Chapter 11. Linear Waves in the Vlasov Plasma

Topics or concepts to learn in Chapter 11:

1. Method of characteristics (or integration over unperturbed orbits)
2. The linear kinetic dispersion relation of plasma with uniform background equilibrium.

Suggested Readings:

(1) Section 6.10~6.12 in Nicholson (1983)

(2) Sections 8.8~8.10 in Krall and Trivelpiece (1973)

(3) Section 7.10 in F. F. Chen (1984)

In this lecture, we consider both electrostatic and electromagnetic linear waves in the Vlasov plasma. Basic equations to be used in this study include the Vlasov equations of the αth species:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} = 0 \tag{11.1}$$

and Maxwell's equations

$$\nabla\cdot\mathbf{E} = -\nabla^2\Phi = \frac{e}{\varepsilon_0}(n_i - n_e) = \frac{e}{\varepsilon_0}\iiint (f_i - f_e)d^3v \tag{11.2}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{11.3}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{11.4}$$

$$\nabla\times\mathbf{B} = \mu_0 e(n_i\mathbf{V}_i - n_e\mathbf{V}_e) + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 e\iiint \mathbf{v}(f_i - f_e)d^3v + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{11.5}$$

Let us consider a uniform background plasma and field. Equilibrium distributions of plasma in a uniform background field satisfy the following conditions

$$\iiint f_{i0}d^3v = \iiint f_{e0}d^3v = n_0 \tag{11.6}$$

$$\iiint \mathbf{v}f_{i0}d^3v = \iiint \mathbf{v}f_{e0}d^3v = n_0\mathbf{V}_0 \tag{11.7}$$

and

$$f_{\alpha 0} = f_{\alpha 0}(\mathbf{v})\,. \tag{11.8}$$

Let $f_\alpha = f_{\alpha 0} + f_{\alpha 1}$, $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, where $f_{\alpha 1} << f_{\alpha 0}$, $E_1 << E_0$, and $B_1 << B_0$. The linearized Vlasov equation (11.1) can be written as

$$L_{\alpha 0} f_{\alpha 1} = -L_{\alpha 1} f_{\alpha 0} \tag{11.9}$$

where

$$L_{\alpha 0} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial}{\partial \mathbf{v}} \tag{11.10}$$

$$L_{\alpha 1} = \frac{e_\alpha}{m_\alpha}(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) \cdot \frac{\partial}{\partial \mathbf{v}} \tag{11.11}$$

are the zeroth order and the first order differential operators, respectively.

Substituting Eqs. (11.10) and (11.11) into Eq. (11.9) yields

$$[\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial}{\partial \mathbf{v}}] f_{\alpha 1} = -\frac{e_\alpha}{m_\alpha}(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} \tag{11.12}$$

Linearizing Maxwell's equations (11.2)~(11.5) yields

$$\nabla \cdot \mathbf{E}_1 = \frac{e}{\varepsilon_0} \iiint (f_{i1} - f_{e1}) d^3 v \tag{11.13}$$

$$\nabla \cdot \mathbf{B}_1 = 0 \tag{11.14}$$

$$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t} \tag{11.15}$$

$$\nabla \times \mathbf{B}_1 = \mu_0 e \iiint \mathbf{v} (f_{i1} - f_{e1}) d^3 v + \frac{1}{c^2} \frac{\partial \mathbf{E}_1}{\partial t} \tag{11.16}$$

Integration transform methods, such as Fourier transform and Laplace transform can reduce linear differential equations to algebraic equations.

11.1. Linear Waves in Field-Free Plasma ($\mathbf{E}_0 = 0, \quad \mathbf{B}_0 = 0$)

From equilibrium Maxwell's equations

$$\nabla \cdot \mathbf{E}_0 = \frac{e}{\varepsilon_0}(n_{i0} - n_{e0}) = 0$$

$$\nabla \times \mathbf{B}_0 = \mu_0 e (n_{i0} \mathbf{V}_{i0} - n_{e0} \mathbf{V}_{e0}) = 0 \text{ ,}$$

it yields

$$n_{i0} = n_{e0} = n_0 \tag{11.17}$$

$\mathbf{V}_{i0} = \mathbf{V}_{e0} = \mathbf{V}_0$

Thus, we can choose a moving frame such that

$$\mathbf{V}_{i0} = \mathbf{V}_{e0} = \mathbf{V}_0 = 0 \tag{11.18}$$

From equilibrium Vlasov equation, $\mathbf{v} \cdot (\partial f_{\alpha 0} / \partial \mathbf{x}) = 0$, it yields, $f_{\alpha 0} = f_{\alpha 0}(\mathbf{v})$.

Eqs. (11.17) and (11.18) yield,

$$\iiint f_{\alpha 0}(\mathbf{v}) d^3 v = n_0 \tag{11.19}$$

$$\iiint \mathbf{v} f_{\alpha 0}(\mathbf{v}) d^3 v = 0 \tag{11.20}$$

Let us consider plane wave solution, i.e.,

$$f_{\alpha 1}(\mathbf{v}, \mathbf{x}, t) = \tilde{f}_{\alpha 1}(\mathbf{v}, \mathbf{k}, \omega) \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{11.21}$$

$$\mathbf{E}_1(\mathbf{x}, t) = \tilde{\mathbf{E}}_1(\mathbf{k}, \omega) \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{11.22}$$

$$\mathbf{B}_1(\mathbf{x}, t) = \tilde{\mathbf{B}}_1(\mathbf{k}, \omega) \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{11.23}$$

Substituting $\mathbf{E}_0 = 0$, $\mathbf{B}_0 = 0$, and Eqs. (11.21)~(11.23) into Eqs. (11.12)~(11.16), it yields

$$[-i(\omega - \mathbf{v} \cdot \mathbf{k})] \tilde{f}_{\alpha 1} = -\frac{e_\alpha}{m_\alpha} (\tilde{\mathbf{E}}_1 + \mathbf{v} \times \tilde{\mathbf{B}}_1) \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} \tag{11.24}$$

$$i\mathbf{k} \cdot \tilde{\mathbf{E}}_1 = \frac{e}{\varepsilon_0} \iiint (\tilde{f}_{i1} - \tilde{f}_{e1}) d^3 v \tag{11.25}$$

$$i\mathbf{k} \cdot \tilde{\mathbf{B}}_1 = 0 \tag{11.26}$$

$$i\mathbf{k} \times \tilde{\mathbf{E}}_1 = i\omega \tilde{\mathbf{B}}_1 \tag{11.27}$$

$$i\mathbf{k} \times \tilde{\mathbf{B}}_1 = \mu_0 e \iiint \mathbf{v} (\tilde{f}_{i1} - \tilde{f}_{e1}) d^3 v + \frac{-i\omega}{c^2} \tilde{\mathbf{E}}_1 \tag{11.28}$$

From $i\mathbf{k} \times (4.27)$, it yields

$$i\mathbf{k} \times (i\mathbf{k} \times \tilde{\mathbf{E}}_1) = k^2 \tilde{\mathbf{E}}_1 - \mathbf{k}\mathbf{k} \cdot \tilde{\mathbf{E}}_1 = i\omega (i\mathbf{k} \times \tilde{\mathbf{B}}_1) \tag{11.29}$$

Substituting Eq. (11.28) into Eq. (11.29) to eliminate $\tilde{\mathbf{B}}_1$, it yields

$$k^2 \tilde{\mathbf{E}}_1 - \mathbf{k}\mathbf{k} \cdot \tilde{\mathbf{E}}_1 = i\omega \mu_0 e \iiint \mathbf{v} (\tilde{f}_{i1} - \tilde{f}_{e1}) d^3 v + \frac{\omega^2}{c^2} \tilde{\mathbf{E}}_1$$

or

$$\frac{\omega^2}{c^2} [(1 - \frac{c^2 k^2}{\omega^2}) \mathbf{1} + \frac{c^2}{\omega^2} \mathbf{k}\mathbf{k}] \cdot \tilde{\mathbf{E}}_1 + i \frac{\omega}{c^2} \sum_\alpha \frac{1}{\varepsilon_0} e_\alpha \iiint \mathbf{v} \tilde{f}_{\alpha 1} d^3 v = 0 \tag{11.30}$$

Substituting Eq. (11.27) into (11.24) to eliminate $\tilde{\mathbf{B}}_1$, it yields

$$\tilde{f}_{\alpha 1} = -\frac{\frac{e_\alpha}{m_\alpha}[\tilde{\mathbf{E}}_1 + \mathbf{v}\times(\frac{\mathbf{k}}{\omega}\times\tilde{\mathbf{E}}_1)]}{[-i(\omega - \mathbf{v}\cdot\mathbf{k})]}\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = -i\frac{\frac{e_\alpha}{m_\alpha}\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}\cdot[(1-\frac{\mathbf{k}\cdot\mathbf{v}}{\omega})\mathbf{1} + \frac{\mathbf{kv}}{\omega}]}{\omega - \mathbf{v}\cdot\mathbf{k}}\cdot\tilde{\mathbf{E}}_1 \tag{11.31}$$

