Chapter 12: Electrostatic Drift Wave in Two-Fluid Plasma

Two-fluid equations and Poisson equation are

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{V}_i) = \frac{\partial n_i}{\partial t} + n_i \nabla \cdot \mathbf{V}_i + \mathbf{V}_i \cdot \nabla n_i = 0 \tag{12.1}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{V}_e) = \frac{\partial n_e}{\partial t} + n_e \nabla \cdot \mathbf{V}_e + \mathbf{V}_e \cdot \nabla n_e = 0 \tag{12.2}$$

$$n_i m_i (\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla)\mathbf{V}_i = e n_i (\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) - \nabla p_i \tag{12.3}$$

$$n_e m_e (\frac{\partial}{\partial t} + \mathbf{V}_e \cdot \nabla)\mathbf{V}_e = -e n_e (\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) - \nabla p_e \tag{12.4}$$

$$\nabla p_i = \frac{\gamma_i p_i}{n_i} \nabla n_i = \gamma_i k_B T_i \nabla n_i \tag{12.5}$$

$$\nabla p_e = \frac{\gamma_{e\parallel} p_e}{n_e} \nabla_\parallel n_e + \frac{\gamma_{e\perp} p_e}{n_e} \nabla_\perp n_e = \gamma_{e\parallel} k_B T_e \nabla_\parallel n_e + \gamma_{e\perp} k_B T_e \nabla_\perp n_e \tag{12.6}$$

$$\nabla \cdot \mathbf{E} = \frac{e}{\varepsilon_0}(n_i - n_e) \tag{12.7}$$

We assume

$\mathbf{B}_0 = B_0 \hat{z}$

$n_{i0} = n_{e0} = n_0 = n_0(x)$

$T_{i0} = 0$

$1 >> \beta > m_e / m_i$

It can be shown that $\beta << 1$ yields $\nabla B / B << \nabla n / n$. Therefore, we can ignore gradient-B drift and assume a uniform background magnetic field.

For low frequency wave and $\beta > m_e / m_i$, we can ignore electron inertial term in Eq. (12.4).

For $T_{i0} = 0$, yields $\mathbf{V}_{i0} = 0$.

For uniform $T_{e0} \neq 0$, the equilibrium state of Eq. (12.4) becomes

$0 = -e n_0 (\mathbf{V}_{e0} \times \mathbf{B}_0) - k_B T_{e0} \nabla n_0$

which yields

$$\mathbf{V}_{e0} = \frac{-k_B T_{e0} \nabla n_0 \times \mathbf{B}_0}{-e n_0 B_0^2} = \frac{k_B T_{e0}}{e n_0 B_0} \frac{dn_0}{dx} (\hat{x} \times \hat{z}) = (-\frac{k_B T_{e0}}{e B_0} \frac{1}{n_0} \frac{dn_0}{dx}) \hat{y} \tag{12.8}$$

For $\mathbf{V}_{e0} = V_{e0} \hat{y}$, Eq. (12.8) yields

$$V_{eo} = -\frac{k_B T_{e0}}{e B_0} \frac{1}{n_0} \frac{dn_0}{dx} \tag{12.8a}$$

If $\frac{dn_0}{dx} < 0$, we have $V_{eo} > 0$.

For electrostatic wave, we have $B_1 = 0$ and $\mathbf{E}_1 = -\nabla \Phi_1$

We consider a "local approximation" in the density gradient region and perturbations in this region can be written in the following wave form $A_1 = \mathrm{Re}(\tilde{A}_1 \exp\{i[(k_y y + k_z z) - \omega t]\})$. As a result, the Fourier and Laplace transform of Eqs. (12.1)-(12.7) yields,

$$-i\omega \tilde{n}_{i1} + n_0 (ik_y \tilde{V}_{i1y} + ik_z \tilde{V}_{i1z}) + \tilde{V}_{i1x} \frac{dn_0}{dx} = 0 \tag{12.1a}$$

$$-i(\omega - V_{e0} k_y) \tilde{n}_{e1} + n_0 (ik_y \tilde{V}_{e1y} + ik_z \tilde{V}_{e1z}) + \tilde{V}_{e1x} \frac{dn_0}{dx} = 0 \tag{12.2a}$$

$$n_0 m_i (-i\omega) \tilde{V}_{i1x} = e n_0 (0 + \tilde{V}_{i1y} B_0) \tag{12.3a}$$

$$n_0 m_i (-i\omega) \tilde{V}_{i1y} = e n_0 (-ik_y \tilde{\Phi}_1 - \tilde{V}_{i1x} B_0) \tag{12.3b}$$

$$n_0 m_i (-i\omega) \tilde{V}_{i1z} = e n_0 (-ik_z \tilde{\Phi}_1) \tag{12.3c}$$

$$0 = -e n_0 (0 + \tilde{V}_{e1y} B_0) - e \tilde{n}_{e1} V_{e0} B_0 - 0 \tag{12.4a}$$

$$0 = -e n_0 (-ik_y \tilde{\Phi}_1 - \tilde{V}_{e1x} B_0) - \gamma_{e\perp} k_B T_{e0} (ik_y \tilde{n}_{e1}) \tag{12.4b}$$

$$0 = -e n_0 (-ik_z \tilde{\Phi}_1 + 0) - \gamma_{e\|} k_B T_{e0} (ik_z \tilde{n}_{e1}) \tag{12.4c}$$

$$k^2 \tilde{\Phi}_1 = \frac{e}{\varepsilon_0} (\tilde{n}_{i1} - \tilde{n}_{e1}) \tag{12.7a}$$

Eq. (12.4c) yields

$$\tilde{n}_{e1} = n_0 \frac{e \tilde{\Phi}_1}{\gamma_{e\|} k_B T_{e0}} \tag{12.4c'}$$

