf{P}_\alpha \cdot \nabla) \cdot \mathbf{V}_\alpha + \nabla \cdot \mathbf{q}_\alpha = 0$$

In summary, Table 3.1 lists governing equations of ion-electron two-fluid plasma, in which both species have reached to a thermal-dynamic-equilibrium state so that $\mathbf{P}_\alpha = \mathbf{1}\, p_\alpha$ and $\mathbf{q}_\alpha = 0$. There are 16 unknowns (n_α, $\mathbf{V}_\alpha$, p_α, $\mathbf{E}$, $\mathbf{B}$) and 16 independent equations in this system.

Table 3.1. The governing equations of the ion-electron two-fluid plasma with an isotropic pressure and zero heat flux

SI Units	Gaussian Units
The electrons' equations: $(\frac{\partial}{\partial t} + \mathbf{V}_e \cdot \nabla) n_e = -n_e \nabla \cdot \mathbf{V}_e$ $m_e n_e (\frac{\partial}{\partial t} + \mathbf{V}_e \cdot \nabla)\mathbf{V}_e = -\nabla p_e - e n_e (\mathbf{E} + \mathbf{V}_e \times \mathbf{B})$ $(\frac{\partial}{\partial t} + \mathbf{V}_e \cdot \nabla)\ln(p_e n_e^{-5/3}) = 0$	The electrons' equations: $(\frac{\partial}{\partial t} + \mathbf{V}_e \cdot \nabla) n_e = -n_e \nabla \cdot \mathbf{V}_e$ $m_e n_e (\frac{\partial}{\partial t} + \mathbf{V}_e \cdot \nabla)\mathbf{V}_e = -\nabla p_e - e n_e (\mathbf{E} + \frac{\mathbf{V}_e \times \mathbf{B}}{c})$ $(\frac{\partial}{\partial t} + \mathbf{V}_e \cdot \nabla)\ln(p_e n_e^{-5/3}) = 0$
The ions' equations: $(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla) n_i = -n_i \nabla \cdot \mathbf{V}_i$ $m_i n_i (\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla)\mathbf{V}_i = -\nabla p_i + e n_i (\mathbf{E} + \mathbf{V}_i \times \mathbf{B})$ $(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla)\ln(p_i n_i^{-5/3}) = 0$	The ions' equations: $(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla) n_i = -n_i \nabla \cdot \mathbf{V}_i$ $m_i n_i (\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla)\mathbf{V}_i = -\nabla p_i + e n_i (\mathbf{E} + \frac{\mathbf{V}_i \times \mathbf{B}}{c})$ $(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla)\ln(p_i n_i^{-5/3}) = 0$
The Maxwell's equations: $\nabla \cdot \mathbf{E} = e(n_i - n_e)/\varepsilon_0$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ $\nabla \times \mathbf{B} = \mu_0 e(n_i \mathbf{V}_i - n_e \mathbf{V}_e) + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$	The Maxwell's equations: $\nabla \cdot \mathbf{E} = 4\pi e(n_i - n_e)$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ $\nabla \times \mathbf{B} = \frac{4\pi}{c} e(n_i \mathbf{V}_i - n_e \mathbf{V}_e) + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$

3.3.3. The One-Fluid Equations in the Conservative Form for Studying Nonlinear Wave in the MHD Plasma

One-fluid equations in the conservative form (summarized in Table 3.2) will be obtained in this section. Equations in conservative form are good for nonlinear wave analysis in the magnetohydrodynamic (MHD) plasma. Before introducing the one-fluid plasma equations, we shall first define one-fluid plasma variables.

3.3.3.1 The One-Fluid Variables

The mass density of the one-fluid plasma is defined by

$$\rho \equiv \sum_{\alpha} (m_\alpha n_\alpha) \tag{3.36}$$

The charge density of the one-fluid plasma is defined by

$$\rho_c \equiv \sum_{\alpha} (e_\alpha n_\alpha) \tag{3.37}$$

The average (bulk) velocity of the one-fluid plasma is defined by

$$\mathbf{V} \equiv \frac{1}{\rho} \sum_{\alpha} (m_\alpha n_\alpha \mathbf{V}_\alpha) \tag{3.38}$$

The momentum of the one-fluid plasma is defined by

$$\rho \mathbf{V} \equiv \sum_{\alpha} (m_\alpha n_\alpha \mathbf{V}_\alpha) \tag{3.39}$$

The electric current density of the one-fluid plasma is defined by

$$\mathbf{J} \equiv \sum_{\alpha} (e_\alpha n_\alpha \mathbf{V}_\alpha) \tag{3.40}$$

The total kinetic pressure of the one-fluid plasma is defined by

$$\rho \mathbf{V}\mathbf{V} + \mathsf{P} \equiv \sum_{\alpha} (m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + \mathsf{P}_\alpha) \tag{3.41}$$

where the thermal pressure tensor of the one-fluid plasma is defined by

$$\mathsf{P} \equiv \sum_{\alpha} (m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + \mathsf{P}_\alpha) - \rho \mathbf{V}\mathbf{V} \tag{3.42}$$

or

$$\mathsf{P}(\mathbf{x},t) \equiv \sum_{\alpha} [\iiint m_\alpha (\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V}) f_\alpha(\mathbf{x},\mathbf{v},t) d^3v] \tag{3.42'}$$

