Chapter 2. Deriving the Vlasov Equation From the Klimontovich Equation

Topics or concepts to learn in Chapter 2:
1. The microscopic plasma distribution: the Klimontovich equation
2. The statistic plasma distribution: the Boltzmann equation and the Vlasov equation

Suggested Reading:
(1) Chapter 3 in Nicholson (1983)

2.1. Klimontovich Equation

Let us define a microscopic distribution function of the \( \alpha \)th species in the six-dimensional phase space

\[
N_\alpha(x, v, t) = \sum_{k=1}^{N_\alpha} \delta[x - x_k(t)]\delta[v - v_k(t)]
\]  
(2.1)

where \( x_k(t) \) and \( v_k(t) \) satisfy the following equations of motion

\[
\frac{dx_k(t)}{dt} = v_k(t)
\]  
(2.2)

\[
\frac{dv_k(t)}{dt} = \frac{e_\alpha}{m_\alpha} \left\{ E^m[x_k(t), t] + v_k(t) \times B^m[x_k(t), t] \right\}
\]  
(2.3)

in which \( E^m(x, t) \) and \( B^m(x, t) \) are the microscopic electric field and magnetic field, respectively. The Klimontovich equation can be obtained by evaluating the time derivative of \( N_\alpha(x, v, t) \).

Taking time derivative of Eq. (2.1) and making use of Eqs. (2.2)~(2.3) and \( a\delta(a - b) = b\delta(a - b) \), it yields
\[
\frac{\partial N_\alpha(x,v,t)}{\partial t} = \frac{\partial}{\partial t} \sum_{k=1}^{N_\alpha} [\delta(x-x_k(t)) \delta(v-v_k(t))]
= \sum_{k=1}^{N_\alpha} \left\{ \frac{\partial}{\partial t} \delta(x-x_k(t)) \right\} \delta(v-v_k(t)) + \sum_{k=1}^{N_\alpha} \delta(x-x_k(t)) \left( \frac{\partial}{\partial v} \delta(v-v_k(t)) \right)
= \sum_{k=1}^{N_\alpha} \left[ \frac{\partial \delta(x-x_k(t))}{\partial x} \right] \cdot \left[ -\frac{dx_k(t)}{dt} \right] \delta(v-v_k(t)) + \sum_{k=1}^{N_\alpha} \delta(x-x_k(t)) \left( \frac{\partial \delta(v-v_k(t))}{\partial v} \right) \cdot \left[ -\frac{dv_k(t)}{dt} \right]
= \sum_{k=1}^{N_\alpha} \delta[v-v_k(t)] \left[ -\frac{dv_k(t)}{dt} \right] \frac{\partial}{\partial x} \delta(x-x_k(t))
+ \sum_{k=1}^{N_\alpha} \delta(x-x_k(t)) \left( -\frac{e_\alpha}{m_\alpha} \right) [E^m(x,t) + v \times B^m(x,t)] \cdot \frac{\partial}{\partial v} \delta(v-v_k(t))
= \left[-v \right] \cdot \frac{\partial}{\partial x} \sum_{k=1}^{N_\alpha} \{\delta[x-x_k(t)] \delta[v-v_k(t)]\}
+ \left( -\frac{e_\alpha}{m_\alpha} \right) [E^m(x,t) + v \times B^m(x,t)] \cdot \frac{\partial}{\partial v} \sum_{k=1}^{N_\alpha} \{\delta[x-x_k(t)] \delta[v-v_k(t)]\}
= -v \cdot \frac{\partial N_\alpha(x,v,t)}{\partial x} - \frac{e_\alpha}{m_\alpha} [E^m(x,t) + v \times B^m(x,t)] \cdot \frac{\partial N_\alpha(x,v,t)}{\partial v}
\]

Eq. (2.4) is the Klimontovich equation of the microscopic distribution function \(N_\alpha(x,v,t)\).