Eqs. (12.4a) and (12.4c') yield

$$\tilde{V}_{e1y} = -\frac{\tilde{n}_{e1}}{n_0} V_{e0} \tag{12.4a'}$$

Eqs. (12.4b) and (12.4c') yield

$$\tilde{V}_{e1x} = -ik_y\left(\frac{\tilde{\Phi}_1}{B_0} - \frac{\gamma_{e\perp}k_B T_{e0}\tilde{n}_{e1}}{en_0 B_0}\right) = -ik_y\tilde{n}_{e1}\frac{k_B T_{e0}}{en_0 B_0}(\gamma_{e\|} - \gamma_{e\perp}) \tag{12.4b'}$$

Substituting (12.4a') and (12.4b') into (12.2a), yields

$$-i(\omega - V_{e0}k_y)\tilde{n}_{e1} + n_0\left(-ik_y\frac{\tilde{n}_{e1}}{n_0}V_{e0} + ik_z\tilde{V}_{e1z}\right) - ik_y\tilde{n}_{e1}\frac{k_B T_{e0}}{en_0 B_0}(\gamma_{e\|} - \gamma_{e\perp})\frac{dn_0}{dx} = 0$$

or

$$\tilde{V}_{e1z} = \frac{\tilde{n}_{e1}}{n_0 k_z}\left[\omega + k_y(\gamma_{e\|} - \gamma_{e\perp})\frac{k_B T_{e0}}{en_0 B_0}\frac{dn_0}{dx}\right] = \frac{\tilde{n}_{e1}}{n_0}\frac{\omega - (\gamma_{e\|} - \gamma_{e\perp})V_{e0}k_y}{k_z} \tag{12.2a'}$$

where Eq. (12.8a) has been used to obtain Eq. (12.2a').

Namely, for a given $\tilde{\Phi}_1$, we can determine $\tilde{n}_{e1}$ from Eq. (12.4c'), and then substituting $\tilde{n}_{e1}$ into Eqs. (12.4b'), (12.4a'), and (12.2a') to determine $\tilde{\mathbf{V}}_{e1} = (\tilde{V}_{e1x}, \tilde{V}_{e1y}, \tilde{V}_{e1z})$.

The Poisson equation (12.7a) can be rewritten as

$$k^2\tilde{\Phi}_1 = \frac{n_0 e}{\varepsilon_0}\left(\frac{\tilde{n}_{i1}}{n_0} - \frac{\tilde{n}_{e1}}{n_0}\right) = \frac{n_0 e}{\varepsilon_0}\left(\frac{\tilde{n}_{i1}}{n_0} - \frac{e\tilde{\Phi}_1}{\gamma_{e\|}k_B T_{e0}}\right)$$

For $[\varepsilon(\mathbf{k},\omega)]k^2\tilde{\Phi}_1 = 0$, we have

$$\varepsilon(\mathbf{k},\omega) = 1 - \frac{n_0 e^2}{k^2 m_i \varepsilon_0}\left(\frac{n_{i1}}{n_0}\frac{m_i}{e\Phi_1} - \frac{m_i}{\gamma_{e\|}k_B T_{e0}}\right) = 1 - \frac{\omega_{pi}^2}{k^2}\left(\frac{n_{i1}}{n_0}\frac{m_i}{e\Phi_1} - \frac{1}{C_S^2}\right) \tag{12.9}$$

where $C_S = \sqrt{\gamma_{e\|}k_B T_{e0}/m_i}$ is the wave speed of ion acoustic wave.

Eq. (12.3a) yields

$$\tilde{V}_{i1x} = i\frac{eB_0}{m_i\omega}\tilde{V}_{i1y} = i\frac{\Omega_{ci}}{\omega}\tilde{V}_{i1y} \tag{12.3a'}$$

Eq. (12.3b) yields

$$\tilde{V}_{i1y} = \frac{ek_y}{m_i\omega}\left(1 - \frac{\Omega_{ci}^2}{\omega^2}\right)^{-1}\tilde{\Phi}_1 \tag{12.3b'}$$

Substituting Eq. (12.3b') into Eq. (12.3a') yields

$$\tilde{V}_{i1x} = i\frac{\Omega_{ci}}{\omega}\frac{ek_y}{m_i\omega}\left(1 - \frac{\Omega_{ci}^2}{\omega^2}\right)^{-1}\tilde{\Phi}_1 \tag{12.3a''}$$

Eq. (12.3c) yields

$$\tilde{V}_{i1z} = \frac{ek_z}{m_i\omega}\tilde{\Phi}_1 \tag{12.3c'}$$

Substituting Eqs. (12.3a"), (12.3b'), (12.3c') into Eq. (12.1a) yields

$$\frac{\tilde{n}_{i1}}{n_0}=[\frac{k_y^2}{\omega^2}(1-\frac{\Omega_{ci}^2}{\omega^2})^{-1}+\frac{k_z^2}{\omega^2}+\frac{\Omega_{ci}}{\omega}\frac{k_y}{\omega^2}(1-\frac{\Omega_{ci}^2}{\omega^2})^{-1}\frac{1}{n_0}\frac{dn_0}{dx}]\frac{e\tilde{\Phi}_1}{m_i}$$

or

$$\frac{\tilde{n}_{i1}}{n_0}=[\frac{-\frac{k_y^2}{\omega^2}\frac{\omega^2}{\Omega_{ci}^2}-\frac{\Omega_{ci}}{\omega}\frac{k_y}{\omega^2}\frac{\omega^2}{\Omega_{ci}^2}\frac{1}{n_0}\frac{dn_0}{dx}}{(1-\frac{\omega^2}{\Omega_{ci}^2})}+\frac{k_z^2}{\omega^2}]\frac{e\tilde{\Phi}_1}{m_i} \tag{12.1a'}$$