The total kinetic pressure flux of the one-fluid plasma is defined by

$$\rho\mathbf{VVV} + (\mathbf{P}\cdot\mathbf{V})^S + \mathbf{Q} \equiv \sum_\alpha [(m_\alpha n_\alpha \mathbf{V}_\alpha\mathbf{V}_\alpha\mathbf{V}_\alpha + (\mathbf{P}_\alpha\cdot\mathbf{V}_\alpha)^S + \mathbf{Q}_\alpha] \tag{3.43}$$

where the heat-flux tensor of the one-fluid plasma is defined by

$$\mathbf{Q} \equiv \sum_\alpha [(m_\alpha n_\alpha \mathbf{V}_\alpha\mathbf{V}_\alpha\mathbf{V}_\alpha + (\mathbf{P}_\alpha\cdot\mathbf{V}_\alpha)^S + \mathbf{Q}_\alpha] - \rho\mathbf{VVV} - (\mathbf{P}\cdot\mathbf{V})^S \tag{3.44}$$

or

$$\mathbf{Q}(\mathbf{x},t) \equiv \sum_\alpha [\iiint m_\alpha(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V}) f_\alpha(\mathbf{x},\mathbf{v},t)d^3v] \tag{3.44'}$$

The total kinetic energy flux of the one-fluid plasma is defined by

$$(\frac{1}{2}\rho V^2 + \frac{3}{2}p)\mathbf{V} + \mathbf{P}\cdot\mathbf{V} + \mathbf{q} \equiv \sum_\alpha [(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha)\mathbf{V}_\alpha + \mathbf{P}_\alpha\cdot\mathbf{V}_\alpha + \mathbf{q}_\alpha] \tag{3.45}$$

where the heat-flux vector of the one-fluid plasma is defined by

$$\mathbf{q} \equiv \sum_\alpha [(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha)\mathbf{V}_\alpha + \mathbf{P}_\alpha\cdot\mathbf{V}_\alpha + \mathbf{q}_\alpha] - (\frac{1}{2}\rho V^2 + \frac{3}{2}p)\mathbf{V} - \mathbf{P}\cdot\mathbf{V} \tag{3.46}$$

or

$$\mathbf{q}(\mathbf{x},t) \equiv \sum_\alpha [\iiint \frac{1}{2}m_\alpha(\mathbf{v}-\mathbf{V})\cdot(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V}) f_\alpha(\mathbf{x},\mathbf{v},t)d^3v] \tag{3.46'}$$

3.3.3.2 The One-Fluid Equations

The one-fluid mass continuity equation, obtained from $\sum_\alpha \iiint m_\alpha (3.1')_\alpha d^3v$, is

$$\frac{\partial}{\partial t}\sum_\alpha (m_\alpha n_\alpha) + \nabla\cdot\sum_\alpha (m_\alpha n_\alpha \mathbf{V}_\alpha) = 0$$

or

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{V}) = 0 \tag{3.47}$$

The one-fluid momentum equation, obtained from $\sum_\alpha \iiint m_\alpha \mathbf{v}(3.1')_\alpha d^3v$, is

$$\frac{\partial}{\partial t}\sum_\alpha (m_\alpha n_\alpha \mathbf{V}_\alpha) + \nabla\cdot\sum_\alpha (m_\alpha n_\alpha \mathbf{V}_\alpha\mathbf{V}_\alpha + \mathbf{P}_\alpha) - \sum_\alpha [e_\alpha n_\alpha(\mathbf{E} + \mathbf{V}_\alpha\times\mathbf{B})] = 0$$

or

$$\frac{\partial}{\partial t}(\rho\mathbf{V}) + \nabla\cdot(\rho\mathbf{VV} + \mathbf{P}) = \rho_c\mathbf{E} + \mathbf{J}\times\mathbf{B} \tag{3.48}$$

Substituting the Maxwell's equations (3.2)~(3.5) into Eq. (3.48) yields

$$\frac{\partial}{\partial t}(\rho\mathbf{V}+\frac{1}{c^2}\frac{\mathbf{E}\times\mathbf{B}}{\mu_0})+\nabla\cdot(\rho\mathbf{V}\mathbf{V}+\mathbf{P}+\varepsilon_0\frac{\mathbf{1}E^2}{2}-\varepsilon_0\mathbf{E}\mathbf{E}+\frac{\mathbf{1}B^2}{2\mu_0}-\frac{\mathbf{B}\mathbf{B}}{\mu_0})=0 \qquad (3.49)$$

where $(\mathbf{E}\times\mathbf{B})/\mu_0$ is the Poynting vector of electromagnetic wave.

Exercise 3.3.

