Chapter 1. Introduction

Topics or concepts to learn in Chapter 1:

1. What is plasma?
2. The two systems of units that are commonly used in the literatures of plasma physics: The Gaussian units and the SI units (Also: The basic equations, the dimension analysis, and the scale analysis)
3. What is the difference between 10^6 °K electrons and 86 eV thermal electrons? Understand the temperature, thermal pressure, and the kinetic thermal energy of a plasma. Understand how special the 0.5 MeV electron is.
4. What is Boltzmann relation?
5. What is Debye shielding? How to determine the Debye length of a plasma?
6. What is plasma parameter?
7. What is the plasma oscillation frequency in a un-magnetized plasma?
8. What are the gyro frequency and the gyro radius (or Larmor radius) of a magnetized charge particle?
9. What is the definition of "collision" in the plasma physics?

Suggested Readings:

(1) Chapters 1 and 9 in Nicholson (1983)

(2) Sections 1.1~1.5, 1.8 and Appendix III in Krall and Trivelpiece (1973)

(3) Chapter 1 in F. F. Chen (1984)

1.1. Definition of Plasma

Plasma is the fourth state of matter. Heating can transfer matter from *solid state* to *liquate state*, then to *gas state*, and then to *plasma state*. Plasma is a fully ionized gas or a partially ionized gas. Gas with only 1% ionization can be considered as plasma. Therefore, 99% of the matter in the universe is in the plasma state.

Plasma is usually a *high-temperature* and *low-density* ionized gas. High temperature and low density are the favorite conditions for *ionization* but not for *recombination*. Without recombination, ionized particles can remain ionized so that the ions (with positive charge) and the electrons will not be recombined into neutral gas.

Plasma can be considered as a *fluid* even though sometimes *it does not reach to thermal dynamic equilibrium state*. It is the collective behavior that makes the plasma behave more like a fluid than independent particles. Like a fluid, there must be a large number of ionized gas particles (N>>1) in a plasma system, so that the number density and the thermal pressure of plasma can be statistically meaningful. Due to low-density nature, the basic scale length of plasma must be large enough in order to contain enough numbers of ionized particles. We shall show that this characteristic scale length in the plasma is roughly the Debye length.

Before we introduce the concept of Debye length, Debye shielding, and plasma parameter, we shall first briefly review the differences between the SI (MKS) units and the Gaussian units. Both of them are commonly used in the plasma research community. We shall also introduce two different units of temperature. Both of them are commonly used in the space plasma observations.

1.2. The SI Units and The Gaussian Units

The SI units are the standard units today for all scientific communities around the world. But the expressions in Gaussian units have also been used for more than 50 years. Many textbooks and theoretical papers written before 1980s are based on the Gaussian units. Since magnetic field and electric field have the same dimension in the Gaussian units, it is easy for theorists to check the correctness of their theoretical derivations. But all instruments are designed based on the SI units, as a result, it is hard to apply the theoretical results (in the Gaussian units) to the space observations (in the SI units). Thus, scientists in space community have tried very hard to change this old habit and try to use SI units in all new textbooks and scientific papers of space plasma physics. Change of the units can make the readers of the new generation hard to follow the contents in the old textbooks and the classical papers written in the Gaussian units. We shall use the SI units in most of the derivations presented in this book. To help the students to read early literatures in plasma physics, we will present the basic equations in both units in Chapters 1 and 3.

Table 1.1 and Table 1.2 list some of the commonly used equations and physical terms in both units, where c is the speed of light, ρ_c and $\mathbf{J}$ are the charge density and the electric current density, respectively. Note that the charge density and the current density appeared in these equations include both free and bounded components. They are in contrast to the Maxwell's equations (in SI units) listed below, in which ρ_{cf} and $\mathbf{J}_f$ are the free charge density and the free current density, respectively.

$$\nabla \cdot \mathbf{D} = \rho_{cf}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Here, the "free" means the distributions of the charge density and the current density will not be affected by the applied electric field and the magnetic field. In most space plasmas, we have $\rho_{cf} = 0$ and $\mathbf{J}_f = 0$. Namely, the charge density and the electric current density in the space plasma will change according to the electric field and the magnetic field.

Additional information on these units can be found in the Dimensions and Units section of *NRL Plasma Formulary*.

