Chapter 2. Deriving the Vlasov Equation From the Klimontovich Equation

Topics or concepts to learn in Chapter 2:

1. The microscopic plasma distribution: the Klimontovich equation
2. The statistic plasma distribution: the Boltzmann equation and the Vlasov equation

Suggested Reading:

(1) Chapter 3 in Nicholson (1983)

2.1. Klimontovich Equation

Let us define a microscopic distribution function of the αth species in the six-dimensional phase space

$$N_\alpha(\mathbf{x},\mathbf{v},t) = \sum_{k=1}^{N_0} \delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{v}-\mathbf{v}_k(t)] \tag{2.1}$$

where $\mathbf{x}_k(t)$ and $\mathbf{v}_k(t)$ satisfy the following equations of motion

$$\frac{d\mathbf{x}_k(t)}{dt} = \mathbf{v}_k(t) \tag{2.2}$$

$$\frac{d\mathbf{v}_k(t)}{dt} = \frac{e_\alpha}{m_\alpha}\{\mathbf{E}^m[\mathbf{x}_k(t),t] + \mathbf{v}_k(t)\times\mathbf{B}^m[\mathbf{x}_k(t),t]\} \tag{2.3}$$

in which $\mathbf{E}^m(\mathbf{x},t)$ and $\mathbf{B}^m(\mathbf{x},t)$ are the microscopic electric field and magnetic field, respectively. The Klimontovich equation can be obtained by evaluating the time derivative of $N_\alpha(\mathbf{x},\mathbf{v},t)$.

Taking time derivative of Eq. (2.1) and making use of Eqs. (2.2)~(2.3) and $a\delta(a-b) = b\delta(a-b)$, it yields

$$\frac{\partial N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} = \frac{\partial}{\partial t}\sum_{k=1}^{N_0}\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{v}-\mathbf{v}_k(t)]$$

$$= \sum_{k=1}^{N_0}\{\frac{\partial}{\partial t}\delta[\mathbf{x}-\mathbf{x}_k(t)]\}\delta[\mathbf{v}-\mathbf{v}_k(t)] + \sum_{k=1}^{N_0}\delta[\mathbf{x}-\mathbf{x}_k(t)]\{\frac{\partial}{\partial t}\delta[\mathbf{v}-\mathbf{v}_k(t)]\}$$

$$= \sum_{k=1}^{N_0}\{\frac{\partial\delta[\mathbf{x}-\mathbf{x}_k(t)]}{\partial \mathbf{x}}\cdot[-\frac{d\mathbf{x}_k(t)}{dt}]\}\delta[\mathbf{v}-\mathbf{v}_k(t)] + \sum_{k=1}^{N_0}\delta[\mathbf{x}-\mathbf{x}_k(t)]\{\frac{\partial\delta[\mathbf{v}-\mathbf{v}_k(t)]}{\partial \mathbf{v}}\cdot[-\frac{d\mathbf{v}_k(t)}{dt}]\}$$

$$= \sum_{k=1}^{N_0}\delta[\mathbf{v}-\mathbf{v}_k(t)][-\mathbf{v}_k(t)]\cdot\frac{\partial}{\partial \mathbf{x}}\delta[\mathbf{x}-\mathbf{x}_k(t)]$$

$$+\sum_{k=1}^{N_0}\delta[\mathbf{x}-\mathbf{x}_k(t)][-\frac{e_\alpha}{m_\alpha}\{\mathbf{E}^m[\mathbf{x}_k(t),t]+\mathbf{v}_k(t)\times\mathbf{B}^m[\mathbf{x}_k(t),t]\}]\cdot\frac{\partial}{\partial \mathbf{v}}\delta[\mathbf{v}-\mathbf{v}_k(t)]$$

$$= \sum_{k=1}^{N_0}\delta[\mathbf{v}-\mathbf{v}_k(t)][-\mathbf{v}]\cdot\frac{\partial}{\partial \mathbf{x}}\delta[\mathbf{x}-\mathbf{x}_k(t)]$$

$$+\sum_{k=1}^{N_0}\delta[\mathbf{x}-\mathbf{x}_k(t)](-\frac{e_\alpha}{m_\alpha})[\mathbf{E}^m(\mathbf{x},t)+\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial}{\partial \mathbf{v}}\delta[\mathbf{v}-\mathbf{v}_k(t)]$$

$$= [-\mathbf{v}]\cdot\frac{\partial}{\partial \mathbf{x}}\sum_{k=1}^{N_0}\{\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{v}-\mathbf{v}_k(t)]\}$$

$$+(-\frac{e_\alpha}{m_\alpha})[\mathbf{E}^m(\mathbf{x},t)+\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial}{\partial \mathbf{v}}\sum_{k=1}^{N_0}\{\delta[\mathbf{x}-\mathbf{x}_k(t)]\delta[\mathbf{v}-\mathbf{v}_k(t)]\}$$

$$= -\mathbf{v}\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{x}} - \frac{e_\alpha}{m_\alpha}[\mathbf{E}^m(\mathbf{x},t)+\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}}$$

or

$$\frac{\partial N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}[\mathbf{E}^m(\mathbf{x},t)+\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}} = 0 \tag{2.4}$$

Eq. (2.4) is the Klimontovich equation of the microscopic distribution function $N_\alpha(\mathbf{x},\mathbf{v},t)$.