Show that $\rho_c\mathbf{E}+\mathbf{J}\times\mathbf{B}=\nabla\cdot(\varepsilon_0\mathbf{E}\mathbf{E}-\varepsilon_0\frac{\mathbf{1}E^2}{2}+\frac{\mathbf{B}\mathbf{B}}{\mu_0}-\frac{\mathbf{1}B^2}{2\mu_0})-\frac{1}{c^2}\frac{\partial}{\partial t}(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0})$

Answer to Exercise 3.3.

$$\begin{aligned}
&\rho_c\mathbf{E}+\mathbf{J}\times\mathbf{B}\\
&=(\varepsilon_0\nabla\cdot\mathbf{E})\mathbf{E}+[\frac{1}{\mu_0}(\nabla\times\mathbf{B})-\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}]\times\mathbf{B}\\
&=(\varepsilon_0\nabla\cdot\mathbf{E})\mathbf{E}+\frac{1}{\mu_0}\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2\mu_0}\nabla B^2-\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\\
&=(\varepsilon_0\nabla\cdot\mathbf{E})\mathbf{E}+\frac{1}{\mu_0}\mathbf{B}\cdot\nabla\mathbf{B}+\frac{1}{\mu_0}\mathbf{B}(\nabla\cdot\mathbf{B})-\frac{1}{2\mu_0}\nabla B^2-\varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})+\varepsilon_0(\mathbf{E}\times\frac{\partial}{\partial t}\mathbf{B})\\
&=(\varepsilon_0\nabla\cdot\mathbf{E})\mathbf{E}+\nabla\cdot(\frac{\mathbf{B}\mathbf{B}}{\mu_0}-\frac{\mathbf{1}B^2}{2\mu_0})-\varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})-\varepsilon_0\mathbf{E}\times(\nabla\times\mathbf{E})\\
&=(\varepsilon_0\nabla\cdot\mathbf{E})\mathbf{E}+\nabla\cdot(\frac{\mathbf{B}\mathbf{B}}{\mu_0}-\frac{\mathbf{1}B^2}{2\mu_0})-\varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})+\varepsilon_0\mathbf{E}\cdot\nabla\mathbf{E}-\frac{\varepsilon_0}{2}\nabla E^2\\
&=\nabla\cdot(\varepsilon_0\mathbf{E}\mathbf{E}-\varepsilon_0\frac{\mathbf{1}E^2}{2}+\frac{\mathbf{B}\mathbf{B}}{\mu_0}-\frac{\mathbf{1}B^2}{2\mu_0})-\frac{1}{c^2}\frac{\partial}{\partial t}(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0})
\end{aligned}$$

The one-fluid energy equation, obtained from $\sum_\alpha\iiint\frac{1}{2}m_\alpha v^2(3.1')_\alpha d^3v$, is

$$\frac{\partial}{\partial t}\sum_\alpha(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2+\frac{3}{2}p_\alpha)+\nabla\cdot\sum_\alpha[(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2+\frac{3}{2}p_\alpha)\mathbf{V}_\alpha+\mathbf{P}_\alpha\cdot\mathbf{V}_\alpha+\mathbf{q}_\alpha]-\sum_\alpha(e_\alpha n_\alpha\mathbf{E}\cdot\mathbf{V}_\alpha)=0$$

or

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho V^2+\frac{3}{2}p)+\nabla\cdot[(\frac{1}{2}\rho V^2+\frac{3}{2}p)\mathbf{V}+\mathbf{P}\cdot\mathbf{V}+\mathbf{q}]=\mathbf{E}\cdot\mathbf{J} \qquad (3.50)$$

Substituting the Maxwell's equations (3.2)~(3.5) into Eq. (3.50), it yields

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho V^2+\frac{3}{2}p+\frac{\varepsilon_0E^2}{2}+\frac{B^2}{2\mu_0})+\nabla\cdot[(\frac{1}{2}\rho V^2+\frac{3}{2}p)\mathbf{V}+\mathbf{P}\cdot\mathbf{V}+\mathbf{q}+\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}]=0 \qquad (3.51)$$

Exercise 3.4.

Show that $\mathbf{E}\cdot\mathbf{J} = -\nabla\cdot(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}) - \frac{\partial}{\partial t}(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0})$

Answer to Exercise 3.4.

$$\mathbf{E}\cdot\mathbf{J} = \mathbf{E}\cdot[\frac{1}{\mu_0}(\nabla\times\mathbf{B}) - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}] = -\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}) + \frac{1}{\mu_0}\mathbf{B}\cdot\nabla\times\mathbf{E} - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t}$$

$$= -\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}) - \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t} = -\nabla\cdot(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}) - \frac{\partial}{\partial t}(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0})$$