(URL: http://www.nrl.navy.mil/ppd/content/nrl-plasma-formulary, cited on 2014-09-15)

Table 1.1 Maxwell's Equations in the SI Units and in the Gaussian Units

SI Units	Gaussian Units
$\nabla \cdot \mathbf{E} = \dfrac{\rho_c}{\varepsilon_0}$	$\nabla \cdot \mathbf{E} = 4\pi\rho_c$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} (= \mu_0 \mathbf{J} + \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t})$	$\nabla \times \mathbf{B} = \dfrac{4\pi}{c}\mathbf{J} + \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t}$

Table 1.2 Force, energy, and frequency in the SI Units and in the Gaussian Units

	SI Units	Gaussian Units
Magnetic force of a charged particle	$q\mathbf{v} \times \mathbf{B}$	$q\dfrac{\mathbf{v} \times \mathbf{B}}{c}$
Magnetic force in fluid plasma per unit volume	$\mathbf{J} \times \mathbf{B}$	$\dfrac{\mathbf{J} \times \mathbf{B}}{c}$
Magnetic energy density (or magnetic pressure)	$\dfrac{B^2}{2\mu_0}$	$\dfrac{B^2}{8\pi}$
Electric energy density (or electric pressure)	$\dfrac{\varepsilon_0 E^2}{2}$	$\dfrac{E^2}{8\pi}$
Alfvén speed	$\dfrac{B}{\sqrt{\mu_0 \rho}}$	$\dfrac{B}{\sqrt{4\pi\rho}}$
Plasma frequency of the αth species	$\sqrt{\dfrac{ne^2}{\varepsilon_0 m_\alpha}}$	$\sqrt{\dfrac{4\pi ne^2}{m_\alpha}}$
gyro frequency of the αth species	$\dfrac{eB}{m_\alpha}$	$\dfrac{eB}{m_\alpha c}$

Show that 1 Tesla *in the SI Units is equivalent to* 10^4 Gauss *in the Gaussian Units:*

Some of the content shown in the conversion table of the SI/Gaussian units in Page 10 of the *NRL Plasma Formulary* are derived below.

Let us consider the dimension of ε_0 in the SI units:

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0} \Rightarrow \frac{E}{L} = \frac{Q/L^3}{\varepsilon_0} \Rightarrow E = \frac{1}{\varepsilon_0}\frac{Q}{L^2}$$

$$QE = M\frac{L}{T^2} \Rightarrow E = \frac{M}{Q}\frac{L}{T^2} = \frac{1}{\varepsilon_0}\frac{Q}{L^2} \Rightarrow \varepsilon_0 = \frac{Q^2T^2}{ML^3}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = \frac{1}{(4\pi\times10^{-7})\times(3\times10^8)^2}\frac{\mathrm{C^2s^2}}{\mathrm{kg\cdot m^3}} = \frac{1}{4\pi\times9\times10^9}\frac{\mathrm{C^2s^2}}{\mathrm{kg\cdot m^3}} = 0.88\times10^{13}\frac{\mathrm{C^2s^2}}{\mathrm{kg\cdot m^3}}$$

Converting ε_0 to the cgs units without introducing statcoulomb (SC), it yields

$$\varepsilon_0 = \frac{1}{4\pi\times9\times10^9}\frac{\mathrm{C^2s^2}}{\mathrm{kg\cdot m^3}} = \frac{1}{4\pi\times9\times10^9}\frac{\mathrm{C^2s^2}}{10^3\mathrm{g}\cdot10^6\mathrm{cm^3}} = \frac{1}{4\pi\times(3\times10^9)^2}\frac{\mathrm{C^2s^2}}{\mathrm{g\cdot cm^3}}$$

The ε_0 in the Gsuaaian units is a dimensionless constant, which is equal to 1 or $(1/4\pi)$. The dimension of the ε_0 in the SI units goes to the charge Q in the Gaussian units. That is, in the Gaussian units, we have

$$\nabla \cdot \mathbf{E} = \frac{4\pi\rho_c}{(\varepsilon_0^* = 1)} \Rightarrow \frac{E}{L} = \frac{4\pi Q}{L^3} \Rightarrow E = \frac{4\pi}{(\varepsilon_0^* = 1)}\frac{Q}{L^2}$$

$$QE = M\frac{L}{T^2} \Rightarrow E = \frac{M}{Q}\frac{L}{T^2} = \frac{4\pi}{(\varepsilon_0^* = 1)}\frac{Q}{L^2} \Rightarrow Q^2 = \frac{(\varepsilon_0^* = 1)}{4\pi}\frac{ML^3}{T^2}$$

Thus, in Gaussian units the charge Q has a dimension $\frac{M^{1/2}L^{3/2}}{T}$, which will be called statcoulomb (SC). Namely,

$$\boxed{1\,\mathrm{SC} = \sqrt{4\pi}\,\frac{\mathrm{g^{1/2}\cdot cm^{3/2}}}{\mathrm{s}}}$$