Substituting Eq. (11.31) into Eq. (11.30) to eliminate $\tilde{f}_{\alpha 1}$, it yields

$$\frac{\omega^2}{c^2}\{(1-\frac{c^2k^2}{\omega^2})\mathbf{1} + \frac{c^2}{\omega^2}\mathbf{kk} + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}\iiint \mathbf{v}\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}\cdot\frac{[(\omega - \mathbf{k}\cdot\mathbf{v})\mathbf{1} + \mathbf{kv}]}{\omega - \mathbf{v}\cdot\mathbf{k}}d^3v\}\cdot\tilde{\mathbf{E}}_1 = 0 \tag{11.32}$$

We can choose a coordinate system such that $\mathbf{k} = \hat{\mathbf{x}}k$. Thus, Eq. (11.32) becomes

$$\frac{\omega^2}{c^2}\{(1-\frac{c^2k^2}{\omega^2})\mathbf{1} + \frac{c^2k^2}{\omega^2}\hat{\mathbf{x}}\hat{\mathbf{x}} + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}\iiint \mathbf{v}\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}\cdot\frac{[(\omega - kv_x)\mathbf{1} + k\hat{\mathbf{x}}\mathbf{v}]}{\omega - kv_x}d^3v\}\cdot\tilde{\mathbf{E}}_1 = 0 \tag{11.33}$$

or

$$\frac{\omega^2}{c^2}\mathbf{D}\cdot\tilde{\mathbf{E}}_1 = 0$$

where

$$\mathbf{D} = (1-\frac{c^2k^2}{\omega^2})\mathbf{1} + \frac{c^2k^2}{\omega^2}\hat{\mathbf{x}}\hat{\mathbf{x}} + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}\iiint \mathbf{v}\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}\cdot\frac{[(\omega - kv_x)\mathbf{1} + k\hat{\mathbf{x}}\mathbf{v}]}{\omega - kv_x}d^3v$$

$$= (1-\frac{c^2k^2}{\omega^2})\mathbf{1} + \frac{c^2k^2}{\omega^2}\hat{\mathbf{x}}\hat{\mathbf{x}} + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint \mathbf{v}\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x}\frac{k\mathbf{vv}}{\omega - kv_x}d^3v]$$

It can be shown that

$$D_{xx} = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_x\frac{\partial f_{\alpha 0}}{\partial v_x}d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x}\frac{kv_xv_x}{\omega - kv_x}d^3v]$$

$$D_{yy} = (1-\frac{c^2k^2}{\omega^2}) + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_y\frac{\partial f_{\alpha 0}}{\partial v_y}d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x}\frac{kv_yv_y}{\omega - kv_x}d^3v]$$

$$D_{zz} = (1-\frac{c^2k^2}{\omega^2}) + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_z\frac{\partial f_{\alpha 0}}{\partial v_z}d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x}\frac{kv_zv_z}{\omega - kv_x}d^3v]$$

$$D_{xy} = \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_x\frac{\partial f_{\alpha 0}}{\partial v_y}d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x}\frac{kv_xv_y}{\omega - kv_x}d^3v]$$

$$D_{xz} = \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_x\frac{\partial f_{\alpha 0}}{\partial v_z}d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x}\frac{kv_xv_z}{\omega - kv_x}d^3v]$$

$$D_{yx} = \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_y\frac{\partial f_{\alpha 0}}{\partial v_x}d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x}\frac{kv_yv_x}{\omega - kv_x}d^3v]$$

$$D_{yz} = \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_y\frac{\partial f_{\alpha 0}}{\partial v_z}d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x}\frac{kv_yv_z}{\omega - kv_x}d^3v]$$

$$D_{zx} = \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_z \frac{\partial f_{\alpha 0}}{\partial v_x} d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{kv_z v_x}{\omega - kv_x} d^3v]$$

$$D_{zy} = \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_z \frac{\partial f_{\alpha 0}}{\partial v_y} d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{kv_z v_y}{\omega - kv_x} d^3v]$$

For $f_{\alpha 0}(v_x, v_y, v_z) = \frac{n_0}{[2\pi(k_B T_{\alpha 0}/m_\alpha)]^{3/2}} \exp[-\frac{v_x^2 + v_y^2 + v_z^2}{2(k_B T_{\alpha 0}/m_\alpha)}]$. It yields

$$D_{xy} = \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_x \frac{\partial f_{\alpha 0}}{\partial v_y} d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{kv_x v_y}{\omega - kv_x} d^3v]$$

$$= \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iint v_x \, dv_x \, dv_z \int \frac{\partial f_{\alpha 0}}{\partial v_y} dv_y + \iint v_y \, dv_y \, dv_z \int_L \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{kv_x}{\omega - kv_x} dv_x]$$

$$= \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iint v_y \, dv_y \, dv_z \int_L \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{kv_x}{\omega - kv_x} dv_x]$$

$$= \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\int dv_z \int_L \frac{kv_x}{\omega - kv_x} (\frac{\partial}{\partial v_x} \int f_{\alpha 0} v_y \, dv_y) dv_x] = 0$$

Likewise, $D_{xz} = D_{yx} = D_{zx} = D_{yz} = D_{zy} = 0$. The D_{xx} must be the same as the dispersion relation obtained in Equation (9.32) in Chapter 9.

$$D_{xx} = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint v_x \frac{\partial f_{\alpha 0}}{\partial v_x} d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{kv_x v_x}{\omega - kv_x} d^3v]$$

$$= 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{v_x(\omega - kv_x) + kv_x v_x}{\omega - kv_x} d^3v] = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\iiint \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{v_x\omega}{\omega - kv_x} d^3v]$$

$$= 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2}[\int_L \frac{dF_{\alpha 0}(v_x)}{dv_x} \frac{v_x(\omega/k)}{v_x - (\omega/k)} dv_x]$$

where $\iiint f_{\alpha 0}(v_x, v_y, v_z) dv_x \, dv_y \, dv_z = \int F_{\alpha 0}(v_x) dv_x$. Let $u = \frac{v_x(\omega/k)}{v_x - (\omega/k)}$ and $dw = \frac{dF_{\alpha 0}(v_x)}{dv_x} dv_x$. It yields $du = \frac{-(\omega/k)^2 dv_x}{[v_x - (\omega/k)]^2}$ and $w = F_{\alpha 0}$. For $\omega_i \neq 0$, we have $uw|_{-\infty}^{+\infty} = 0$. But for $\omega_i = 0$, we need to consider

$$\lim_{\varepsilon \to 0}[uw|_{-\infty}^{(\omega/k)-\varepsilon} + uw|_{(\omega/k)+\varepsilon}^{+\infty}] = \lim_{\varepsilon \to 0}[uw|_{(\omega/k)+\varepsilon}^{(\omega/k)-\varepsilon}] = \lim_{\varepsilon \to 0} F_{\alpha 0}(v_x = \frac{\omega}{k})(\frac{[\frac{\omega}{k} - \varepsilon]\frac{\omega}{k}}{-\varepsilon} - \frac{[\frac{\omega}{k} + \varepsilon]\frac{\omega}{k}}{+\varepsilon})$$

$$= \lim_{\varepsilon \to 0} \frac{\omega^2}{k^2} F_{\alpha 0}(v_x = \frac{\omega}{k})(\frac{1}{-\varepsilon} - \frac{1}{+\varepsilon}) = \lim_{\varepsilon \to 0} \frac{\omega^2}{k^2} F_{\alpha 0}(v_x = \frac{\omega}{k})[\frac{1}{v_x - (\omega/k)}]_{(\omega/k)+\varepsilon}^{(\omega/k)-\varepsilon}$$

$$= -\frac{\omega^2}{k^2} F_{\alpha 0}(v_x = \frac{\omega}{k}) p \int \frac{1}{[v_x - (\omega/k)]^2} dv_x$$

Thus, the integration by parts of the Landau integration yields

$$D_{xx} = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2} \begin{cases} \int_{-\infty}^{+\infty} F_{\alpha 0} \dfrac{(\omega/k)^2}{[v_x - (\omega/k)]^2} dv_x & \text{if } \omega_i > 0 \\ p\int [F_{\alpha 0}(v_x) - F_{\alpha 0}(v_x = \dfrac{\omega}{k})] \dfrac{(\omega/k)^2}{[v_x - (\omega/k)]^2} dv_x \quad + \pi i (\dfrac{\omega^2}{k^2}) \dfrac{dF_{\alpha 0}}{dv_x}\Big|_{v_x = \frac{\omega}{k}} & \text{if } \omega_i = 0 \\ \int_{-\infty}^{+\infty} F_{\alpha 0} \dfrac{(\omega/k)^2}{[v_x - (\omega/k)]^2} dv_x \quad + 2\pi i (\dfrac{\omega^2}{k^2}) \dfrac{dF_{\alpha 0}}{dv_x}\Big|_{v_x = \frac{\omega}{k}} & \text{if } \omega_i < 0 \end{cases}$$

$$= 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0 k^2} [\int_L \frac{dF_{\alpha 0}}{dv_x} \frac{1}{v_x - (\omega/k)} dv_x]$$

$$D_{yy} = (1 - \frac{c^2k^2}{\omega^2}) + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2} [\iiint v_y \frac{\partial f_{\alpha 0}}{\partial v_y} d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{k v_y v_y}{\omega - k v_x} d^3v]$$

$$= 1 - \frac{c^2k^2}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2} [-n_0 - \int_L \frac{k}{kv_x - \omega} (\iint \frac{\partial f_{\alpha 0}}{\partial v_x} v_y^2\, dv_y\, dv_z) dv_x]$$

$$= 1 - \frac{c^2k^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} (1 + \frac{k_B T_{\alpha 0}}{n_0 m_\alpha} \int_L \frac{\partial F_{\alpha 0}}{\partial v_x} \frac{1}{v_x - (\omega/k)} dv_x)$$