Substituting Eq. (12.1a') into Eq. (12.9) yields

$$\varepsilon(\mathbf{k},\omega)=1-\frac{\omega_{pi}^2}{k^2}([\frac{-\frac{k_y^2}{\omega^2}\frac{\omega^2}{\Omega_{ci}^2}-\frac{\Omega_{ci}}{\omega}\frac{k_y}{\omega^2}\frac{\omega^2}{\Omega_{ci}^2}\frac{1}{n_0}\frac{dn_0}{dx}}{(1-\frac{\omega^2}{\Omega_{ci}^2})}+\frac{k_z^2}{\omega^2}]-\frac{1}{C_S^2})$$

$$=1+\frac{\omega_{pi}^2}{C_S^2k^2}[\frac{+\frac{k_y^2}{\omega^2}\frac{\omega^2}{\Omega_{ci}^2}+\frac{\Omega_{ci}}{\omega}\frac{k_y}{\Omega_{ci}^2}\frac{1}{n_0}\frac{dn_0}{dx}}{(1-\frac{\omega^2}{\Omega_{ci}^2})}C_S^2-\frac{k_z^2C_S^2}{\omega^2}+1]$$

$$=1+\frac{\omega_{pi}^2}{C_S^2k^2}[\frac{+\frac{\omega^2}{\Omega_{ci}^2}k_y^2C_S^2(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}+\frac{\omega}{\Omega_{ci}}k_y\frac{1}{n_0}\frac{dn_0}{dx}C_S^2(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}-k_z^2C_S^2+\omega^2}{\omega^2}]$$

$$=1+\frac{\omega_{pi}^2}{C_S^2k^2}[\frac{+\frac{\omega^2}{\Omega_{ci}^2}k_y^2C_S^2(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}-\omega k_yV_{e0}(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}-k_z^2C_S^2+\omega^2}{\omega^2}]$$

or

$$\varepsilon(\mathbf{k},\omega)=\frac{\omega_{pi}^2}{\omega^2}\frac{1}{C_S^2k^2}\{\frac{\omega^2}{\omega_{pi}^2}C_S^2k^2+\frac{\omega^2}{\Omega_{ci}^2}k_y^2C_S^2(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}-\omega k_yV_{e0}(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}-k_z^2C_S^2+\omega^2\} \tag{12.9a}$$

For $\omega^2 \ll \omega_{pi}^2$ and $\omega^2 \ll \Omega_{ci}^2$, dispersion relation $\varepsilon(\mathbf{k},\omega)=0$ yields

$$\omega^2-\omega k_yV_{e0}-k_z^2C_S^2=0 \tag{12.10}$$

The roots of Eq. (12.10) are

$$\omega=\frac{1}{2}\{k_yV_{e0}\pm\sqrt{(k_yV_{e0})^2+4k_z^2C_S^2}\} \tag{12.10a}$$

or

$$\frac{\omega}{k}=\frac{1}{2}\{V_{e0}\sin\theta\pm\sqrt{V_{e0}^2\sin^2\theta+4C_S^2\cos^2\theta}\} \tag{12.10b}$$

where θ is the angle between background magnetic field and wave propagation direction. Figure 12.1 shows plots of drift wave speed in polar coordinate $(r,\theta) = (\omega / kV_0, \theta)$ with $T_{i0} = 0$. Panel (a) is for $C_S = 2V_{e0}$. Panel (b) is for $C_S = V_{e0}$. Panel (c) is for $C_S = 0.5V_{e0}$. Panel (d) is for $C_S = 0.25V_{e0}$.