**Exercise 2.1**

Show that
\[
\frac{\partial}{\partial t} \delta(x-x_k(t)) = \frac{\partial \delta(x-x_k(t))}{\partial x} \cdot \left[ -\frac{dx_k(t)}{dt} \right]
\]
Answer to Exercise 2.1

\[
\begin{align*}
\frac{\partial}{\partial t} \delta[\mathbf{x} - \mathbf{x}_k(t)] &= \frac{\partial}{\partial t} \{\delta[x - x_k(t)]\delta[y - y_k(t)]\delta[z - z_k(t)]\} \\
&= \frac{\partial}{\partial t} \{\delta[x - x_k(t)]\} \delta[y - y_k(t)]\delta[z - z_k(t)] \\
&\quad + \delta[x - x_k(t)] \frac{\partial}{\partial t} \{\delta[y - y_k(t)]\} \delta[z - z_k(t)] \\
&\quad + \delta[x - x_k(t)] \delta[y - y_k(t)] \frac{\partial}{\partial t} \{\delta[z - z_k(t)]\} \\
&= \left(\frac{d\delta[x - x_k(t)]}{d[x - x_k(t)]}\right) \delta[y - y_k(t)]\delta[z - z_k(t)] \\
&\quad + \delta[x - x_k(t)] \left(\frac{d\delta[y - y_k(t)]}{d[y - y_k(t)]}\right) \frac{dy_k(t)}{dt} \delta[z - z_k(t)] \\
&\quad + \delta[x - x_k(t)] \delta[y - y_k(t)] \left(\frac{d\delta[z - z_k(t)]}{d[z - z_k(t)]}\right) \frac{dz_k(t)}{dt} \\
&= \frac{\partial \delta[\mathbf{x} - \mathbf{x}_k(t)]}{\partial \mathbf{x}} \cdot \left(-\frac{d\mathbf{x}_k(t)}{dt}\right) = \{\nabla_x \delta[\mathbf{x} - \mathbf{x}_k(t)]\} \cdot \left(-\frac{d\mathbf{x}_k(t)}{dt}\right)
\end{align*}
\]

where

\[
\frac{\partial \delta[\mathbf{x} - \mathbf{x}_k(t)]}{\partial \mathbf{x}} = \{\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}\}(\delta[x - x_k(t)]\delta[y - y_k(t)]\delta[z - z_k(t)])
\]

and

\[
\frac{dx_k(t)}{dt} = \dot{x} \frac{dx_k(t)}{dt} + \dot{y} \frac{dy_k(t)}{dt} + \dot{z} \frac{dz_k(t)}{dt}
\]

Exercise 2.2

Show that \(\frac{\partial \delta[x - x_k(t)]}{\partial t} = \frac{\partial \delta[x - x_k(t)]}{\partial x} \left(-\frac{dx_k(t)}{dt}\right)\)
Answer to Exercise 2.2

Let \( f \) be a functional of a function \( W(x,t) \), i.e., \( f = f[W(x,t)] \). Then

\[
\frac{\partial f}{\partial t} = \frac{df}{dW} \frac{\partial W}{\partial t}
\]

If \( \partial W / \partial x = 1 \), then

\[
\frac{\partial f}{\partial x} = \frac{df}{dW} \frac{\partial W}{\partial x} = \frac{df}{dW}
\]

Thus, for \( \partial W / \partial x = 1 \), we have

\[
\frac{\partial f}{\partial t} = \frac{df}{dW} \frac{\partial W}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial W}{\partial t}
\]

This is the reason why

\[
\frac{\partial \delta[x - x_k(t)]}{\partial t} = \frac{\partial \delta[x - x_k(t)]}{\partial x} \frac{\partial [x - x_k(t)]}{\partial t} = \frac{\partial \delta[x - x_k(t)]}{\partial x} [-\frac{dx_k(t)}{dt}]
\]

Exercise 2.3

Show that

\[
\sum_{k=1}^{N_\alpha} \delta[x - x_k(t)] (v_k(t) \times B^m(x_k(t), t)) \cdot \frac{\partial}{\partial v} \delta[v - v_k(t)]
\]

\[
= \sum_{k=1}^{N_\alpha} \delta[x - x_k(t)] [v \times B^m(x, t)] \cdot \frac{\partial}{\partial v} \delta[v - v_k(t)]
\]