Likewise, we can show that

$$D_{zz} = (1 - \frac{c^2k^2}{\omega^2}) + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2} [\iiint v_z \frac{\partial f_{\alpha 0}}{\partial v_z} d^3v + \iiint \frac{\partial f_{\alpha 0}}{\partial v_x} \frac{k v_z v_z}{\omega - k v_x} d^3v]$$

$$= 1 - \frac{c^2k^2}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0\omega^2} [-n_0 - \int_L \frac{k}{kv_x - \omega} (\iint \frac{\partial f_{\alpha 0}}{\partial v_x} v_z^2\, dv_y\, dv_z) dv_x]$$

$$= 1 - \frac{c^2k^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} (1 + \frac{k_B T_{\alpha 0}}{n_0 m_\alpha} \int_L \frac{\partial F_{\alpha 0}}{\partial v_x} \frac{1}{v_x - (\omega/k)} dv_x)$$

Since the wave speed is approximately equal to the speed of light, if the thermal speed is much less than the speed of light, we can assume that $T_{\alpha 0} \approx 0$, which yields

$$D_{yy} = D_{zz} = 1 - \frac{c^2k^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \tag{11.34}$$

Equation (11.34) is the same as the equation (8.9.14) in Krall & Trivelpiece (1973).

In summary, for field-free plasma ($\mathbf{E}_0 = \mathbf{B}_0 = 0$) and $\mathbf{k} = \hat{\mathbf{x}}k$, we have $\mathbf{D} \cdot \tilde{\mathbf{E}}_1 = 0$, where

$$D_{xx} = 1 - \frac{1}{k^2} \sum_\alpha \frac{\omega_{p\alpha}^2}{n_0} [\int_L \frac{\partial F_{\alpha 0}}{\partial v_x} \frac{1}{v_x - (\omega/k)} dv_x] \tag{11.35}$$

$$D_{yy} = D_{zz} = 1 - \frac{c^2k^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} (1 + \frac{k_B T_{\alpha 0}}{n_0 m_\alpha} \int_L \frac{\partial F_{\alpha 0}}{\partial v_x} \frac{1}{v_x - (\omega/k)} dv_x) \tag{11.36}$$

$$D_{xy} = D_{xz} = D_{yx} = D_{yz} = D_{zx} = D_{zy} = 0 \tag{11.37}$$

11.2. Linear Waves in Magnetized Plasma With Uniform Background $\mathbf{B}_0$

From equilibrium Maxwell's equations

$$\nabla \cdot \mathbf{E}_0 = \frac{e}{\varepsilon_0}(n_{i0} - n_{e0}) = 0$$

$$\nabla \times \mathbf{B}_0 = \mu_0 e(n_{i0}\mathbf{V}_{i0} - n_{e0}\mathbf{V}_{e0}) = 0,$$

it yields

$$n_{i0} = n_{e0} = n_0 \tag{11.17}$$

$$\mathbf{V}_{i0} = \mathbf{V}_{e0} = \mathbf{V}_0$$

Thus, we can choose a moving frame such that

$$\mathbf{V}_{i0} = \mathbf{V}_{e0} = \mathbf{V}_0 = 0 \tag{11.18}$$

From equilibrium Vlasov equation

$$[\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial}{\partial \mathbf{v}}] f_{\alpha 0} = 0,$$

it yields,

$$f_{\alpha 0} = f_{\alpha 0}(v_\perp^2, v_\parallel)$$

Eqs. (11.17) and (11.18) yield,

$$\iiint f_{\alpha 0}(v_\perp^2, v_\parallel) d^3 v = n_0 \tag{11.19a}$$

$$\iiint \mathbf{v} f_{\alpha 0}(v_\perp^2, v_\parallel) d^3 v = 0 \tag{11.20a}$$

Let us consider plane wave solution, i.e.,

$$f_{\alpha 1}(\mathbf{v}, \mathbf{x}, t) = \tilde{f}_{\alpha 1}(\mathbf{v}, \mathbf{k}, \omega) \exp[-i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{11.21}$$

$$\mathbf{E}_1(\mathbf{x}, t) = \tilde{\mathbf{E}}_1(\mathbf{k}, \omega) \exp[-i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{11.22}$$

$$\mathbf{B}_1(\mathbf{x}, t) = \tilde{\mathbf{B}}_1(\mathbf{k}, \omega) \exp[-i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{11.23}$$

Substituting $\mathbf{E}_0 = 0$, $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$, and equations (11.21)~(11.23) into equation (11.12), it yields

$$[-i(\omega - \mathbf{v} \cdot \mathbf{k}) + \frac{e_\alpha}{m_\alpha}(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial}{\partial \mathbf{v}}] \tilde{f}_{\alpha 1} = -\frac{e_\alpha}{m_\alpha}(\tilde{\mathbf{E}}_1 + \mathbf{v} \times \tilde{\mathbf{B}}_1) \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} \tag{11.24a}$$

As we can see, we have removed the two differential operators $\frac{\partial}{\partial t}$ and $\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}}$ from equation (11.12) by taking the Fourier transfer and Laplace transfer of equation (11.12).

But the differential operator $\frac{e_\alpha}{m_\alpha}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial}{\partial\mathbf{v}}$ in equation (11.24a) cannot be removed due to nonzero $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. The solution of $\tilde{f}_{\alpha 1}$ in equation (11.24a) can be written as

$$\tilde{f}_{\alpha 1} = [-i(\omega - \mathbf{v}\cdot\mathbf{k}) + \frac{e_\alpha}{m_\alpha}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial}{\partial\mathbf{v}}]^{-1} - \frac{e_\alpha}{m_\alpha}(\tilde{\mathbf{E}}_1 + \mathbf{v}\times\tilde{\mathbf{B}}_1)\cdot\frac{\partial f_{\alpha 0}}{\partial\mathbf{v}} \tag{11.31a}$$

It is difficult to find analytic solution of $\tilde{f}_{\alpha 1}$ from either equation (11.24a) or equation (11.31a). To overcome this difficulty, we shall introduce a method of characteristics, also called "integrating over unperturbed orbit," to help us to solve this type of problems.

Define new variables $\mathbf{x}'(t')$, $\mathbf{v}'(t')$, and t', which satisfy the following ordinary differential equations,

$$\frac{d\mathbf{x}'}{dt'} = \mathbf{v}' \tag{11.38}$$

$$\frac{d\mathbf{v}'}{dt'} = \frac{e_\alpha}{m_\alpha}(\mathbf{E}_0 + \mathbf{v}'\times\mathbf{B}_0) \tag{11.39}$$

with boundary conditions

$$\mathbf{x}'(t'=t) = \mathbf{x}, \text{ and } \mathbf{v}'(t'=t) = \mathbf{v}. \tag{11.40}$$

It yields

$$\frac{df_{\alpha 1}}{dt'} = \frac{\partial f_{\alpha 1}}{\partial t'} + \frac{\partial f_{\alpha 1}}{\partial\mathbf{x}'}\cdot\frac{d\mathbf{x}'}{dt'} + \frac{\partial f_{\alpha 1}}{\partial\mathbf{v}'}\cdot\frac{d\mathbf{v}'}{dt'} = [\frac{\partial}{\partial t'} + \mathbf{v}'\cdot\frac{\partial}{\partial\mathbf{x}'} + \frac{e_\alpha}{m_\alpha}(\mathbf{E}_0 + \mathbf{v}'\times\mathbf{B}_0)\cdot\frac{\partial}{\partial\mathbf{v}'}]f_{\alpha 1} \tag{11.41}$$

Equations (11.12) and (11.41) yield

$$\frac{df_{\alpha 1}}{dt'} = [\frac{\partial}{\partial t'} + \mathbf{v}'\cdot\frac{\partial}{\partial\mathbf{x}'} + \frac{e_\alpha}{m_\alpha}(\mathbf{E}_0 + \mathbf{v}'\times\mathbf{B}_0)\cdot\frac{\partial}{\partial\mathbf{v}'}]f_{\alpha 1} = -\frac{e_\alpha}{m_\alpha}(\mathbf{E}_1 + \mathbf{v}'\times\mathbf{B}_1)\cdot\frac{\partial f_{\alpha 0}}{\partial\mathbf{v}'} \tag{11.42}$$

In contrast,

$$\frac{df_{\alpha 0}}{dt'} = [\mathbf{v}'\cdot\frac{\partial}{\partial\mathbf{x}'} + \frac{e_\alpha}{m_\alpha}(\mathbf{E}_0 + \mathbf{v}'\times\mathbf{B}_0)\cdot\frac{\partial}{\partial\mathbf{v}'}]f_{\alpha 0} = 0 \tag{11.43}$$