The charge continuity, obtained from $\sum_\alpha \iiint e_\alpha (3.1')_\alpha d^3v$, is

$$\frac{\partial}{\partial t}\sum_\alpha (e_\alpha n_\alpha) + \nabla\cdot\sum_\alpha (e_\alpha n_\alpha \mathbf{V}_\alpha) = 0$$

or

$$\frac{\partial}{\partial t}\rho_c + \nabla\cdot\mathbf{J} = 0 \tag{3.52}$$

The one-fluid Ohm's law can be obtained from $\sum_\alpha \iiint e_\alpha \mathbf{v}(3.1')_\alpha d^3v$. There are two ways to expand the integration $\sum_\alpha \iiint e_\alpha \mathbf{v}\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}} f_\alpha d^3v$, i.e.,

$$\sum_\alpha \iiint e_\alpha \mathbf{v}\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}} f_\alpha d^3v$$

$$= \nabla\cdot\sum_\alpha \iiint e_\alpha[(\mathbf{v}-\mathbf{V}_\alpha)+\mathbf{V}_\alpha][(\mathbf{v}-\mathbf{V}_\alpha)+\mathbf{V}_\alpha] f_\alpha d^3v$$

$$= \nabla\cdot\sum_\alpha (\frac{e_\alpha}{m_\alpha}\mathsf{P}_\alpha + e_\alpha n_\alpha \mathbf{V}_\alpha\mathbf{V}_\alpha)$$

$$= \nabla\cdot\{\frac{e}{m_i}\mathsf{P}_i - \frac{e}{m_e}\mathsf{P}_e + \frac{\mathbf{V}\mathbf{J} + \mathbf{J}\mathbf{V} - \rho_c\mathbf{V}\mathbf{V} - \frac{\mathbf{J}\mathbf{J}}{en}[\frac{m_i - m_e}{m_i + m_e} + \frac{m_i m_e}{(m_i + m_e)^2}\frac{\rho_c}{en}]}{1 - \frac{m_i - m_e}{m_i + m_e}\frac{\rho_c}{en} - \frac{m_i m_e}{(m_i + m_e)^2}(\frac{\rho_c}{en})^2}\}$$

or

$$\sum_\alpha \iiint e_\alpha \mathbf{v}\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} f_\alpha d^3v$$
$$=\nabla\cdot\sum_\alpha \iiint e_\alpha[(\mathbf{v}-\mathbf{V})+\mathbf{V}][(\mathbf{v}-\mathbf{V})+\mathbf{V}] f_\alpha d^3v$$
$$=\nabla\cdot\sum_\alpha[\frac{e_\alpha}{m_\alpha}\mathsf{P}_\alpha^{\text{C.M.}}+(e_\alpha\mathbf{V}n_\alpha\mathbf{V}_\alpha-e_\alpha n_\alpha\mathbf{V}\mathbf{V})+(e_\alpha n_\alpha\mathbf{V}_\alpha\mathbf{V}-e_\alpha n_\alpha\mathbf{V}\mathbf{V})+e_\alpha n_\alpha\mathbf{V}\mathbf{V}]$$
$$=\nabla\cdot[\frac{e}{m_i}\mathsf{P}_i^{\text{C.M.}}-\frac{e}{m_e}\mathsf{P}_e^{\text{C.M.}}+\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}-\rho_c\mathbf{V}\mathbf{V}]$$

where $n\equiv\rho/(m_i+m_e)$, and $\mathsf{P}_\alpha^{\text{C.M.}}\equiv\iiint m_\alpha(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})f_\alpha(\mathbf{x},\mathbf{v},t)d^3v$.

Thus, the one-fluid Ohm's law can be written as

$$\frac{\partial}{\partial t}\mathbf{J}+\nabla\cdot\{\frac{e}{m_i}\mathsf{P}_i-\frac{e}{m_e}\mathsf{P}_e+\frac{\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}-\rho_c\mathbf{V}\mathbf{V}-\frac{\mathbf{J}\mathbf{J}}{en}[\frac{m_i-m_e}{m_i+m_e}+\frac{m_im_e}{(m_i+m_e)^2}\frac{\rho_c}{en}]}{1-\frac{m_i-m_e}{m_i+m_e}\frac{\rho_c}{en}-\frac{m_im_e}{(m_i+m_e)^2}(\frac{\rho_c}{en})^2}\}$$
$$-\frac{\rho e^2}{m_im_e}(\mathbf{E}+\mathbf{V}\times\mathbf{B})+\frac{e(m_i-m_e)}{m_im_e}(\rho_c\mathbf{E}+\mathbf{J}\times\mathbf{B})=0 \tag{3.53}$$

The one-fluid Ohm's law can also be written as

$$\frac{\partial}{\partial t}\mathbf{J}+\nabla\cdot[\frac{e}{m_i}\mathsf{P}_i^{\text{C.M.}}-\frac{e}{m_e}\mathsf{P}_e^{\text{C.M.}}+\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}-\rho_c\mathbf{V}\mathbf{V}]$$
$$-\frac{\rho e^2}{m_im_e}(\mathbf{E}+\mathbf{V}\times\mathbf{B})+\frac{e(m_i-m_e)}{m_im_e}(\rho_c\mathbf{E}+\mathbf{J}\times\mathbf{B})=0 \tag{3.54}$$