Let one coulomb (C) in the SI units is equivalent to x statcoulomb (SC) in the Gaussian units. For

$$\varepsilon_0 = \frac{1}{4\pi\times(3\times10^9)^2}\frac{\mathrm{C^2s^2}}{\mathrm{g\cdot cm^3}}$$

it yields

$$\varepsilon_0^* = 1 = \frac{1}{4\pi \times (3\times 10^9)^2} \frac{(x\cdot \mathrm{SC})^2 \mathrm{s}^2}{\mathrm{g}\cdot \mathrm{cm}^3}$$

$$= \frac{x^2}{4\pi \times (3\times 10^9)^2} (\sqrt{4\pi}\ \frac{\mathrm{g}^{1/2}\cdot \mathrm{cm}^{3/2}}{\mathrm{s}})^2 \frac{\mathrm{s}^2}{\mathrm{g}\cdot \mathrm{cm}^3} = \frac{x^2}{(3\times 10^9)^2}$$

Thus, we can obtain $x = 3\times 10^9$. Namely,

1C in the SI units is equivalent to 3×10^9SC in the Gaussian units.

The electric field in the Gaussian units has a dimension

$$\boxed{\frac{\mathrm{statvolt}}{\mathrm{cm}} = \frac{1}{4\pi} \frac{\mathrm{SC}}{\mathrm{cm}^2}}.$$

The conversion of the electric field between the SI units and the Gaussian units can be obtained by considering adding an electric force of 1 dyne on a particle with charge 1SC. To achieve the 1 dyne force, the electric field in the Gaussian units should be

$$E = \frac{\mathrm{Force}}{Q} = \frac{\mathrm{g}\cdot(\frac{\mathrm{cm}}{\mathrm{s}^2})}{\mathrm{SC}} = \frac{\mathrm{g}\cdot(\frac{\mathrm{cm}}{\mathrm{s}^2})}{\sqrt{4\pi\cdot \mathrm{g}\cdot(\frac{\mathrm{cm}^3}{\mathrm{s}^2})}} = \frac{\sqrt{4\pi\cdot \mathrm{g}\cdot(\frac{\mathrm{cm}^3}{\mathrm{s}^2})}}{4\pi\cdot \mathrm{cm}^2} = \frac{1}{4\pi} \frac{\mathrm{SC}}{\mathrm{cm}^2} = \frac{\mathrm{statvolt}}{\mathrm{cm}}$$

To achieve the 1 dyne force, the electric field in the SI units should be

$$E = \frac{\mathrm{Force}}{Q} = \frac{\mathrm{g}\cdot(\frac{\mathrm{cm}}{\mathrm{s}^2})}{\mathrm{SC}} = \frac{10^{-5}\times \mathrm{kg}\cdot(\frac{\mathrm{m}}{\mathrm{s}^2})}{\frac{\mathrm{C}}{3\times 10^9}} = 3\times 10^4 \frac{\mathrm{kg}\cdot \mathrm{m}}{\mathrm{C}\cdot \mathrm{s}^2} = 3\times 10^4 \frac{\mathrm{Volt}}{\mathrm{m}}$$

Thus, we have

1(statvolt/cm) in the Gaussian units is equivalent to 3×10^4(Volt/m) in the SI units.

Note that

$$1\mathrm{dyne} = 1\mathrm{SC}\cdot \frac{\mathrm{statvolt}}{\mathrm{cm}} = 10^{-5}\mathrm{Nt} = 10^{-5}\mathrm{C}\cdot \frac{\mathrm{Volt}}{\mathrm{m}}$$

The Lorentz force in the Gaussian units can be written as

$$\mathbf{F} = Q(\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B})$$

Thus, the magnetic field in the Gaussian units has the same dimension as the electric field. Namely, the magnetic field in the Gaussian units has a dimension

$$\boxed{1\mathrm{Gauss} = \frac{1}{4\pi} \frac{\mathrm{SC}}{\mathrm{cm}^2} = \frac{\mathrm{statvolt}}{\mathrm{cm}}}.$$

The conversion of the magnetic field between the SI units and the Gaussian units can be obtained by considering adding a magnetic force of 1 dyne on a particle with charge 1SC, and perpendicular velocity 1 cm/s. To achieve the 1 dyne force, the magnetic field in the Gaussian units should be

$$B = \frac{\text{Force}}{\frac{V}{c}Q} = \frac{\mathrm{g}\cdot(\frac{\mathrm{cm}}{\mathrm{s}^2})}{\frac{1}{3\times10^{10}}\sqrt{4\pi\cdot\mathrm{g}\cdot(\frac{\mathrm{cm}^3}{\mathrm{s}^2})}} = 3\times10^{10}\frac{\sqrt{4\pi\cdot\mathrm{g}\cdot(\frac{\mathrm{cm}^3}{\mathrm{s}^2})}}{4\pi\cdot\mathrm{cm}^2}$$

$$= \frac{3\times10^{10}}{4\pi}\frac{\mathrm{SC}}{\mathrm{cm}^2} = 3\times10^{10}\frac{\mathrm{statvolt}}{\mathrm{cm}} = 3\times10^{10}\mathrm{Gauss}$$