The orbits, which satisfy equations (11.38) and (11.39), are the characteristic curves of the equilibrium distribution function $f_{\alpha 0}$. We can conclude from equations (11.43) and (11.42) that the magnitude of $f_{\alpha 0}$ is constant along each characteristic curve, but the magnitude of $f_{\alpha 1}$ vary with time along each characteristic curve of $f_{\alpha 0}$. That is

$$\int_{-\infty}^{t} dt'\frac{df_{\alpha 1}}{dt'} = \int_{-\infty}^{t} dt'[-\frac{e_\alpha}{m_\alpha}(\mathbf{E}_1 + \mathbf{v}'\times\mathbf{B}_1)\cdot\frac{\partial f_{\alpha 0}}{\partial\mathbf{v}'}] \tag{11.44}$$

We choose the boundary conditions, such that

$$f_{\alpha 1}\big|_{t'\to-\infty} = 0, \quad \mathbf{x}'(t'=t) = \mathbf{x}, \text{ and } \mathbf{v}'(t'=t) = \mathbf{v} \tag{11.45}$$

Substituting the boundary conditions into equation (11.44), it yields

$$f_{\alpha 1}(\mathbf{x},\mathbf{v},t)=\int_{-\infty}^{t}dt'[-\frac{e_\alpha}{m_\alpha}(\mathbf{E}_1+\mathbf{v}'\times\mathbf{B}_1)\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}'}] \quad (11.46)$$

Now, we can apply the Fourier transfer and Laplace transfer on equation (11.46). It yields

$$\tilde{f}_{\alpha 1}(\mathbf{k},\mathbf{v},\omega)\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]=\int_{-\infty}^{t}dt'\exp[i(\mathbf{k}\cdot\mathbf{x}'-\omega t')][-\frac{e_\alpha}{m_\alpha}(\tilde{\mathbf{E}}_1+\mathbf{v}'\times\tilde{\mathbf{B}}_1)\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}'}] \quad (11.47)$$

or

$$\tilde{f}_{\alpha 1}(\mathbf{k},\mathbf{v},\omega)=\int_{-\infty}^{t}dt'\exp\{i[\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})-\omega(t'-t)]\}[-\frac{e_\alpha}{m_\alpha}(\tilde{\mathbf{E}}_1+\mathbf{v}'\times\tilde{\mathbf{B}}_1)\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}'}] \quad (11.48)$$

Let

$$\mathbf{x}'-\mathbf{x}=\mathbf{X} \quad (11.49)$$

$$t'-t=\tau \quad (11.50)$$

Equation (11.48) can be rewritten as

$$\tilde{f}_{\alpha 1}(\mathbf{k},\mathbf{v},\omega)=\int_{-\infty}^{0}d\tau\exp[i(\mathbf{k}\cdot\mathbf{X}-\omega\tau)]\{-\frac{e_\alpha}{m_\alpha}\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}'}\cdot[(1-\frac{\mathbf{k}\cdot\mathbf{v}'}{\omega})\mathbf{1}+\frac{\mathbf{k}\mathbf{v}'}{\omega}]\cdot\tilde{\mathbf{E}}_1\} \quad (11.51)$$

where equation (11.27) has been used to eliminate $\tilde{\mathbf{B}}_1$ in equation (11.48).

Solving equations (11.38) and (11.39) with boundary conditions given in (11.40), it yields

$$v'_x=v_\perp\cos(\phi-\Omega_{c\alpha}\tau) \quad (11.52)$$

$$v'_y=v_\perp\sin(\phi-\Omega_{c\alpha}\tau) \quad (11.53)$$

$$v'_z=v_\parallel \quad (11.54)$$

$$\mathbf{X}\cdot\mathbf{e}_x=x'-x=\frac{v_\perp}{\Omega_{c\alpha}}[-\sin(\phi-\Omega_{c\alpha}\tau)+\sin\phi] \quad (11.55)$$

$$\mathbf{X}\cdot\mathbf{e}_y=y'-y=\frac{v_\perp}{\Omega_{c\alpha}}[+\cos(\phi-\Omega_{c\alpha}\tau)-\cos\phi] \quad (11.56)$$

$$\mathbf{X}\cdot\mathbf{e}_z=z'-z=v_\parallel\tau \quad (11.57)$$

where equations (11.49) and (11.50) have been used to eliminate $\mathbf{x}'-\mathbf{x}$ and $t'-t$ in the above equations (11.52)~(11.57). Note that, as discussed below, the $\Omega_{c\alpha}$ in equations (11.52), (11.53), (11.55), and (11.56) is defined by $\Omega_{c\alpha}=e_\alpha B_0/m_\alpha$. Namely $\Omega_{c\alpha}<0$ if $e_\alpha<0$.

Proof equations (11.52), (11.53), (11.55), and (11.56):

$$\frac{dv'_x}{dt'}=\frac{e_\alpha}{m_\alpha}v'_y B_0$$

$$\frac{dv'_y}{dt'} = -\frac{e_\alpha}{m_\alpha} v'_x B_0$$

$$\Rightarrow \frac{d^2 v'_x}{dt'^2} = -(\frac{e_\alpha}{m_\alpha} B_0)^2 v'_x$$

$$\Rightarrow v'_x = C_1 \cos\Omega_\alpha t' + C_2 \sin\Omega_\alpha t'$$

where $\Omega_\alpha = \sqrt{(e_\alpha B_0 / m_\alpha)^2} = |e_\alpha B_0 / m_\alpha|$

$$\Rightarrow v'_y = (\frac{e_\alpha}{m_\alpha} B_0)^{-1} \frac{dv'_x}{dt'} = \frac{e_\alpha}{|e_\alpha|}(-C_1 \sin\Omega_\alpha t' + C_2 \cos\Omega_\alpha t')$$

The boundary conditions are

$$v'_x(t' = t) = v_x = v_\perp \cos\phi = C_1 \cos\Omega_\alpha t + C_2 \sin\Omega_\alpha t$$

$$v'_y(t' = t) = v_y = v_\perp \sin\phi = \frac{e_\alpha}{|e_\alpha|}(-C_1 \sin\Omega_\alpha t + C_2 \cos\Omega_\alpha t)$$

Solve for C_1 and C_2, it yields

$$C_1 = v_\perp \cos\Omega_\alpha t \cos\phi - \frac{e_\alpha}{|e_\alpha|} v_\perp \sin\Omega_\alpha t \sin\phi = v_\perp \cos(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t)$$

$$C_2 \frac{e_\alpha}{|e_\alpha|} = \frac{e_\alpha}{|e_\alpha|} v_\perp \sin\Omega_\alpha t \cos\phi + v_\perp \cos\Omega_\alpha t \sin\phi = v_\perp \sin(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t)$$

or

$$C_2 = \frac{e_\alpha}{|e_\alpha|} v_\perp \sin(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t)$$

Substituting C_1 and C_2 into v'_x and v'_y, it yields

$$\begin{aligned}
v'_x &= C_1 \cos\Omega_\alpha t' + C_2 \sin\Omega_\alpha t' \\
&= v_\perp \cos(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t)\cos\Omega_\alpha t' + \frac{e_\alpha}{|e_\alpha|} v_\perp \sin(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t)\sin\Omega_\alpha t' \\
&= v_\perp \cos(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t - \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t') \\
&= v_\perp \cos[\phi - \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha (t' - t)]
\end{aligned}$$

$$v'_y = \frac{e_\alpha}{|e_\alpha|}(-C_1 \sin\Omega_\alpha t' + C_2 \cos\Omega_\alpha t')$$

$$= \frac{e_\alpha}{|e_\alpha|}[-v_\perp \cos(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t)\sin\Omega_\alpha t' + \frac{e_\alpha}{|e_\alpha|} v_\perp \sin(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t)\cos\Omega_\alpha t']$$

$$= v_\perp \sin(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t)\cos\Omega_\alpha t' - \frac{e_\alpha}{|e_\alpha|} v_\perp \cos(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t)\sin\Omega_\alpha t'$$

$$= v_\perp \sin(\phi + \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t - \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha t')$$

$$= v_\perp \sin[\phi - \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha (t' - t)]$$

Substituting v'_x and v'_y into the following equations,

$$\frac{dx'}{dt'} = v'_x \text{ and } \frac{dy'}{dt'} = v'_y$$

it yields

$$x' = -\frac{e_\alpha}{|e_\alpha|}\frac{v_\perp}{\Omega_\alpha}\sin[\phi - \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha (t' - t)] + C_3$$

$$y' = \frac{e_\alpha}{|e_\alpha|}\frac{v_\perp}{\Omega_\alpha}\cos[\phi - \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha (t' - t)] + C_4$$

The integration constants C_3 and C_4 can be solved from the boundary conditions.

$$x'(t' = t) = x = -\frac{e_\alpha}{|e_\alpha|}\frac{v_\perp}{\Omega_\alpha}\sin\phi + C_3$$

$$\Rightarrow C_3 = x + \frac{e_\alpha}{|e_\alpha|}\frac{v_\perp}{\Omega_\alpha}\sin\phi$$

$$y'(t' = t) = y = \frac{e_\alpha}{|e_\alpha|}\frac{v_\perp}{\Omega_\alpha}\cos\phi + C_4$$

$$\Rightarrow C_4 = y - \frac{e_\alpha}{|e_\alpha|}\frac{v_\perp}{\Omega_\alpha}\cos\phi$$

Substituting C_3 and C_4 into the equations of x' and y', it yields

$$x' - x = \frac{e_\alpha}{|e_\alpha|}\frac{v_\perp}{\Omega_\alpha}\{-\sin[\phi - \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha (t' - t)] + \sin\phi\}$$

$$y' - y = \frac{e_\alpha}{|e_\alpha|}\frac{v_\perp}{\Omega_\alpha}\{\cos[\phi - \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha (t' - t)] - \cos\phi\}$$

where $\Omega_\alpha = \sqrt{(e_\alpha B_0 / m_\alpha)^2} = |e_\alpha B_0 / m_\alpha|$

For convenience, we shall define

$$\Omega_{c\alpha} = \frac{e_\alpha}{|e_\alpha|}\Omega_\alpha = \frac{e_\alpha B_0}{m_\alpha}$$

Thus, we have

$$v'_x = v_\perp \cos[\phi - \Omega_{c\alpha}(t'-t)]$$

$$v'_y = v_\perp \sin[\phi - \Omega_{c\alpha}(t'-t)]$$

$$x'-x = \frac{v_\perp}{\Omega_{c\alpha}}\{-\sin[\phi - \Omega_{c\alpha}(t'-t)] + \sin\phi\}$$

$$y'-y = \frac{v_\perp}{\Omega_{c\alpha}}\{\cos[\phi - \Omega_{c\alpha}(t'-t)] - \cos\phi\}$$

where $\Omega_{c\alpha} < 0$ if $e_\alpha < 0$.