2.2. Vlasov Equation

Let \( f_\alpha(x,v,t) \), \( E(x,t) \), and \( B(x,t) \) be the ensemble average of \( N_\alpha(x,v,t) \), \( E^m(x,t) \), and \( B^m(x,t) \), respectively. Let

\[
N_\alpha(x,v,t) = f_\alpha(x,v,t) + \delta N_\alpha(x,v,t)
\]

\[
E^m(x,t) = E(x,t) + \delta E^m(x,t)
\]

\[
B^m(x,t) = B(x,t) + \delta B^m(x,t)
\]

If we use \( \langle A \rangle \) to denote the ensemble average of \( A \), then we have

\[
\langle N_\alpha(x,v,t) \rangle = f_\alpha(x,v,t)
\]

\[
\langle E^m(x,t) \rangle = E(x,t)
\]
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\[ \langle B^m(x, t) \rangle = B(x, t) \]

and

\[ \langle \delta N_\alpha(x, v, t) \rangle = 0 \]

\[ \langle \delta E^m(x, t) \rangle = 0 \]

\[ \langle \delta B^m(x, t) \rangle = 0 \]

Taking the ensemble average of Eq. (2.4), it yields

\[
\left\langle \frac{\partial N_\alpha(x, v, t)}{\partial t} + v \cdot \frac{\partial N_\alpha(x, v, t)}{\partial x} + \frac{e_\alpha}{m_\alpha} [E^m(x, t) + v \times B^m(x, t)] \cdot \frac{\partial N_\alpha(x, v, t)}{\partial v} \right\rangle = 0
\]

or

\[
\frac{\partial f_\alpha(x, v, t)}{\partial t} + v \cdot \frac{\partial f_\alpha(x, v, t)}{\partial x} + \frac{e_\alpha}{m_\alpha} [E(x, t) + v \times B(x, t)] \cdot \frac{\partial f_\alpha(x, v, t)}{\partial v} + \frac{e_\alpha}{m_\alpha} \left\{ [\delta E^m(x, t) + v \times \delta B^m(x, t)] \cdot \frac{\partial \delta N_\alpha(x, v, t)}{\partial v} \right\} = 0
\]

(2.5)

Let \( Df_\alpha(x, v, t) / Dt \) denote the time derivative of the distribution function \( f_\alpha(x, v, t) \) along its characteristic curve in the \( (x, v) \) phase space, then Eq. (2.5) can be rewritten as

\[
\frac{Df_\alpha(x, v, t)}{Dt} = \frac{\partial f_\alpha(x, v, t)}{\partial t} + v \cdot \frac{\partial f_\alpha(x, v, t)}{\partial x} + \frac{e_\alpha}{m_\alpha} [E(x, t) + v \times B(x, t)] \cdot \frac{\partial f_\alpha(x, v, t)}{\partial v} + \frac{e_\alpha}{m_\alpha} \left\{ [\delta E^m(x, t) + v \times \delta B^m(x, t)] \cdot \frac{\partial \delta N_\alpha(x, v, t)}{\partial v} \right\} = \frac{\delta f_\alpha(x, v, t)}{\delta t \text{ collision}}
\]

(2.6)

For

\[- \frac{e_\alpha}{m_\alpha} \left\{ [\delta E^m(x, t) + v \times \delta B^m(x, t)] \cdot \frac{\partial \delta N_\alpha(x, v, t)}{\partial v} \right\} = \frac{\delta f_\alpha(x, v, t)}{\delta t \text{ collision}} = 0,\]

the Boltzmann equation, Eq. (2.6), is reduced to the Vlasov equation (Vlasov, 1945):

\[
\frac{\partial f_\alpha(x, v, t)}{\partial t} + v \cdot \frac{\partial f_\alpha(x, v, t)}{\partial x} + \frac{e_\alpha}{m_\alpha} [E(x, t) + v \times B(x, t)] \cdot \frac{\partial f_\alpha(x, v, t)}{\partial v} = 0
\]

(2.7)

References