Exercise 2.1

Show that

$$\frac{\partial}{\partial t}\delta[\mathbf{x}-\mathbf{x}_k(t)] = \frac{\partial\delta[\mathbf{x}-\mathbf{x}_k(t)]}{\partial \mathbf{x}}\cdot[-\frac{d\mathbf{x}_k(t)}{dt}]$$

Answer to Exercise 2.1

$$\frac{\partial}{\partial t}\delta[\mathbf{x}-\mathbf{x}_k(t)] = \frac{\partial}{\partial t}\{\delta[x-x_k(t)]\delta[y-y_k(t)]\delta[z-z_k(t)]\}$$

$$= \frac{\partial}{\partial t}\{\delta[x-x_k(t)]\}\delta[y-y_k(t)]\delta[z-z_k(t)]$$

$$+\delta[x-x_k(t)]\frac{\partial}{\partial t}\{\delta[y-y_k(t)]\}\delta[z-z_k(t)]$$

$$+\delta[x-x_k(t)]\delta[y-y_k(t)]\frac{\partial}{\partial t}\{\delta[z-z_k(t)]\}$$

$$= \{\frac{d\delta[x-x_k(t)]}{d[x-x_k(t)]}\frac{\partial[x-x_k(t)]}{\partial t}\}\delta[y-y_k(t)]\delta[z-z_k(t)]$$

$$+\delta[x-x_k(t)]\{\frac{d\delta[y-y_k(t)]}{d[y-y_k(t)]}\frac{\partial[y-y_k(t)]}{\partial t}\}\delta[z-z_k(t)]$$

$$+\delta[x-x_k(t)]\delta[y-y_k(t)]\{\frac{d\delta[z-z_k(t)]}{d[z-z_k(t)]}\frac{\partial[z-z_k(t)]}{\partial t}\}$$

$$= \{\frac{\partial\delta[x-x_k(t)]}{\partial x}(-\frac{dx_k(t)}{dt})\}\delta[y-y_k(t)]\delta[z-z_k(t)]$$

$$+\delta[x-x_k(t)]\{\frac{\partial\delta[y-y_k(t)]}{\partial y}(-\frac{dy_k(t)}{dt})\}\delta[z-z_k(t)]$$

$$+\delta[x-x_k(t)]\delta[y-y_k(t)]\{\frac{\partial\delta[z-z_k(t)]}{\partial z}(-\frac{dz_k(t)}{dt})\}$$

$$= \frac{\partial\delta[\mathbf{x}-\mathbf{x}_k(t)]}{\partial\mathbf{x}}\cdot(-\frac{d\mathbf{x}_k(t)}{dt}) = \{\nabla_{\mathbf{x}}\delta[\mathbf{x}-\mathbf{x}_k(t)]\}\cdot(-\frac{d\mathbf{x}_k(t)}{dt})$$

where

$$\frac{\partial\delta[\mathbf{x}-\mathbf{x}_k(t)]}{\partial\mathbf{x}} = \{(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z})(\delta[x-x_k(t)]\delta[y-y_k(t)]\delta[z-z_k(t)])\}$$

and

$$\frac{d\mathbf{x}_k(t)}{dt} = \hat{x}\frac{dx_k(t)}{dt}+\hat{y}\frac{dy_k(t)}{dt}+\hat{z}\frac{dz_k(t)}{dt}$$

Exercise 2.2

Show that $\dfrac{\partial\delta[x-x_k(t)]}{\partial t} = \dfrac{\partial\delta[x-x_k(t)]}{\partial x}[-\dfrac{dx_k(t)}{dt}]$

Answer to Exercise 2.2

Let f be a functional of a function $W(x,t)$, i.e, $f = f[W(x,t)]$. Then

$$\frac{\partial f}{\partial t} = \frac{df}{dW}\frac{\partial W}{\partial t}$$

If $\partial W/\partial x = 1$, then

$$\frac{\partial f}{\partial x} = \frac{df}{dW}\frac{\partial W}{\partial x} = \frac{df}{dW}$$

Thus, for $\partial W/\partial x = 1$, we have

$$\frac{\partial f}{\partial t} = \frac{df}{dW}\frac{\partial W}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial W}{\partial t}$$