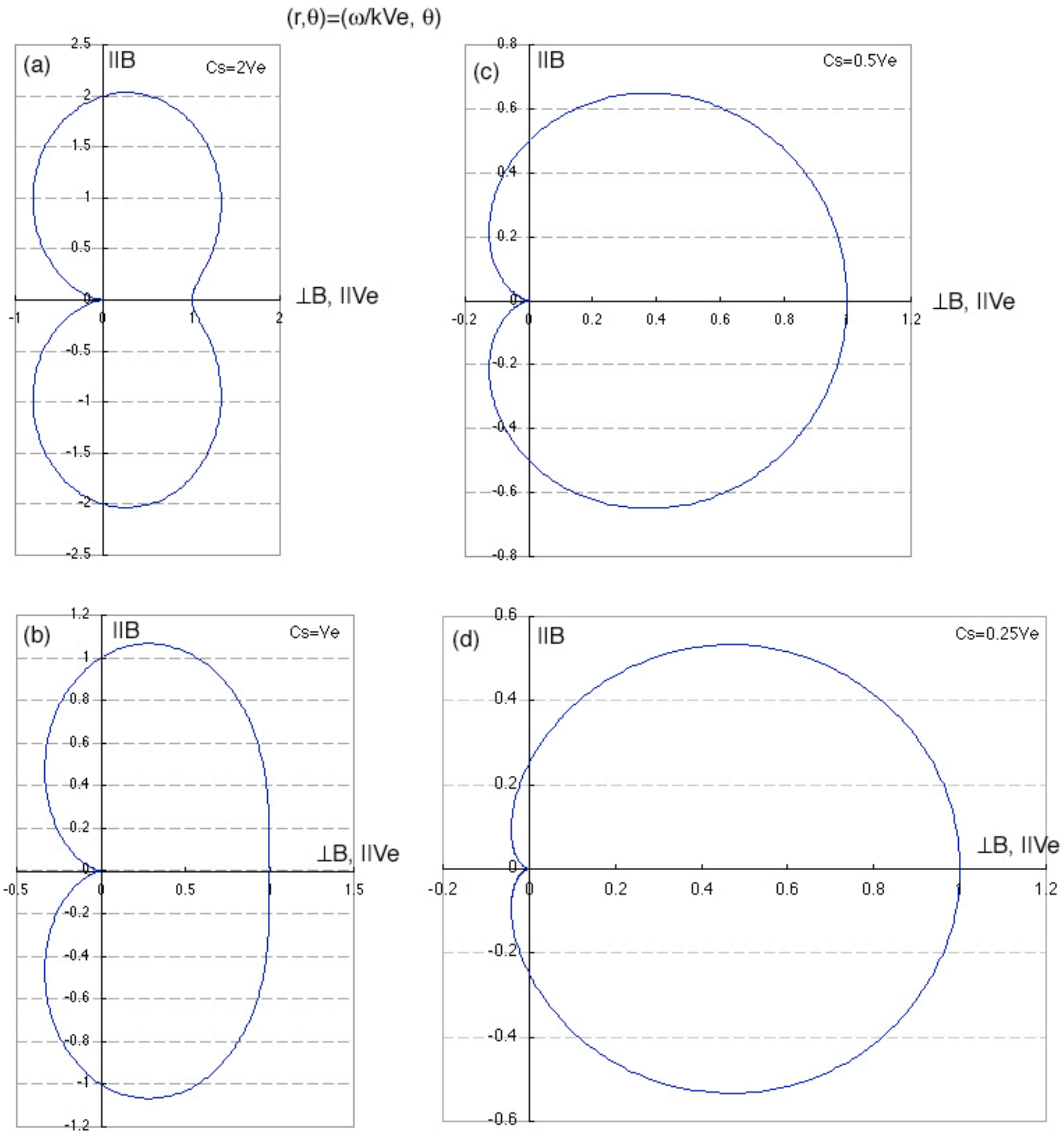

Figure 12.1. Plots of drift wave speed in polar coordinate $(r,\theta) = (\omega / kV_{e0}, \theta)$ with $T_{i0} = 0$. Panel (a) is for $C_S = 2V_{e0}$. Panel (b) is for $C_S = V_{e0}$. Panel (c) is for $C_S = 0.5V_{e0}$. Panel (d) is for $C_S = 0.25V_{e0}$.

In summary, we have

$$\tilde{V}_{i1x} = -i\frac{ek_y}{\Omega_{ci}m_i}(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}\tilde{\Phi}_1 \qquad (12.3a'')$$

$$\tilde{V}_{i1y} = -\frac{\omega^2}{\Omega_{ci}^2}\frac{ek_y}{m_i\omega}(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}\tilde{\Phi}_1 \qquad (12.3b')$$

or $\tilde{\dot{V}}_{i1y} = -i\omega\tilde{V}_{i1y} = i\frac{\omega^2}{\Omega_{ci}^2}\frac{ek_y}{m_i}(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}\tilde{\Phi}_1 \to 0$ (12.3b")

$$\tilde{V}_{i1z} = \frac{ek_z}{m_i\omega}\tilde{\Phi}_1 \qquad (12.3c')$$

or $\tilde{\dot{V}}_{i1z} = -i\omega\tilde{V}_{i1z} = -i\frac{ek_z}{m_i}\tilde{\Phi}_1$ (12.3c")

$$\frac{\tilde{n}_{i1}}{n_0} = [-k_y^2\frac{\omega^2}{\Omega_{ci}^2}(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1} + k_y\omega V_{e0}(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1} + k_z^2]\frac{e\tilde{\Phi}_1}{\omega^2 m_i} \qquad (12.1a'')$$

$$\tilde{n}_{e1} = n_0\frac{e\tilde{\Phi}_1}{\gamma_{e\parallel}k_B T_{e0}} \qquad (12.4c')$$

$$\tilde{V}_{e1x} = -ik_y\tilde{n}_{e1}\frac{k_B T_{e0}}{en_0B_0}(\gamma_{e\parallel}-\gamma_{e\perp}) = -ik_y(\gamma_{e\parallel}-\gamma_{e\perp})\frac{\tilde{\Phi}_1}{B_0} \qquad (12.4b')$$

$$\tilde{V}_{e1y} = -\frac{\tilde{n}_{e1}}{n_0}V_{e0} = -\frac{e\tilde{\Phi}_1}{\gamma_{e\parallel}k_B T_{e0}}V_{e0} \qquad (12.4a')$$

$$\tilde{V}_{e1z} = \frac{\tilde{n}_{e1}}{n_0}\frac{\omega-(\gamma_{e\parallel}-\gamma_{e\perp})V_{e0}k_y}{k_z} = \frac{\omega-(\gamma_{e\parallel}-\gamma_{e\perp})V_{e0}k_y}{k_z}\frac{e\tilde{\Phi}_1}{m_iC_S^2} \qquad (12.2a')$$

and

$$\tilde{E}_{1y} = -ik_y\tilde{\Phi}_1 \qquad (12.11)$$

$$\tilde{E}_{1z} = -ik_z\tilde{\Phi}_1 \qquad (12.12)$$

Particularly, from (12.3a") and (12.4b') yield

$$\delta\tilde{x}_i = \frac{\tilde{V}_{i1x}}{-i\omega} = \frac{ek_y}{\Omega_{ci}m_i\omega}(1-\frac{\omega^2}{\Omega_{ci}^2})^{-1}\tilde{\Phi}_1 \qquad (12.13)$$

$$\delta\tilde{x}_e = \frac{\tilde{V}_{e1x}}{-i\omega} = \frac{k_y(\gamma_{e\parallel}-\gamma_{e\perp})}{\omega}\frac{\tilde{\Phi}_1}{B_0} \qquad (12.14)$$

We can understand the asymmetric wave speed distribution in +y and -y directions shown in Figure 12.1 by studying phase changes of different variables list above.