For $f_{\alpha 0} = f_{\alpha 0}(v_\perp^2, v_\parallel)$, it yields

$$\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}'} = \mathbf{e}_x \frac{\partial f_{\alpha 0}}{\partial v'_x} + \mathbf{e}_y \frac{\partial f_{\alpha 0}}{\partial v'_y} + \mathbf{e}_z \frac{\partial f_{\alpha 0}}{\partial v'_z} = \mathbf{e}_x \frac{\partial f_{\alpha 0}}{\partial v_\perp^2}\frac{\partial v_\perp^2}{\partial v'_x} + \mathbf{e}_y \frac{\partial f_{\alpha 0}}{\partial v_\perp^2}\frac{\partial v_\perp^2}{\partial v'_y} + \mathbf{e}_z \frac{\partial f_{\alpha 0}}{\partial v_\parallel}$$

or

$$\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}'} = 2(\mathbf{e}_x v'_x + \mathbf{e}_y v'_y)\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + 2\mathbf{e}_z v_\parallel \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2} \tag{11.58}$$

Substituting equation (11.58) into equation (11.51), it yields

$$\tilde{f}_{\alpha 1}(\mathbf{k},\mathbf{v},\omega)$$
$$= \int_{-\infty}^{0} d\tau \exp[i(\mathbf{k}\cdot\mathbf{X} - \omega\tau)]\{-\frac{e_\alpha}{m_\alpha}[2(\mathbf{e}_x v'_x + \mathbf{e}_y v'_y)\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + 2\mathbf{e}_z v_\parallel \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}]\cdot[(1-\frac{\mathbf{k}\cdot\mathbf{v}'}{\omega})\mathbf{1} + \frac{\mathbf{k}\mathbf{v}'}{\omega}]\cdot\tilde{\mathbf{E}}_1\} \tag{11.59}$$

For convenience, we choose the coordinate system such that $\mathbf{k}$ lies on the x-z plane.

Define

$$\mathbf{k} = \mathbf{e}_x k_\perp + \mathbf{e}_z k_\parallel. \tag{11.60}$$

Substituting equation (11.60) and equations (11.52)~(11.54) into the expression

$\{-\frac{e_\alpha}{m_\alpha}[2(\mathbf{e}_x v'_x + \mathbf{e}_y v'_y)\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + 2\mathbf{e}_z v_\parallel \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}]\cdot[(1-\frac{\mathbf{k}\cdot\mathbf{v}'}{\omega})\mathbf{1} + \frac{\mathbf{k}\mathbf{v}'}{\omega}]\cdot\tilde{\mathbf{E}}_1\}$, it yields

$$\{-\frac{e_\alpha}{m_\alpha}[2(\mathbf{e}_x v_x' + \mathbf{e}_y v_y')\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + 2\mathbf{e}_z v_\parallel \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}]\cdot[(1-\frac{\mathbf{k}\cdot\mathbf{v}'}{\omega})\mathbf{1} + \frac{\mathbf{k}\mathbf{v}'}{\omega}]\cdot\tilde{\mathbf{E}}_1\}$$

$$= -2\frac{e_\alpha}{m_\alpha}[v_x'\frac{\partial f_{\alpha 0}}{\partial v_\perp^2},\quad v_y'\frac{\partial f_{\alpha 0}}{\partial v_\perp^2},\quad v_\parallel\frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}]\cdot\begin{bmatrix} 1-\frac{k_\parallel v_\parallel}{\omega} & \frac{k_\perp v_y'}{\omega} & \frac{k_\perp v_\parallel}{\omega} \\ 0 & 1-\frac{k_\perp v_x'}{\omega}-\frac{k_\parallel v_\parallel}{\omega} & 0 \\ \frac{k_\parallel v_x'}{\omega} & \frac{k_\parallel v_y'}{\omega} & 1-\frac{k_\perp v_x'}{\omega}\end{bmatrix}\cdot\begin{bmatrix}\tilde{E}_{1x}\\ \tilde{E}_{1y}\\ \tilde{E}_{1z}\end{bmatrix} \quad (11.61)$$

$$= -2\frac{e_\alpha}{m_\alpha}[U_\alpha,\quad V_\alpha,\quad W_\alpha]\begin{bmatrix}\tilde{E}_{1x}\\ \tilde{E}_{1y}\\ \tilde{E}_{1z}\end{bmatrix}$$

where

$$U_\alpha = v_x'\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(1-\frac{k_\parallel v_\parallel}{\omega}) + v_\parallel\frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}\frac{k_\parallel v_x'}{\omega} = v_x'\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + v_x'\frac{k_\parallel v_\parallel}{\omega}(\frac{\partial f_{\alpha 0}}{\partial v_\parallel^2} - \frac{\partial f_{\alpha 0}}{\partial v_\perp^2})$$

$$V_\alpha = v_x'\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}\frac{k_\perp v_y'}{\omega} + v_y'\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(1-\frac{k_\perp v_x'}{\omega}-\frac{k_\parallel v_\parallel}{\omega}) + v_\parallel\frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}\frac{k_\parallel v_y'}{\omega} = v_y'\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + v_y'\frac{k_\parallel v_\parallel}{\omega}(\frac{\partial f_{\alpha 0}}{\partial v_\parallel^2} - \frac{\partial f_{\alpha 0}}{\partial v_\perp^2})$$

$$W_\alpha = v_x'\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}\frac{k_\perp v_\parallel}{\omega} + v_\parallel\frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}(1-\frac{k_\perp v_x'}{\omega}) = v_x'\frac{k_\perp v_\parallel}{\omega}(\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} - \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}) + v_\parallel\frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}$$

Equation (11.61) can be rewritten as

$$\{-\frac{e_\alpha}{m_\alpha}[2(\mathbf{e}_x v_x' + \mathbf{e}_y v_y')\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + 2\mathbf{e}_z v_\parallel \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}]\cdot[(1-\frac{\mathbf{k}\cdot\mathbf{v}'}{\omega})\mathbf{1} + \frac{\mathbf{k}\mathbf{v}'}{\omega}]\cdot\tilde{\mathbf{E}}_1\}$$

$$= -2\frac{e_\alpha}{m_\alpha}[v_x'(\tau),\quad v_y'(\tau),\quad v_\parallel]\cdot\begin{bmatrix}\chi_\alpha & 0 & \gamma_\alpha\\ 0 & \chi_\alpha & 0\\ 0 & 0 & \varepsilon_\alpha\end{bmatrix}\cdot\begin{bmatrix}\tilde{E}_{1x}\\ \tilde{E}_{1y}\\ \tilde{E}_{1z}\end{bmatrix} \quad (11.62)$$

where

$$\chi_\alpha = \frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + \frac{k_\parallel v_\parallel}{\omega}(\frac{\partial f_{\alpha 0}}{\partial v_\parallel^2} - \frac{\partial f_{\alpha 0}}{\partial v_\perp^2})$$

$$\gamma_\alpha = \frac{k_\perp v_\parallel}{\omega}(\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} - \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2})$$

$$\varepsilon_\alpha = \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}$$

Substituting equation (11.62) into equation (11.59), it yields

$$\tilde{f}_{\alpha 1}(\mathbf{k},\mathbf{v},\omega) = -2\frac{e_\alpha}{m_\alpha}\{\int_{-\infty}^{0} d\tau \exp[i(\mathbf{k}\cdot\mathbf{X}-\omega\tau)][v_x'(\tau),\quad v_y'(\tau),\quad v_\parallel]\}\cdot\begin{bmatrix}\chi_\alpha & 0 & \gamma_\alpha\\ 0 & \chi_\alpha & 0\\ 0 & 0 & \varepsilon_\alpha\end{bmatrix}\cdot\begin{bmatrix}\tilde{E}_{1x}\\ \tilde{E}_{1y}\\ \tilde{E}_{1z}\end{bmatrix}$$

(11.63)