Substituting the result of exercise 3.3 into equation (3.54), it yields

$$\frac{\partial}{\partial t}[\mathbf{J}-\frac{e(m_i-m_e)}{m_im_e}\frac{1}{c^2}\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}]+\nabla\cdot[\frac{e}{m_i}\mathsf{P}_i^{\text{C.M.}}-\frac{e}{m_e}\mathsf{P}_e^{\text{C.M.}}+\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}-\rho_c\mathbf{V}\mathbf{V}$$
$$+\frac{e(m_i-m_e)}{m_im_e}(\varepsilon_0\mathbf{E}\mathbf{E}-\varepsilon_0\frac{\mathsf{1}E^2}{2}+\frac{\mathbf{B}\mathbf{B}}{\mu_0}-\frac{\mathsf{1}B^2}{2\mu_0})]=\frac{\rho e^2}{m_im_e}(\mathbf{E}+\mathbf{V}\times\mathbf{B}) \tag{3.54'}$$

Eq. (3.54) is commonly called the *generalized Ohm's law*. Please see Appendix B for discussion of the generalized Omh's Law in detail.

Exercise 3.5.

Verify Eqs. (3.53).

Hint: $\rho=m_in_i+m_en_e$, $\rho_c=e(n_i-n_e)$, $\rho\mathbf{V}=m_in_i\mathbf{V}_i+m_en_e\mathbf{V}_e$, and $\mathbf{J}=e(n_i\mathbf{V}_i-n_e\mathbf{V}_e)$

yields $n_{i,e}=(\rho+\frac{m_{e,i}}{e_{i,e}}\rho_c)/(m_i+m_e)$, $\mathbf{V}_{i,e}=(\rho\mathbf{V}+\frac{m_{e.i}}{e_{i,e}}\mathbf{J})/(\rho+\frac{m_{e,i}}{e_{i,e}}\rho_c)$

Table 3.2. The one-fluid equations in the conservative form

SI Units	Gaussian Units
The mass continuity equation $\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\mathbf{V})=0$	The mass continuity equation $\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\mathbf{V})=0$
The momentum equation $\frac{\partial}{\partial t}(\rho\mathbf{V}+\frac{1}{c^2}\frac{\mathbf{E}\times\mathbf{B}}{\mu_0})$ $+\nabla\cdot[\rho\mathbf{V}\mathbf{V}+\mathbf{P}+\varepsilon_0(\frac{\mathbf{1}E^2}{2}-\mathbf{E}\mathbf{E})+\frac{\mathbf{1}B^2}{2\mu_0}-\frac{\mathbf{B}\mathbf{B}}{\mu_0}]=0$	The momentum equation $\frac{\partial}{\partial t}(\rho\mathbf{V}+\frac{1}{c^2}\frac{\mathbf{E}\times\mathbf{B}}{4\pi})$ $+\nabla\cdot[\rho\mathbf{V}\mathbf{V}+\mathbf{P}+\frac{\mathbf{1}E^2}{8\pi}-\frac{\mathbf{E}\mathbf{E}}{4\pi}+\frac{\mathbf{1}B^2}{8\pi}-\frac{\mathbf{B}\mathbf{B}}{4\pi}]=0$
The energy equation $\frac{\partial}{\partial t}(\frac{1}{2}\rho V^2+\frac{3}{2}p+\frac{\varepsilon_0 E^2}{2}+\frac{B^2}{2\mu_0})$ $+\nabla\cdot[(\frac{1}{2}\rho V^2+\frac{3}{2}p)\mathbf{V}+\mathbf{P}\cdot\mathbf{V}+\mathbf{q}+\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}]=0$	The energy equation $\frac{\partial}{\partial t}(\frac{1}{2}\rho V^2+\frac{3}{2}p+\frac{E^2}{8\pi}+\frac{B^2}{8\pi})$ $+\nabla\cdot[(\frac{1}{2}\rho V^2+\frac{3}{2}p)\mathbf{V}+\mathbf{P}\cdot\mathbf{V}+\mathbf{q}+\frac{\mathbf{E}\times\mathbf{B}}{4\pi}]=0$
The charge continuity equation $\frac{\partial}{\partial t}\rho_c+\nabla\cdot\mathbf{J}=0$	The charge continuity equation $\frac{\partial}{\partial t}\rho_c+\nabla\cdot\mathbf{J}=0$
The generalized Ohm's law $\frac{\partial}{\partial t}[\mathbf{J}-\frac{e(m_i-m_e)}{m_i m_e}\frac{1}{c^2}\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}]+$ $\nabla\cdot[\frac{e}{m_i}\mathbf{P}_i^{\text{C.M.}}-\frac{e}{m_e}\mathbf{P}_e^{\text{C.M.}}+\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}-\rho_c\mathbf{V}\mathbf{V}$ $+\frac{e(m_i-m_e)}{m_i m_e}(\varepsilon_0\mathbf{E}\mathbf{E}-\varepsilon_0\frac{\mathbf{1}E^2}{2}+\frac{\mathbf{B}\mathbf{B}}{\mu_0}-\frac{\mathbf{1}B^2}{2\mu_0})]$ $=\frac{\rho e^2}{m_i m_e}(\mathbf{E}+\mathbf{V}\times\mathbf{B})$	The generalized Ohm's law $\frac{\partial}{\partial t}[\mathbf{J}-\frac{e(m_i-m_e)}{m_i m_e}\frac{1}{c^2}\frac{\mathbf{E}\times\mathbf{B}}{4\pi}]+$ $\nabla\cdot[\frac{e}{m_i}\mathbf{P}_i^{\text{C.M.}}-\frac{e}{m_e}\mathbf{P}_e^{\text{C.M.}}+\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}-\rho_c\mathbf{V}\mathbf{V}$ $+\frac{e(m_i-m_e)}{m_i m_e}(\frac{\mathbf{E}\mathbf{E}}{4\pi}-\frac{\mathbf{1}E^2}{8\pi}+\frac{\mathbf{B}\mathbf{B}}{4\pi}-\frac{\mathbf{1}B^2}{8\pi})]$ $=\frac{\rho e^2}{m_i m_e}(\mathbf{E}+\frac{\mathbf{V}\times\mathbf{B}}{c})$