Figure 12.2 sketches phase changes of (a) n_{i1}, n_{e1}, Φ_1, (b) E_{1y}, E_{1z}, V_{i1x}, V_{e1x}, $\dot{V}_{i1z}$, and

(c) three-dimensional sketches of δx_i and δx_e for $k_y > 0$ and $\gamma_{e\|} - \gamma_{e\perp} > 0$. As we can see for $k_y > 0$, the background density gradient can enhance density variations in the ion acoustic wave. Thus, wave speed increases if waves propagate in the same direction along electron gradient drift direction. Note that for $\gamma = (f+2)/f$, where f is the degree of freedom, the strong uniform background magnetic filed yields $\gamma_{e\|} = 3$ ($f_\| = 1$) and $\gamma_{e\perp} = 2$ ($f_\perp = 2$). Thus, the assumption $\gamma_{e\|} - \gamma_{e\perp} > 0$ is a reasonable assumption.

Figure 12.3 sketches phase changes of (a) n_{i1}, n_{e1}, Φ_1, (b) E_{1y}, E_{1z}, V_{i1x}, V_{e1x}, $\dot{V}_{i1z}$, and (c) three-dimensional sketches of δx_i and δx_e for $k_y < 0$ and $\gamma_{e\|} - \gamma_{e\perp} > 0$. As we can see for $k_y < 0$, the background density gradient can reduce density variations in the ion acoustic wave. Thus, wave speed decrease if waves propagate direction is opposite to electron gradient drift direction.

Since $\mathbf{V}_{i1} = (V_{i1x}, V_{i1y}, V_{i1z})$ and $\mathbf{V}_{e1} = (V_{e1x}, V_{e0} + V_{e1y}, V_{e1z})$, localized two-stream instabilities can take place along both y and z directions due to electrons' gradient drift in y-direction and due to fast field-aligned motion of electrons (V_{e1z}) in nearly perpendicular propagation wave ($|k_z| << |k_y|$). Electrons' field-aligned motion is a result of electrons trying to neutralize ions' density perturbations in the ion acoustic wave. These two-stream instabilities can modify specific heats of electrons in both parallel ($\gamma_{e\|}$) and perpendicular ($\gamma_{e\perp}$) direction.

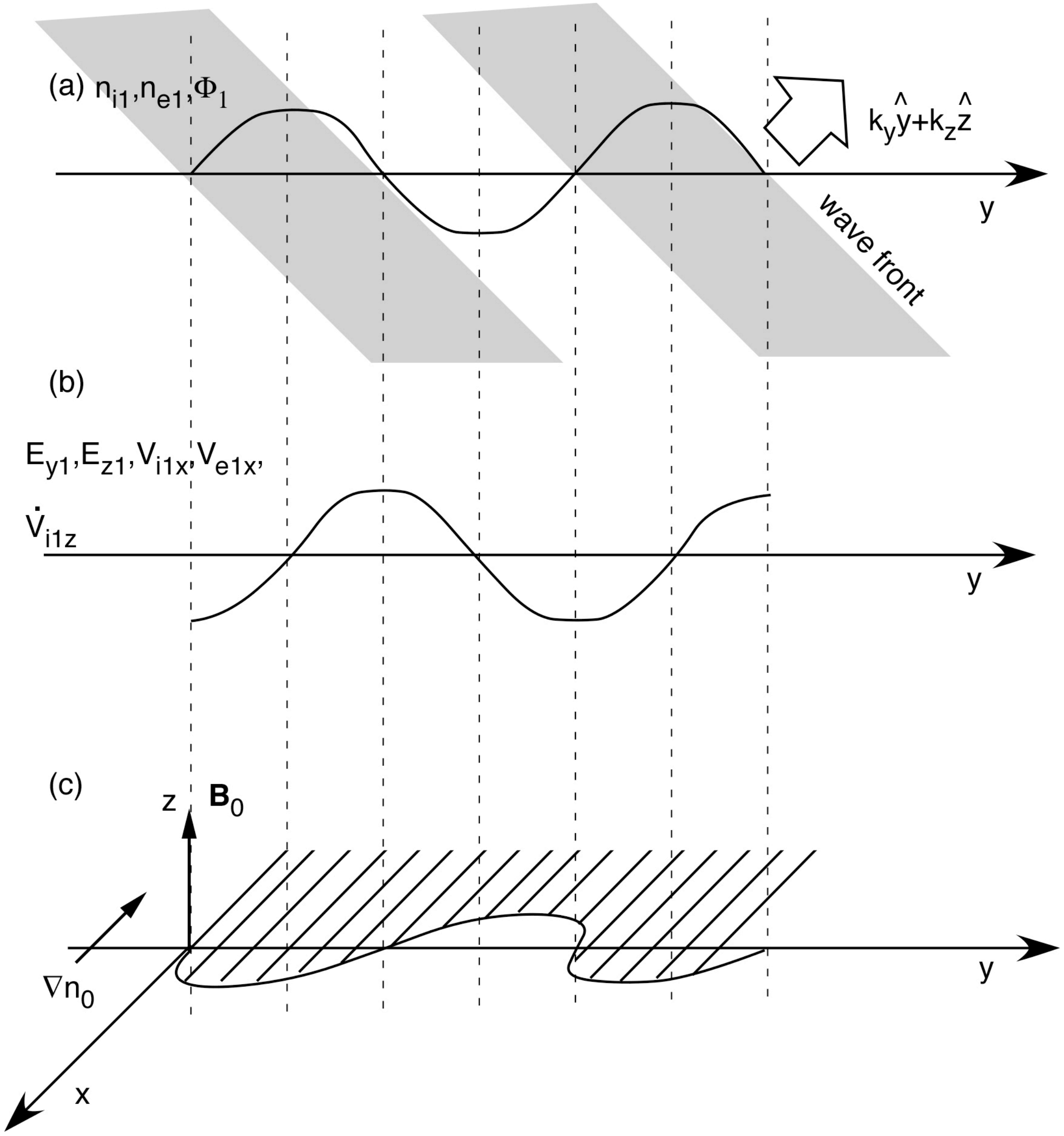

Figure 12.2. Sketches of phase changes of (a) n_{i1}, n_{e1}, Φ_1, (b) E_{1y}, E_{1z}, V_{i1x}, V_{e1x}, $\dot{V}_{i1z}$, and (c) three-dimensional sketches of δx_i and δx_e for $k_y > 0$.