If we rewrite equation (11.63) into the following form,

$$\tilde{f}_{\alpha 1}(\mathbf{k},\mathbf{v},\omega)=\int_{-\infty}^{0}d\tau\exp[i(\mathbf{k}\cdot\mathbf{X}-\omega\tau)]\{-\frac{e_\alpha}{m_\alpha}[2(Xv'_x+Yv'_y+Zv_\parallel)]\} \tag{11.63a}$$

Then, we have

$$X=\chi_\alpha\tilde{E}_{1x}+\gamma_\alpha\tilde{E}_{1z}$$

$$Y=\chi_\alpha\tilde{E}_{1y}$$

$$Z=\varepsilon_\alpha\tilde{E}_{1z}$$

Substituting equations (11.55)~(11.57) into the expression $\exp[i(\mathbf{k}\cdot\mathbf{X}-\omega\tau)]$, it yields

$$\begin{aligned}&\exp[i(\mathbf{k}\cdot\mathbf{X}-\omega\tau)]\\&=\exp\{i[\frac{k_\perp v_\perp}{\Omega_{c\alpha}}[-\sin(\phi-\Omega_{c\alpha}\tau)+\sin\phi]-(\omega-k_\parallel v_\parallel)\tau]\}\\&=\exp[-i\frac{k_\perp v_\perp}{\Omega_{c\alpha}}\sin(\phi-\Omega_{c\alpha}\tau)]\exp[i\frac{k_\perp v_\perp}{\Omega_{c\alpha}}\sin\phi]\exp[-i(\omega-k_\parallel v_\parallel)\tau]\\&=[\sum_{l=-\infty}^{+\infty}e^{-il(\phi-\Omega_{c\alpha}\tau)}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})][\sum_{n=-\infty}^{+\infty}e^{in\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})][e^{-i(\omega-k_\parallel v_\parallel)\tau}]\end{aligned} \tag{11.64}$$

where the following generating function of the Bessel function has been used to obtain equation (11.64).

$$\sum_{n=-\infty}^{+\infty}e^{in\theta}J_n(z)=\exp[iz\sin\theta] \tag{11.65}$$

Differentiating equation (11.65) once with respect to z, it yields

$$2J'_n(z)=J_{n-1}(z)-J_{n+1}(z) \tag{11.65a}$$

Differentiating equation (11.65) once with respect to θ, it yields

$$\frac{2n}{z}J_n(z)=J_{n-1}(z)+J_{n+1}(z) \tag{11.65b}$$

Equations (11.52) and (11.53) can be rewritten as

$$v'_x=v_\perp\cos(\phi-\Omega_{c\alpha}\tau)=v_\perp\frac{e^{i(\phi-\Omega_{c\alpha}\tau)}+e^{-i(\phi-\Omega_{c\alpha}\tau)}}{2} \tag{11.52a}$$

$$v'_y=v_\perp\sin(\phi-\Omega_{c\alpha}\tau)=v_\perp\frac{e^{i(\phi-\Omega_{c\alpha}\tau)}-e^{-i(\phi-\Omega_{c\alpha}\tau)}}{2i} \tag{11.53a}$$

Substituting equations (11.64), (11.52a), and (11.53a) into the expression $\exp[i(\mathbf{k}\cdot\mathbf{X}-\omega\tau)][v'_x(\tau),\quad v'_y(\tau),\quad v_\parallel]$, it yields

$$\exp[i(\mathbf{k}\cdot\mathbf{X}-\omega\tau)][v'_x(\tau),\quad v'_y(\tau),\quad v_\parallel]$$

$$=[\sum_{l=-\infty}^{+\infty}e^{-il(\phi-\Omega_{c\alpha}\tau)}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})][\sum_{n=-\infty}^{+\infty}e^{in\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})][e^{-i(\omega-k_\parallel v_\parallel)\tau}]$$

$$[v_\perp\frac{e^{i(\phi-\Omega_{c\alpha}\tau)}+e^{-i(\phi-\Omega_{c\alpha}\tau)}}{2},\quad v_\perp\frac{e^{i(\phi-\Omega_{c\alpha}\tau)}-e^{-i(\phi-\Omega_{c\alpha}\tau)}}{2i},\quad v_\parallel]$$

$$=\begin{bmatrix}\frac{v_\perp}{2}\{\sum_{l=-\infty}^{+\infty}e^{-i(l-1)(\phi-\Omega_{c\alpha}\tau)}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})+\sum_{l=-\infty}^{+\infty}e^{-i(l+1)(\phi-\Omega_{c\alpha}\tau)}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})\}\\ \frac{v_\perp}{2i}\{\sum_{l=-\infty}^{+\infty}e^{-i(l-1)(\phi-\Omega_{c\alpha}\tau)}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})-\sum_{l=-\infty}^{+\infty}e^{-i(l+1)(\phi-\Omega_{c\alpha}\tau)}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})\}\\ v_\parallel\sum_{l=-\infty}^{+\infty}e^{-il(\phi-\Omega_{c\alpha}\tau)}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})\end{bmatrix}^T[\sum_{n=-\infty}^{+\infty}e^{in\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})][e^{-i(\omega-k_\parallel v_\parallel)\tau}]$$

$$=\begin{bmatrix}\frac{v_\perp}{2}\sum_{l=-\infty}^{+\infty}e^{-il(\phi-\Omega_{c\alpha}\tau)}\{J_{l+1}(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})+J_{l-1}(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})\}\\ \frac{v_\perp}{2i}\sum_{l=-\infty}^{+\infty}e^{-il(\phi-\Omega_{c\alpha}\tau)}\{J_{l+1}(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})-J_{l-1}(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})\}\\ v_\parallel\sum_{l=-\infty}^{+\infty}e^{-il(\phi-\Omega_{c\alpha}\tau)}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})\end{bmatrix}^T[\sum_{n=-\infty}^{+\infty}e^{in\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})][e^{-i(\omega-k_\parallel v_\parallel)\tau}]$$

$$=\sum_{l=-\infty}^{+\infty}e^{-i(\omega-k_\parallel v_\parallel-l\Omega_{c\alpha})\tau}[\frac{v_\perp}{2}\{J_{l+1}(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})+J_{l-1}(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})\},\qquad \frac{v_\perp}{2i}\{J_{l+1}(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})-J_{l-1}(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})\},$$
$$v_\parallel J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})][\sum_{n=-\infty}^{+\infty}e^{i(n-l)\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]\tag{11.66}$$

Substituting equations (11.65a) and (11.65b) into equation (11.66), it yields

$$\exp[i(\mathbf{k}\cdot\mathbf{X}-\omega\tau)][v'_x(\tau),\quad v'_y(\tau),\quad v_\parallel]$$
$$=\sum_{l=-\infty}^{+\infty}e^{-i(\omega-k_\parallel v_\parallel-l\Omega_{c\alpha})\tau}[\frac{l\Omega_{c\alpha}}{k_\perp}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad iv_\perp J'_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad v_\parallel J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})][\sum_{n=-\infty}^{+\infty}e^{i(n-l)\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]\tag{11.67}$$

Substituting equation (11.67) into equation (11.63), it yields

$$\tilde{f}_{\alpha1}(\mathbf{k},\mathbf{v},\omega)=-2\frac{e_\alpha}{m_\alpha}\{\sum_{l=-\infty}^{+\infty}\int_{-\infty}^{0}d\tau\, e^{-i(\omega-k_\parallel v_\parallel-l\Omega_{c\alpha})\tau}[\sum_{n=-\infty}^{+\infty}e^{i(n-l)\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$
$$[\frac{l\Omega_{c\alpha}}{k_\perp}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad iv_\perp J'_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad v_\parallel J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]\cdot\begin{bmatrix}\chi_\alpha & 0 & \gamma_\alpha\\ 0 & \chi_\alpha & 0\\ 0 & 0 & \varepsilon_\alpha\end{bmatrix}\cdot\begin{bmatrix}\tilde{E}_{1x}\\ \tilde{E}_{1y}\\ \tilde{E}_{1z}\end{bmatrix}\}\tag{11.68}$$

or

$$\tilde{f}_{\alpha 1}(\mathbf{k},\mathbf{v},\omega)=\frac{e_\alpha}{m_\alpha}\{\sum_{n=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\frac{e^{i(n-l)\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})}{i(\omega-k_\parallel v_\parallel-l\Omega_{c\alpha})}$$

$$2[\frac{l\Omega_{c\alpha}}{k_\perp}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad iv_\perp J_l'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad v_\parallel J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]\cdot\begin{bmatrix}\chi_\alpha & 0 & \gamma_\alpha\\ 0 & \chi_\alpha & 0\\ 0 & 0 & \varepsilon_\alpha\end{bmatrix}\cdot\begin{bmatrix}\tilde{E}_{1x}\\ \tilde{E}_{1y}\\ \tilde{E}_{1z}\end{bmatrix}\}$$

(11.68a)