3.3.4. The One-Fluid Equations in the Convective-Time-Derivative Form

Equations obtained in the last section 3.3.3 can be rewritten in a convective-time-derivative form. The convective-time-derivative term is a second-order small term in the linear wave analysis of waves in a uniform background plasma. Thus, equations obtained in this section are particularly useful in linear wave analysis.

The one-fluid continuity equation (3.47) can be rewritten as

$$(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\rho=-\rho\nabla\cdot\mathbf{V} \tag{3.55}$$

The one-fluid momentum equation (3.48) can be rewritten as

$$\rho(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\mathbf{V}=-\nabla\cdot\mathsf{P}+\rho_c\mathbf{E}+\mathbf{J}\times\mathbf{B} \tag{3.56}$$

The one-fluid energy equation (3.50) can be rewritten as

$$\frac{3}{2}[(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)p]+\frac{3}{2}p(\nabla\cdot\mathbf{V})+(\mathsf{P}\cdot\nabla)\cdot\mathbf{V}+\nabla\cdot\mathbf{q}+\rho_c\mathbf{E}\cdot\mathbf{V}-\mathbf{J}\cdot(\mathbf{E}+\mathbf{V}\times\mathbf{B})=0 \tag{3.57}$$

Exercise 3.6.

Verify Eqs. (3.56) and (3.57).

Answer to Exercise 3.6.

Proof of Eq. (3.56):

Substituting Eq. (3.47) into Eq. (3.48), it yields

$$\frac{\partial}{\partial t}(\rho\mathbf{V})+\nabla\cdot(\rho\mathbf{V}\mathbf{V}+\mathsf{P})-\rho_c\mathbf{E}+\mathbf{J}\times\mathbf{B}$$

$$=\mathbf{V}[\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\mathbf{V})]+\rho(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\mathbf{V}+\nabla\cdot\mathsf{P}-\rho_c\mathbf{E}-\mathbf{J}\times\mathbf{B}$$

$$=0+\rho(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\mathbf{V}+\nabla\cdot\mathsf{P}-\rho_c\mathbf{E}-\mathbf{J}\times\mathbf{B}=0$$

Proof of Eq. (3.57):

Substituting Eqs. (3.47) and (3.56) into Eq. (3.50), it yields

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\frac{1}{2}\rho V^2+\frac{3}{2}p)+\nabla\cdot[(\frac{1}{2}\rho V^2+\frac{3}{2}p)\mathbf{V}+\mathbf{P}\cdot\mathbf{V}+\mathbf{q}]-\mathbf{E}\cdot\mathbf{J}\\
&=\frac{1}{2}V^2[\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\mathbf{V})]+\mathbf{V}\cdot[\rho(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\mathbf{V}+\nabla\cdot\mathbf{P}-\rho_c\mathbf{E}-\mathbf{J}\times\mathbf{B}]\\
&\quad+\frac{3}{2}[(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)p]+\frac{3}{2}p(\nabla\cdot\mathbf{V})+(\mathbf{P}\cdot\nabla)\cdot\mathbf{V}+\nabla\cdot\mathbf{q}+\mathbf{V}\cdot(\rho_c\mathbf{E}+\mathbf{J}\times\mathbf{B})-\mathbf{E}\cdot\mathbf{J}\\
&=0+0+\frac{3}{2}[(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)p]+\frac{3}{2}p(\nabla\cdot\mathbf{V})+(\mathbf{P}\cdot\nabla)\cdot\mathbf{V}+\nabla\cdot\mathbf{q}+\rho_c\mathbf{E}\cdot\mathbf{V}-\mathbf{J}\cdot(\mathbf{E}+\mathbf{V}\times\mathbf{B})=0
\end{aligned}
$$

Magnetohydrodynamic (MHD) phenomena are very low frequency and very long wavelength phenomena in the plasma. The time scale of the MHD phenomena is equal or greater than 10^3 ions' characteristic time scale, such as the ion gyro period ($2\pi/\Omega_{ci}$), or the ions' plasma oscillation period ($2\pi/\omega_{pi}$). The spatial scale of the MHD phenomena is equal or greater than 10^3 ions' characteristic length, such as the ion gyro radius (v_0/Ω_{ci}), or the ions' inertial length (c/ω_{pi}).