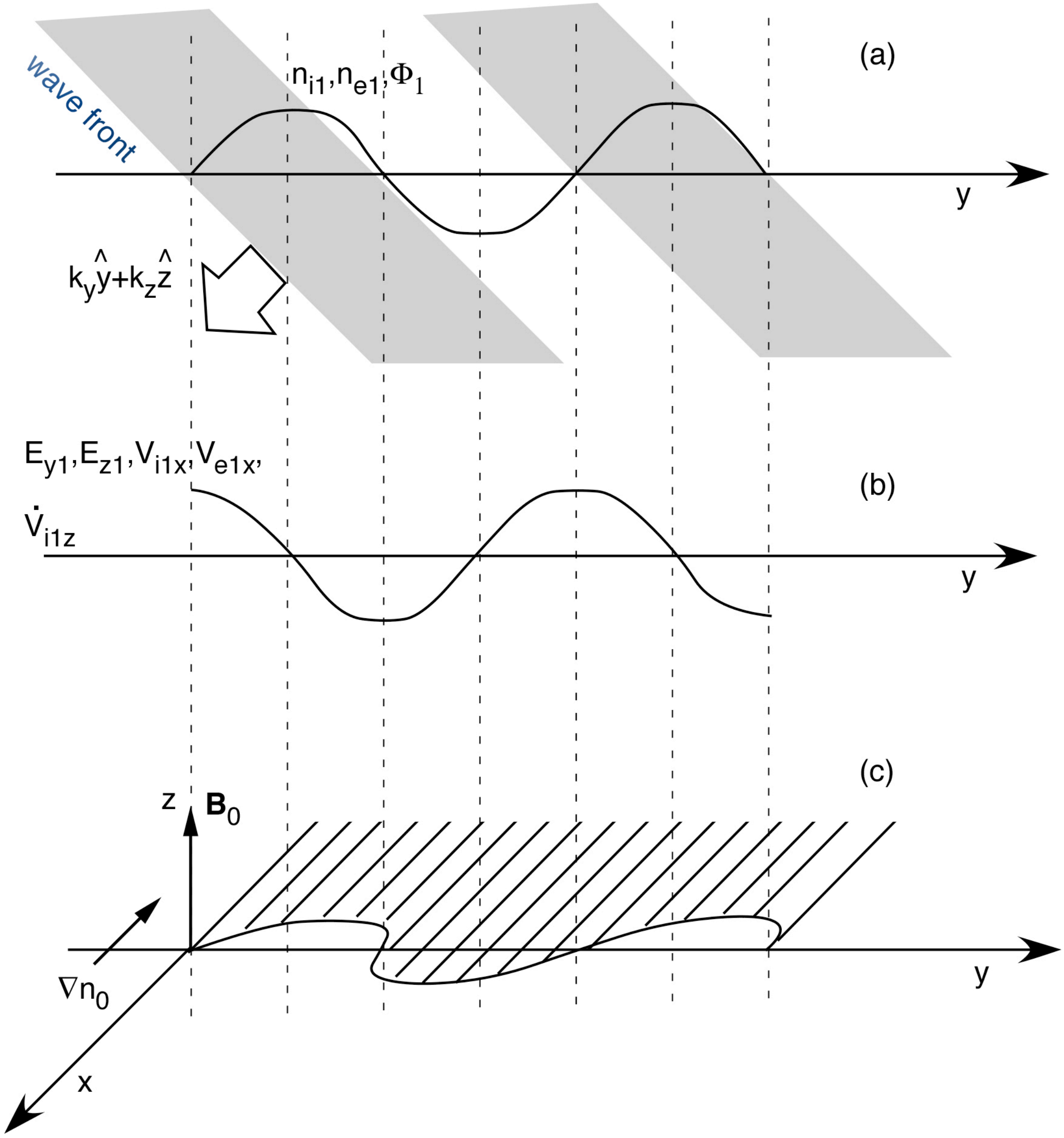

Figure 12.3. Sketches of phase changes of (a) n_{i1}, n_{e1}, Φ_1, (b) E_{1y}, E_{1z}, V_{i1x}, V_{e1x}, $\dot{V}_{i1z}$, and (c) three-dimensional sketches of δx_i and δx_e for $k_y < 0$.

Let the ambient magnetic field along the z direction ($\mathbf{B}_0 = B_0\hat{z}$). Let ambient density gradient $dn_0 / dx < 0$. Let $T_{i0} = 0$ but $T_{e0} \neq 0$. The electron gradient drift will be along the y direction with drift speed equal to

$$V_{eo} = -\frac{k_B T_{e0}}{eB_0}\frac{1}{n_0}\frac{dn_0}{dx}$$

Let C_S be the ion sound speed. The dispersion relation of the low frequency electrostatic wave is

$$\omega^2 - \omega k_y V_{e0} - k_z^2 C_S^2 = 0 \tag{12.10}$$

Let $\mathbf{k} = k(\hat{z}\cos\theta + \hat{y}\sin\theta)$. The dispersion relation can be rewritten as

$$\omega^2 - \omega k \sin\theta V_{e0} - k^2 \cos^2\theta C_S^2 = 0 \tag{12.11}$$

It yields

$$\frac{1}{C_S^2}\frac{\omega^2}{k^2} - \frac{1}{C_S}\frac{\omega}{k}\sin\theta\frac{V_{e0}}{C_S} - \cos^2\theta = 0 \tag{12.12}$$

Define dimensionless variables $V_{e0}^* = V_{e0} / C_S$ and $V_{ph}^* = (\omega / k) / C_S$. Equation (12.12) can be written as

$$(V_{ph}^*)^2 - V_{ph}^* V_{e0}^* \sin\theta - \cos^2\theta = 0 \tag{12.13}$$

Thus, we have

$$V_{ph}^* = \frac{1}{2}\left\{V_{e0}^*\sin\theta + \sqrt{V_{e0}^{*2}\sin^2\theta + 4\cos^2\theta}\right\} \tag{12.14}$$