Substituting equation (11.68a) into equation (11.30), it yields

$$0=\frac{\omega^2}{c^2}[(1-\frac{c^2k^2}{\omega^2})\mathbf{1}+\frac{c^2}{\omega^2}\mathbf{kk}]\cdot\tilde{\mathbf{E}}_1+i\frac{\omega}{c^2}\sum_\alpha\frac{1}{\varepsilon_0}e_\alpha\iiint\mathbf{v}\tilde{f}_{\alpha 1}d^3v$$

$$=\{\frac{\omega^2}{c^2}[(1-\frac{c^2k^2}{\omega^2})\mathbf{1}+\frac{c^2}{\omega^2}\mathbf{kk}]+i\frac{\omega}{c^2}\sum_\alpha\frac{\omega_{p\alpha0}^2}{n_0}\int dv_\parallel\int v_\perp dv_\perp\int d\phi\begin{bmatrix}v_\perp\cos\phi\\ v_\perp\sin\phi\\ v_\parallel\end{bmatrix}$$

$$\sum_{n=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\frac{e^{i(n-l)\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})}{i(\omega-k_\parallel v_\parallel-l\Omega_{c\alpha})}2[\frac{l\Omega_{c\alpha}}{k_\perp}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad iv_\perp J_l'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad v_\parallel J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]\cdot\begin{bmatrix}\chi_\alpha & 0 & \gamma_\alpha\\ 0 & \chi_\alpha & 0\\ 0 & 0 & \varepsilon_\alpha\end{bmatrix}\}\cdot\tilde{\mathbf{E}}_1$$

(11.69)

or

$$\frac{\omega^2}{c^2}\mathbf{D}\cdot\tilde{\mathbf{E}}_1=0 \qquad (11.69a)$$

where

$$\mathbf{D}=\begin{bmatrix}1-\frac{c^2k_\parallel^2}{\omega^2} & 0 & \frac{c^2k_\perp k_\parallel}{\omega^2}\\ 0 & 1-\frac{c^2k^2}{\omega^2} & 0\\ \frac{c^2k_\perp k_\parallel}{\omega^2} & 0 & 1-\frac{c^2k_\perp^2}{\omega^2}\end{bmatrix}+\sum_\alpha\frac{\omega_{p\alpha0}^2}{n_0\omega^2}\omega\int dv_\parallel\int v_\perp dv_\perp\int d\phi\begin{bmatrix}v_\perp\cos\phi\\ v_\perp\sin\phi\\ v_\parallel\end{bmatrix}$$

$$\sum_{n=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\frac{e^{i(n-l)\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})}{(\omega-k_\parallel v_\parallel-l\Omega_{c\alpha})}2[\frac{l\Omega_{c\alpha}}{k_\perp}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad iv_\perp J_l'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}),\quad v_\parallel J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]\cdot\begin{bmatrix}\chi_\alpha & 0 & \gamma_\alpha\\ 0 & \chi_\alpha & 0\\ 0 & 0 & \varepsilon_\alpha\end{bmatrix}$$

(11.70)

For $\cos\phi=(e^{i\phi}+e^{-i\phi})/2$ and $\sin\phi=(e^{i\phi}-e^{-i\phi})/2i$, it can be shown that

$$\sum_{n=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\int_0^{2\pi} d\phi \frac{e^{i(n-l)\phi}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})}{(\omega-k_\parallel v_\parallel - l\Omega_{c\alpha})} 2\begin{bmatrix} v_\perp\cos\phi \\ v_\perp \sin\phi \\ v_\parallel \end{bmatrix}[\frac{l\Omega_{c\alpha}}{k_\perp}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}), \quad iv_\perp J_l'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}), \quad v_\parallel J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$= \sum_{n=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\frac{\int_0^{2\pi} d\phi\, e^{i(n-l)\phi}}{(\omega-k_\parallel v_\parallel - l\Omega_{c\alpha})} 2\begin{bmatrix} \frac{n\Omega_{c\alpha}}{k_\perp}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) \\ -iv_\perp J_n'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) \\ v_\parallel J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) \end{bmatrix}[\frac{l\Omega_{c\alpha}}{k_\perp}J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}), \quad iv_\perp J_l'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}), \quad v_\parallel J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$= 2\sum_{n=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\frac{2\pi\delta(n-l)}{(\omega-k_\parallel v_\parallel - l\Omega_{c\alpha})}$$

$$\begin{bmatrix} \frac{nl\Omega_{c\alpha}^2}{k_\perp^2}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) & iv_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_l'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) & v_\parallel\frac{n\Omega_{c\alpha}}{k_\perp}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) \\ -iv_\perp\frac{l\Omega_{c\alpha}}{k_\perp}J_n'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_l(\frac{k_\perp v_\perp}{\Omega_{\alpha}}) & v_\perp^2 J_n'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_l'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) & -iv_\parallel v_\perp J_n'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) \\ v_\parallel\frac{l\Omega_{c\alpha}}{k_\perp}J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) & iv_\parallel v_\perp J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_l'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) & v_\parallel^2 J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_l(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) \end{bmatrix}$$

$$= 2\sum_{n=-\infty}^{+\infty}\frac{2\pi}{(\omega-k_\parallel v_\parallel - n\Omega_{c\alpha})}\begin{bmatrix} \frac{n^2\Omega_{c\alpha}^2}{k_\perp^2}J_n^2 & iv_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n'J_n & v_\parallel\frac{n\Omega_{c\alpha}}{k_\perp}J_n^2 \\ -iv_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n'J_n & v_\perp^2 J_n'^2 & -iv_\parallel v_\perp J_n'J_n \\ v_\parallel\frac{n\Omega_{c\alpha}}{k_\perp}J_n^2 & iv_\parallel v_\perp J_n'J_n & v_\parallel^2 J_n^2 \end{bmatrix} \tag{11.71}$$

where equations (11.65a), (11.65b), and $\int_0^{2\pi} d\phi\, e^{i(n-l)\phi} = 2\pi\delta(n-l)$ have been used to obtain equation (11.71). Substituting equation (11.71) into equation (11.70), it yields

$$\mathbf{D} = \begin{bmatrix} 1-\frac{c^2k_\parallel^2}{\omega^2} & 0 & \frac{c^2k_\perp k_\parallel}{\omega^2} \\ 0 & 1-\frac{c^2k^2}{\omega^2} & 0 \\ \frac{c^2k_\perp k_\parallel}{\omega^2} & 0 & 1-\frac{c^2k_\perp^2}{\omega^2} \end{bmatrix} + \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0\omega^2}\int dv_\parallel \int v_\perp dv_\perp$$

$$2\sum_{n=-\infty}^{+\infty}\frac{2\pi\omega}{(\omega-k_\parallel v_\parallel - n\Omega_{c\alpha})}\begin{bmatrix} \frac{n^2\Omega_{c\alpha}^2}{k_\perp^2}J_n^2 & iv_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n'J_n & v_\parallel\frac{n\Omega_{c\alpha}}{k_\perp}J_n^2 \\ -iv_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n'J_n & v_\perp^2 J_n'^2 & -iv_\parallel v_\perp J_n'J_n \\ v_\parallel\frac{n\Omega_{c\alpha}}{k_\perp}J_n^2 & iv_\parallel v_\perp J_n'J_n & v_\parallel^2 J_n^2 \end{bmatrix}\cdot\begin{bmatrix} \chi_\alpha & 0 & \gamma_\alpha \\ 0 & \chi_\alpha & 0 \\ 0 & 0 & \varepsilon_\alpha \end{bmatrix}$$

or

$$\mathbf{D}=\begin{bmatrix}1-\frac{c^2k_\parallel^2}{\omega^2} & 0 & \frac{c^2k_\perp k_\parallel}{\omega^2}\\ 0 & 1-\frac{c^2k^2}{\omega^2} & 0\\ \frac{c^2k_\perp k_\parallel}{\omega^2} & 0 & 1-\frac{c^2k_\perp^2}{\omega^2}\end{bmatrix}+\sum_\alpha\frac{\omega_{p\alpha0}^2}{n_0\omega^2}\int_L dv_\parallel\int_0^\infty 2v_\perp dv_\perp\sum_{n=-\infty}^{+\infty}\frac{2\pi\omega}{(\omega-k_\parallel v_\parallel-n\Omega_{c\alpha})}$$

$$\begin{bmatrix}\chi_\alpha\frac{n^2\Omega_{c\alpha}^2}{k_\perp^2}J_n^2 & i\chi_\alpha v_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n'J_n & \gamma_\alpha\frac{n^2\Omega_{c\alpha}^2}{k_\perp^2}J_n^2+\varepsilon_\alpha v_\parallel\frac{n\Omega_{c\alpha}}{k_\perp}J_n^2\\ -i\chi_\alpha v_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n'J_n & \chi_\alpha v_\perp^2J_n'^2 & -i\gamma_\alpha v_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n'J_n-i\varepsilon_\alpha v_\parallel v_\perp J_n'J_n\\ \chi_\alpha v_\parallel\frac{n\Omega_{c\alpha}}{k_\perp}J_n^2 & i\chi_\alpha v_\parallel v_\perp J_n'J_n & \gamma_\alpha v_\parallel\frac{n\Omega_{c\alpha}}{k_\perp}J_n^2+\varepsilon_\alpha v_\parallel^2J_n^2\end{bmatrix}$$