This is the reason why

$$\frac{\partial \delta[x - x_k(t)]}{\partial t} = \frac{\partial \delta[x - x_k(t)]}{\partial x}\frac{\partial [x - x_k(t)]}{\partial t} = \frac{\partial \delta[x - x_k(t)]}{\partial x}[-\frac{dx_k(t)}{dt}]$$

Exercise 2.3

Show that

$$\sum_{k=1}^{N_0} \delta[\mathbf{x} - \mathbf{x}_k(t)]\left(\mathbf{v}_k(t) \times \mathbf{B}^m[\mathbf{x}_k(t),t]\right) \cdot \frac{\partial}{\partial \mathbf{v}} \delta[\mathbf{v} - \mathbf{v}_k(t)]$$
$$= \sum_{k=1}^{N_0} \delta[\mathbf{x} - \mathbf{x}_k(t)][\mathbf{v} \times \mathbf{B}^m(\mathbf{x},t)] \cdot \frac{\partial}{\partial \mathbf{v}} \delta[\mathbf{v} - \mathbf{v}_k(t)]$$

2.2. Vlasov Equation

Let $f_\alpha(\mathbf{x},\mathbf{v},t)$, $\mathbf{E}(\mathbf{x},t)$, and $\mathbf{B}(\mathbf{x},t)$ be the ensemble average of $N_\alpha(\mathbf{x},\mathbf{v},t)$, $\mathbf{E}^m(\mathbf{x},t)$, and $\mathbf{B}^m(\mathbf{x},t)$, respectively. Let

$$N_\alpha(\mathbf{x},\mathbf{v},t) = f_\alpha(\mathbf{x},\mathbf{v},t) + \delta N_\alpha(\mathbf{x},\mathbf{v},t)$$

$$\mathbf{E}^m(\mathbf{x},t) = \mathbf{E}(\mathbf{x},t) + \delta \mathbf{E}^m(\mathbf{x},t)$$

$$\mathbf{B}^m(\mathbf{x},t) = \mathbf{B}(\mathbf{x},t) + \delta \mathbf{B}^m(\mathbf{x},t)$$

If we use $\langle A \rangle$ to denote the ensemble average of A, then we have

$$\langle N_\alpha(\mathbf{x},\mathbf{v},t) \rangle = f_\alpha(\mathbf{x},\mathbf{v},t)$$

$$\langle \mathbf{E}^m(\mathbf{x},t) \rangle = \mathbf{E}(\mathbf{x},t)$$

$\left\langle \mathbf{B}^m(\mathbf{x},t) \right\rangle = \mathbf{B}(\mathbf{x},t)$

and

$\left\langle \delta N_\alpha(\mathbf{x},\mathbf{v},t) \right\rangle = 0$

$\left\langle \delta \mathbf{E}^m(\mathbf{x},t) \right\rangle = 0$

$\left\langle \delta \mathbf{B}^m(\mathbf{x},t) \right\rangle = 0$

Taking the ensemble average of Eq. (2.4), it yields

$$\left\langle \frac{\partial N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}[\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}} \right\rangle = 0$$

or

$$\boxed{\begin{aligned}&\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}[\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)]\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}}\\ &+\frac{e_\alpha}{m_\alpha}\left\langle [\delta\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}\times\delta\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial \delta N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}} \right\rangle = 0\end{aligned}} \tag{2.5}$$

Let $Df_\alpha(\mathbf{x},\mathbf{v},t)/Dt$ denote the time derivative of the distribution function $f_\alpha(\mathbf{x},\mathbf{v},t)$ along its characteristic curve in the $(\mathbf{x},\mathbf{v})$ phase space, then Eq. (2.5) can be rewritten as

$$\boxed{\begin{aligned}&\frac{Df_\alpha(\mathbf{x},\mathbf{v},t)}{Dt} = \frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}[\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)]\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}}\\ &= -\frac{e_\alpha}{m_\alpha}\left\langle [\delta\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}\times\delta\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial \delta N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}} \right\rangle = \left.\frac{\delta f_\alpha(\mathbf{x},\mathbf{v},t)}{\delta t}\right|_{collision}\end{aligned}} \tag{2.6}$$

For

$$-\frac{e_\alpha}{m_\alpha}\left\langle [\delta\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}\times\delta\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial \delta N_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}} \right\rangle = \left.\frac{\delta f_\alpha(\mathbf{x},\mathbf{v},t)}{\delta t}\right|_{collision} = 0\,,$$

the Boltzmann equation, Eq. (2.6), is reduced to the Vlasov equation (Vlasov, 1945):

$$\boxed{\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}[\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)]\cdot\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}} = 0} \tag{2.7}$$

References

Nicholson, D. R. (1983), *Introduction to Plasma Theory*, John Wiley & Sons, New York.

Vlasov, A. A. (1945), *J. Phys. (U.S.S.R.), 9*, 25.