For $-\pi < \theta < \pi$, we can define

$$\hat{k} = \hat{z}\cos\theta + \hat{y}\sin\theta \tag{12.15}$$

$$\hat{\theta} = \hat{z}(-\sin\theta) + \hat{y}\cos\theta \tag{12.16}$$

The normalized phase velocity $\mathbf{V}_{ph}^* = \mathbf{V}_{ph} / C_S$ is equal to

$$\mathbf{V}_{ph}^* = \hat{k}V_{ph}^* = (\hat{z}\cos\theta + \hat{y}\sin\theta)V_{ph}^* \tag{12.17}$$

The normalized group velocity $\mathbf{V}_g^* = \mathbf{V}_g / C_S$ is equal to

$$\mathbf{V}_g^* = \hat{k}\frac{1}{C_S}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{C_S}\frac{1}{k}\frac{\partial\omega}{\partial\theta} \tag{12.18}$$

To determine the k component in Eq. (12.18), we take derivatives $\partial(12.11)/\partial k$. It yields

$$2\omega\frac{\partial\omega}{\partial k} - \frac{\partial\omega}{\partial k}k\sin\theta V_{e0} - \omega\sin\theta V_{e0} - 2k\cos^2\theta C_S^2 = 0 \tag{12.19}$$

Eq. (12.19) yields

$$\frac{1}{C_S}\frac{\partial\omega}{\partial k} = \frac{V_{ph}^* V_{eo}^* \sin\theta + 2\cos^2\theta}{2V_{ph}^* - V_{e0}^*\sin\theta} \tag{12.20}$$

To determine the θ component in Eq. (12.18), we take derivatives $\partial(12.13)/\partial\theta$. It yields

$$2V_{ph}^*\frac{1}{C_S}\frac{1}{k}\frac{\partial\omega}{\partial\theta} - \frac{1}{C_S}\frac{1}{k}\frac{\partial\omega}{\partial\theta}V_{e0}^*\sin\theta - V_{ph}^*V_{e0}^*\cos\theta + 2\cos\theta\sin\theta = 0 \tag{12.21}$$

Eq. (12.21) yields

$$\frac{1}{C_S}\frac{1}{k}\frac{\partial\omega}{\partial\theta} = \frac{V_{ph}^*V_{e0}^*\cos\theta - 2\cos\theta\sin\theta}{2V_{ph}^* - V_{e0}^*\sin\theta} \tag{12.22}$$

Substituting Equations (12.15), (12.16), (12.20), (12.22) into Equation (12.18), it yields

$$\mathbf{V}_g^* = \frac{\hat{z}\,2\cos\theta + \hat{y}V_{ph}^*V_{e0}^*}{2V_{ph}^* - V_{e0}^*\sin\theta} \tag{12.23}$$

Let us consider the special case with $V_{e0}^* = 0$. Equations (12.17) becomes $\mathbf{V}_{ph}^* = \hat{k}\cos\theta$ and Equation (12.23) becomes $\mathbf{V}_g^* = \pm\hat{z}$. Thus, we have $\mathbf{V}_{ph} = \hat{k}C_S\cos\theta$ and $\mathbf{V}_g = \pm\hat{z}C_S$ if $V_{e0} = 0$.

Figure 12.4 shows the phase velocity distribution and the group velocity distribution of the electrostatic drift waves in the ion-electron two fluid plasma. As we can see, the group velocity distribution changes from two points at $\mathbf{V}_g = \pm\hat{z}C_S$ when $V_{e0} = 0$ to an ellipse with the axis along the z direction equal to $2C_S$ and the axis along the y direction equal to V_{e0} when $V_{e0} > 0$. The red curves from top are for $V_{e0}^* =$ 2, 1, 0.5, and 0.1, and for phase velocity parallel to the electron drift direction. The blue curves from top are for $V_{e0}^* =$ 2, 1, 0.5, and 0.1 and for phase velocity anti-parallel to the electron drift direction.