(11.72)

where

$$\chi_\alpha=\frac{\partial f_{\alpha0}}{\partial v_\perp^2}+\frac{k_\parallel v_\parallel}{\omega}\left(\frac{\partial f_{\alpha0}}{\partial v_\parallel^2}-\frac{\partial f_{\alpha0}}{\partial v_\perp^2}\right)$$

$$\gamma_\alpha=\frac{k_\perp v_\parallel}{\omega}\left(\frac{\partial f_{\alpha0}}{\partial v_\perp^2}-\frac{\partial f_{\alpha0}}{\partial v_\parallel^2}\right)$$

$$\varepsilon_\alpha=\frac{\partial f_{\alpha0}}{\partial v_\parallel^2}$$

For $\mathbf{D}=\begin{bmatrix}D_{xx} & D_{xy} & D_{xz}\\ D_{yx} & D_{yy} & D_{yz}\\ D_{zx} & D_{zy} & D_{zz}\end{bmatrix}$, it can be shown that

$$D_{xx}=(1-\frac{c^2k_\parallel^2}{\omega^2})+\sum_\alpha\frac{\omega_{p\alpha0}^2}{n_0\omega^2}\int_L dv_\parallel\int_0^\infty 2v_\perp\,dv_\perp\sum_{n=-\infty}^{+\infty}\frac{2\pi\omega}{(\omega-k_\parallel v_\parallel-n\Omega_{c\alpha})}[\chi_\alpha\frac{n^2\Omega_{c\alpha}^2}{k_\perp^2}J_n^2(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$D_{yx}=\sum_\alpha\frac{\omega_{p\alpha0}^2}{n_0\omega^2}\int_L dv_\parallel\int_0^\infty 2v_\perp\,dv_\perp\sum_{n=-\infty}^{+\infty}\frac{2\pi\omega}{(\omega-k_\parallel v_\parallel-n\Omega_{c\alpha})}[-i\chi_\alpha v_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$D_{zx}=\frac{c^2k_\perp k_\parallel}{\omega^2}+\sum_\alpha\frac{\omega_{p\alpha0}^2}{n_0\omega^2}\int_L dv_\parallel\int_0^\infty 2v_\perp\,dv_\perp\sum_{n=-\infty}^{+\infty}\frac{2\pi\omega}{(\omega-k_\parallel v_\parallel-n\Omega_{c\alpha})}[\chi_\alpha v_\parallel\frac{n\Omega_{c\alpha}}{k_\perp}J_n^2(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$D_{xy}=\sum_\alpha\frac{\omega_{p\alpha0}^2}{n_0\omega^2}\int_L dv_\parallel\int_0^\infty 2v_\perp\,dv_\perp\sum_{n=-\infty}^{+\infty}\frac{2\pi\omega}{(\omega-k_\parallel v_\parallel-n\Omega_{c\alpha})}[i\chi_\alpha v_\perp\frac{n\Omega_{c\alpha}}{k_\perp}J_n'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$D_{yy}=(1-\frac{c^2k^2}{\omega^2})+\sum_\alpha\frac{\omega_{p\alpha0}^2}{n_0\omega^2}\int_L dv_\parallel\int_0^\infty 2v_\perp\,dv_\perp\sum_{n=-\infty}^{+\infty}\frac{2\pi\omega}{(\omega-k_\parallel v_\parallel-n\Omega_{c\alpha})}[\chi_\alpha v_\perp^2J_n'^2(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$D_{zy}=\sum_\alpha\frac{\omega_{p\alpha0}^2}{n_0\omega^2}\int_L dv_\parallel\int_0^\infty 2v_\perp\,dv_\perp\sum_{n=-\infty}^{+\infty}\frac{2\pi\omega}{(\omega-k_\parallel v_\parallel-n\Omega_{c\alpha})}[i\chi_\alpha v_\parallel v_\perp J_n'(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})J_n(\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$D_{xz} = \frac{c^2 k_\perp k_\parallel}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0 \omega^2} \int_L dv_\parallel \int_0^\infty 2v_\perp \, dv_\perp \sum_{n=-\infty}^{+\infty} \frac{2\pi\omega}{(\omega - k_\parallel v_\parallel - n\Omega_{c\alpha})} [\gamma_\alpha \frac{n^2 \Omega_{c\alpha}^2}{k_\perp^2} J_n^2 + \varepsilon_\alpha v_\parallel \frac{n\Omega_{c\alpha}}{k_\perp} J_n^2]$$

$$= \frac{c^2 k_\perp k_\parallel}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0 \omega^2} \int_L dv_\parallel \int_0^\infty 2v_\perp \, dv_\perp \sum_{n=-\infty}^{+\infty} \frac{2\pi\omega}{(\omega - k_\parallel v_\parallel - n\Omega_{c\alpha})} [(\gamma_\alpha \frac{n\Omega_{c\alpha}}{k_\perp v_\parallel} + \varepsilon_\alpha) v_\parallel \frac{n\Omega_{c\alpha}}{k_\perp} J_n^2]$$

$$D_{yz} = \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0 \omega^2} \int_L dv_\parallel \int_0^\infty 2v_\perp \, dv_\perp \sum_{n=-\infty}^{+\infty} \frac{2\pi\omega}{(\omega - k_\parallel v_\parallel - n\Omega_{c\alpha})} [-i\gamma_\alpha v_\perp \frac{n\Omega_{c\alpha}}{k_\perp} J_n' J_n - i\varepsilon_\alpha v_\parallel v_\perp J_n' J_n]$$

$$= \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0 \omega^2} \int_L dv_\parallel \int_0^\infty 2v_\perp \, dv_\perp \sum_{n=-\infty}^{+\infty} \frac{2\pi\omega}{(\omega - k_\parallel v_\parallel - n\Omega_{c\alpha})} [-(\gamma_\alpha \frac{n\Omega_{c\alpha}}{k_\perp v_\parallel} + \varepsilon_\alpha) i v_\parallel v_\perp J_n' J_n]$$

$$D_{zz} = (1 - \frac{c^2 k_\perp^2}{\omega^2}) + \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0 \omega^2} \int_L dv_\parallel \int_0^\infty 2v_\perp \, dv_\perp \sum_{n=-\infty}^{+\infty} \frac{2\pi\omega}{(\omega - k_\parallel v_\parallel - n\Omega_{c\alpha})} [\gamma_\alpha v_\parallel \frac{n\Omega_{c\alpha}}{k_\perp} J_n^2 + \varepsilon_\alpha v_\parallel^2 J_n^2]$$

$$= (1 - \frac{c^2 k_\perp^2}{\omega^2}) + \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0 \omega^2} \int_L dv_\parallel \int_0^\infty 2v_\perp \, dv_\perp \sum_{n=-\infty}^{+\infty} \frac{2\pi\omega}{(\omega - k_\parallel v_\parallel - n\Omega_{c\alpha})} [(\gamma_\alpha \frac{n\Omega_{c\alpha}}{k_\perp v_\parallel} + \varepsilon_\alpha) v_\parallel^2 J_n^2]$$

where

$$\chi_\alpha = \frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + \frac{k_\parallel v_\parallel}{\omega} (\frac{\partial f_{\alpha 0}}{\partial v_\parallel^2} - \frac{\partial f_{\alpha 0}}{\partial v_\perp^2})$$

$$\gamma_\alpha = \frac{k_\perp v_\parallel}{\omega} (\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} - \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2})$$

$$\varepsilon_\alpha = \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}$$

If we define

$$\Lambda_\alpha = \gamma_\alpha \frac{n\Omega_{c\alpha}}{k_\perp v_\parallel} + \varepsilon_\alpha = \frac{n\Omega_{c\alpha}}{\omega} (\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} - \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}) + \frac{\partial f_{\alpha 0}}{\partial v_\parallel^2}$$

then D_{xz}, D_{yz}, and D_{zz} can be rewritten as

$$D_{xz} = \frac{c^2 k_\perp k_\parallel}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0 \omega^2} \int_L dv_\parallel \int_0^\infty 2v_\perp \, dv_\perp \sum_{n=-\infty}^{+\infty} \frac{2\pi\omega}{(\omega - k_\parallel v_\parallel - n\Omega_{c\alpha})} [\Lambda_\alpha v_\parallel \frac{n\Omega_{c\alpha}}{k_\perp} J_n^2 (\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$D_{yz} = \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0 \omega^2} \int_L dv_\parallel \int_0^\infty 2v_\perp \, dv_\perp \sum_{n=-\infty}^{+\infty} \frac{2\pi\omega}{(\omega - k_\parallel v_\parallel - n\Omega_{c\alpha})} [-i\Lambda_\alpha v_\parallel v_\perp J_n' (\frac{k_\perp v_\perp}{\Omega_{c\alpha}}) J_n (\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

$$D_{zz} = (1 - \frac{c^2 k_\perp^2}{\omega^2}) + \sum_\alpha \frac{\omega_{p\alpha 0}^2}{n_0 \omega^2} \int_L dv_\parallel \int_0^\infty 2v_\perp \, dv_\perp \sum_{n=-\infty}^{+\infty} \frac{2\pi\omega}{(\omega - k_\parallel v_\parallel - n\Omega_{c\alpha})} [\Lambda_\alpha v_\parallel^2 J_n^2 (\frac{k_\perp v_\perp}{\Omega_{c\alpha}})]$$

Note that, as discussed after equation (11.57), the $\Omega_{c\alpha}$ in $D_{xx} \sim D_{zz}$ is defined by $\Omega_{c\alpha} = e_\alpha B_0 / m_\alpha$. Namely $\Omega_{c\alpha} < 0$ if $e_\alpha < 0$.

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