The quasi-neutrality assumption

$$\left|\frac{\rho_c}{ne}\right|=\frac{|n_i-n_e|}{n}\to 0$$

is applicable to the MHD plasma. Thus, we can assume $\rho_c=0$ for the MHD plasma. As a result, for MHD plasma, the charge continuity equation (3.52) can be reduced to

$$\nabla\cdot\mathbf{J}=0 \tag{3.58}$$

Since the curl of Eq. (3.5) is the charge continuity equation (3.52), the displacement current in the Maxwell's equation can be ignored if $\partial\rho_c/\partial t=0$. Thus, for the MHD plasma, Eq. (3.5) is reduced to

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J} \tag{3.59}$$

In the low-frequency and long-wavelength limit, the generalized Ohm's law is reduced to the *MHD Ohm's law* (see Appendix B),

$$\mathbf{E}+\mathbf{V}\times\mathbf{B}=0 \tag{3.60}$$

The *MHD Ohm's law* leads to the frozen-in flux in the MHD Plasma (see Appendix C).

For MHD plasma (which satisfies the *MHD Ohm's Law*) with zero heat flux, the energy equation (3.57) can be reduced to the adiabatic equation of state.

Case 1: The adiabatic equation of state of an isotropic MHD plasma

If $\mathbf{P} = \mathbf{1}p$, $\nabla \cdot \mathbf{q} = 0$, $\rho_c = 0$, and $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$, the energy equation (3.57) is reduced to the following adiabatic equation of state

$$\frac{3}{2}[\frac{d}{dt}\ln(p\rho^{-5/3})] = \frac{3}{2}[(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla)\ln(p\rho^{-5/3})] = 0 \tag{3.61}$$

Case 2: The adiabatic condition of an anisotropic MHD plasma

If $\mathbf{P} = \mathbf{e}_{\parallel}\mathbf{e}_{\parallel} p_{\parallel} + (\mathbf{1} - \mathbf{e}_{\parallel}\mathbf{e}_{\parallel})p_{\perp} = \mathbf{1}p_{\perp} + \mathbf{e}_{\parallel}\mathbf{e}_{\parallel}(p_{\parallel} - p_{\perp})$, $\nabla \cdot \mathbf{q} = 0$, $\rho_c = 0$, and $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$, where $\mathbf{e}_{\parallel} = \mathbf{B}/B$, the energy equation (3.57) is reduced to the following form

$$\frac{p_{\parallel}}{2}[\frac{d}{dt}\ln(\frac{p_{\parallel}B^2}{\rho^3})] + p_{\perp}[\frac{d}{dt}\ln(\frac{p_{\perp}}{\rho B})] = 0 \tag{3.62}$$

or

$$\frac{p_{\parallel}}{2}[(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla)\ln(\frac{p_{\parallel}B^2}{\rho^3})] + p_{\perp}[(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla)\ln(\frac{p_{\perp}}{\rho B})] = 0 \tag{3.62'}$$

Under an assumption that no momentum exchange in the directions parallel to and perpendicular to the local magnetic field, Chew et al. (1956) obtained the well-known Chew-Goldberger-Low theory or the CGL double adiabatic equation of states, i.e.,

$$\frac{d(p_{\parallel}B^2/\rho^3)}{dt} = 0 \tag{3.63}$$

and

$$\frac{d(p_{\perp}/\rho B)}{dt} = 0 \tag{3.64}$$

Eqs. (3.63) and (3.64) are a special set of solutions of equation (3.62). It can be shown that Chew-Goldberger-Low theory is only applicable to a system with uniform magnetic field strength, so that the mirror motion is prohibited in the system. Professor J. K. Chao has found an example from magnetosheath observations along the Sun-Earth line, which shows that Eq. (3.62) is a more general adiabatic condition than the Chew-Goldberger-Low theory.