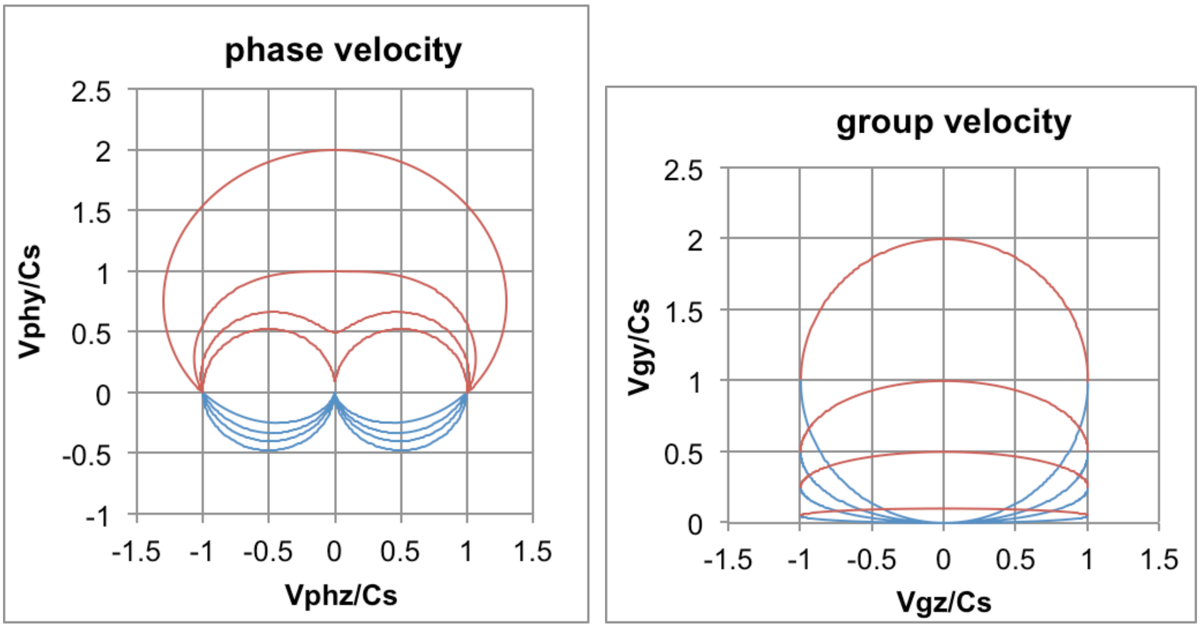

Figure 12.4. Phase velocity distribution and group velocity distribution of the low-frequency drift wave in the ion-electron two-fluid plasma. The red curves from top are for $V_{e0}^* =$ 2, 1, 0.5, and 0.1, and for phase velocity parallel to the electron drift direction. The blue curves from top are for $V_{e0}^* =$ 2, 1, 0.5, and 0.1 and for phase velocity anti-parallel to the electron drift direction.

Figure 12.5 shows the phase velocity distribution and the group velocity distribution of the electrostatic drift waves in the ion-electron two fluid plasma similar to the one shown in Figure 12.4 but for $V_{e0}^{*}=$ 8, 2, 1, 0.5, and 0.1. The distribution of the wave group velocities indicates that the wave energy should be stronger along the major axis of the ellipse.

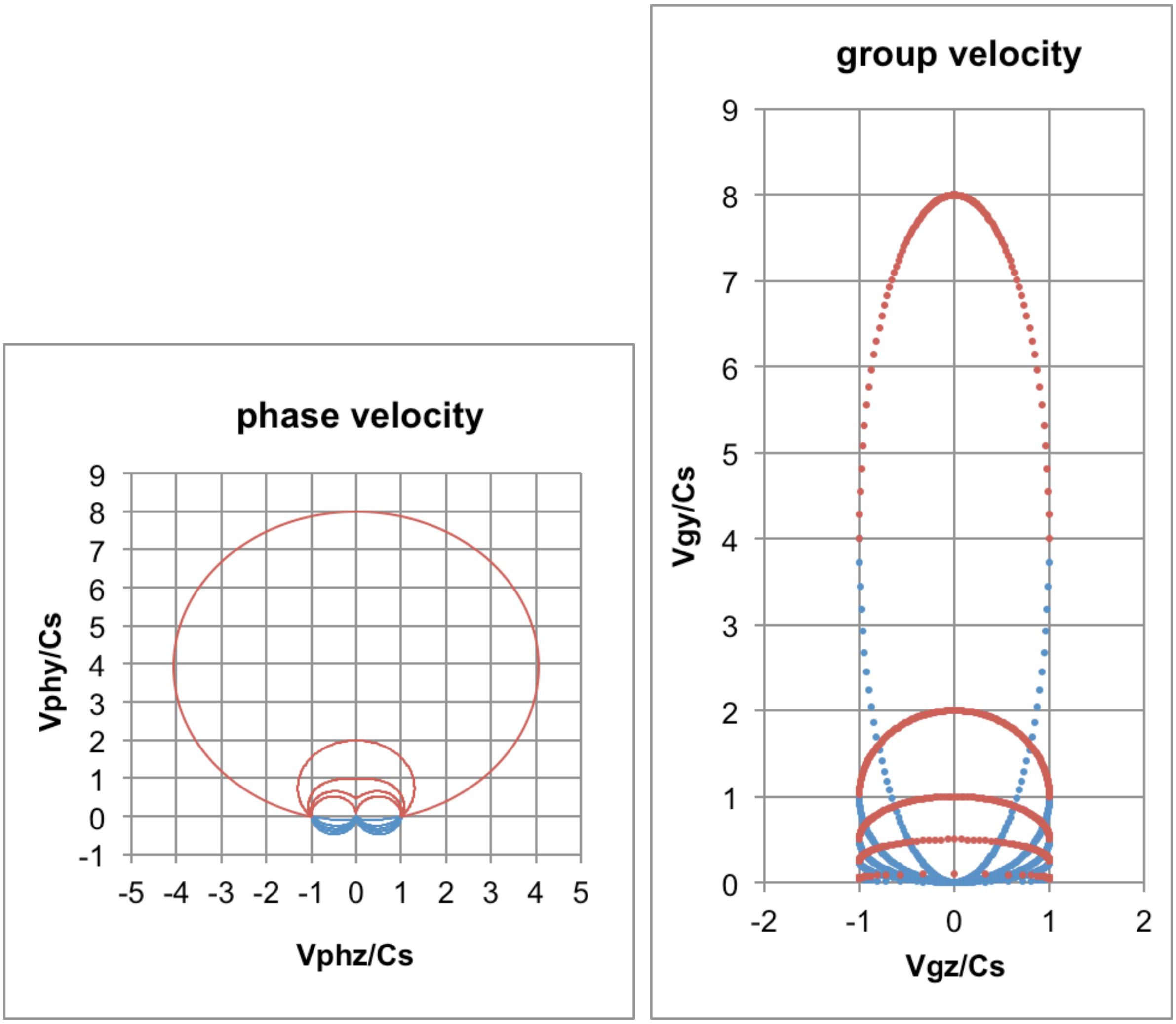

Figure 12.5. Phase velocity distribution and the group velocity distribution of the electrostatic drift waves in the ion-electron two fluid plasma similar to the one shown in Figure 12.4 but for $V_{e0}^{*}=$ 8, 2, 1, 0.5, and 0.1. The distribution of the wave group velocities indicates that the wave energy should be stronger along the major axis of the ellipse.