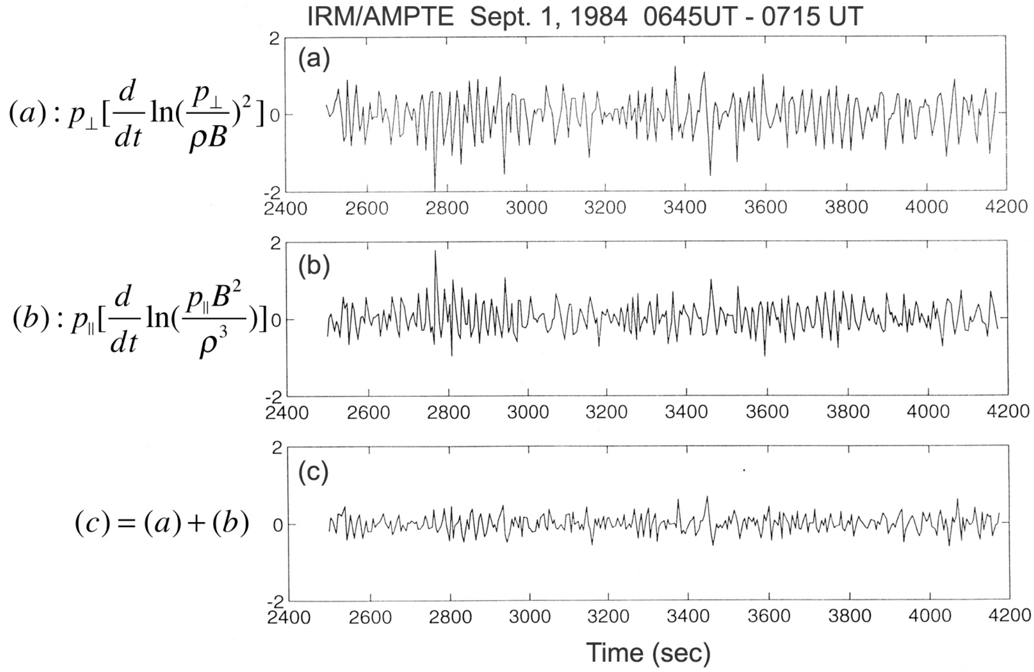

Figure 3.1. Magnetosheath observations along the Sun-Earth line. Panel (a) is a plot of $p_{\perp}[d\ln(p_{\perp}/\rho B)^2/dt]$. Panel (b) is a plot of $(p_{\parallel})[d\ln(p_{\parallel}B^2/\rho^3)/dt]$. Panel (c) is a plot of $p_{\perp}[d\ln(p_{\perp}/\rho B)^2/dt]+p_{\parallel}[d\ln(p_{\parallel}B^2/\rho^3)/dt]$. Wave amplitude in Panel (c) is much smaller than the wave amplitude in Panels (a) and (b). Results shown in this figure indicate that Eq. (3.62) is a more general adiabatic condition than the Chew-Goldberger-Low theory or the so-called CGL double adiabatic equation of states (Chew et al., 1956). Observational data shown in this figure are obtained from IRM/AMPTE at 0645UT-0715UT on September 1, 1984. (Courtesy of Professor J. K. Chao)

Figure 3.1 shows the magnetosheath observations along the Sun-Earth line. Panel (a) is a plot of $p_{\perp}[d\ln(p_{\perp}/\rho B)^2/dt]$. Panel (b) is a plot of $(p_{\parallel})[d\ln(p_{\parallel}B^2/\rho^3)/dt]$. Panel (c) is a plot of $p_{\perp}[d\ln(p_{\perp}/\rho B)^2/dt]+p_{\parallel}[d\ln(p_{\parallel}B^2/\rho^3)/dt]$. Wave amplitude in Panel (c) is much smaller than the wave amplitude in Panels (a) and (b). These results indicate that Eq. (3.62) is a more general adiabatic condition than the CGL double adiabatic equation of states (Chew et al., 1956).

In summary, Table 3.3 lists the governing equations of the MHD plasma with an isotropic pressure ($\mathbf{P}=\mathbf{1}\,p$) and zero heat flux ($\mathbf{q}=0$). There are 14 unknowns ($n,\mathbf{V},p,\mathbf{J},\mathbf{E},\mathbf{B}$) and 14 independent equations in this system.

Table 3.3. Governing equations of MHD plasma with isotropic pressure and zero heat flux

SI Units	Gaussian Units
The mass continuity equation $$(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\rho = -\rho\nabla\cdot\mathbf{V}$$	The mass continuity equation $$(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\rho = -\rho\nabla\cdot\mathbf{V}$$
The MHD momentum equation $$\rho(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p + \mathbf{J}\times\mathbf{B}$$	The MHD momentum equation $$\rho(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p + \frac{\mathbf{J}\times\mathbf{B}}{c}$$
The MHD energy equation $$\frac{3}{2}[(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\ln(p\rho^{-5/3})] = 0$$	The MHD energy equation $$\frac{3}{2}[(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\ln(p\rho^{-5/3})] = 0$$
The MHD charge continuity equation $$\nabla\cdot\mathbf{J} = 0$$	The MHD charge continuity equation $$\nabla\cdot\mathbf{J} = 0$$
The MHD Ohm's law $$\mathbf{E} + \mathbf{V}\times\mathbf{B} = 0$$	The MHD Ohm's law $$\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c} = 0$$
The Maxwell's equations: $$\nabla\cdot\mathbf{E}\to 0$$ $$\nabla\cdot\mathbf{B} = 0$$ $$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$ $$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$$	The Maxwell's equations: $$\nabla\cdot\mathbf{E}\to 0$$ $$\nabla\cdot\mathbf{B} = 0$$ $$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$ $$\nabla\times\mathbf{B} = (4\pi/c)\mathbf{J}$$

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