To achieve the 1 dyne force, the magnetic field in the SI units should be

$$B = \frac{\text{Force}}{VQ} = \frac{\mathrm{g}\cdot(\frac{\mathrm{cm}}{\mathrm{s}^2})}{\frac{\mathrm{cm}}{\mathrm{s}}\frac{\mathrm{C}}{3\times10^9}} = 3\times10^9\frac{\mathrm{g}}{\mathrm{s}\cdot\mathrm{C}} = 3\times10^9\frac{10^{-3}\mathrm{kg}}{\mathrm{s}\cdot\mathrm{C}} = 3\times10^6\mathrm{Tesla}$$

Thus, we have "1 Tesla in the SI units is equivalent to 10^4 Gauss in the Gaussian units."

Although, the dimension of "Tesla" and the dimension of "Gauss" are indeed different from each other, the following "equations" have been commonly used in the space plasma observations

$$\boxed{\begin{aligned} &1\,\mathrm{Tesla} = 10^4\,\mathrm{Gauss} \\ &1\,\mathrm{Gauss} = 10^5\,\gamma \\ &1\,\mathrm{Tesla} = 10^9\,\gamma \\ &1\,\mathrm{Tesla} = 10^9\,\mathrm{nT} \\ &1\gamma = 1\,\mathrm{nT} \end{aligned}}$$

1.3. Temperature in Units of ºK and eV

In gas dynamics, thermal pressure p can be written as $p = nk_B T$, where k_B is the Boltzmann constant, n is the number density of the gas, and T is the temperature of the gas in units of $°\mathrm{K}$. Since dimension of $k_B T$ is equal to the dimension of energy, one can use either electron Volts (eV) or Joules to denote the plasma temperature without introducing the Boltzmann constant.

Exercise 1.1.

Make a table with columns of (1) T_α in $°K$, (2) T_α in eV, (3) $v_{th\alpha}$ with $m_\alpha = m_e$, and (4) $v_{th\alpha}$ with $m_\alpha = m_i$, where T_α and $v_{th\alpha}$ are temperature and thermal speed of the αth species, respectively.

Exercise 1.2.

Boltzmann constant $k_B = 1.38 \times 10^{-23}$ Joule/°K,

The charge of a unit charge $e = 1.6 \times 10^{-19}$C

Electron mass $m_e = 0.9 \times 10^{-30}$kg

Proton/electron mass ratio $m_p / m_e = 1836$

Speed of the light $c = 3 \times 10^8$m/s

(1) Space scientists use both ºK and eV to describe the plasma temperature. For instance, the temperature of the plasma in the Earth's magnetosphere is around 10^6 ºK. What is the kinetic energy (in unit of eV) of the thermal particle with temperature 10^6 ºK (i.e., with speed equal to the thermal speed and with average velocity equal to zero)?

(2) What is the temperature (in unit of ºK) of an 100 keV plasma ?

(3) The kinetic energy of the energetic electrons in the discrete aurora is about 1~100 keV. What is the velocity of an 100 keV electron?

(4) What is the velocity of a 4 MeV electron (the electrons in the radiation belt)?

(5) What is the velocity of a 400 MeV proton?

(6) What is the velocity of a 10 GeV proton (cosmic ray)?

It will be shown in Chapter 4 that, for non-relativistic plasma, the scalar thermal pressure p_α of the αth species is defined by

$$p_\alpha = n_\alpha k_B T_\alpha \equiv \frac{1}{3}\mathrm{trace}[\iiint m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)f_\alpha d^3v]$$
$$= \frac{1}{3}\iiint m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)\cdot(\mathbf{v}-\mathbf{V}_\alpha)f_\alpha d^3v$$

The thermal energy density E_α of the αth species in a non-relativistic plasma is defined by

$$\mathsf{E}_\alpha = \iiint \frac{1}{2}m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)\cdot(\mathbf{v}-\mathbf{V}_\alpha)f_\alpha d^3v$$

Thus, for non-relativistic plasma, we have $\mathsf{E}_\alpha = (3/2)n_\alpha k_B T_\alpha$. For non-relativistic plasma, the thermal speed $v_{th\alpha}$ of the αth species can be defined by $\mathsf{E}_\alpha = (1/2)n_\alpha m_\alpha v_{th\alpha}^2$. Thus, we have

$$v_{th\alpha} = \sqrt{3k_B T_\alpha / m_\alpha}$$

We can also define a characteristic speed based on the standard deviation of the distribution function in each velocity direction, i.e.,

$$(\sigma_x)_\alpha = \sqrt{k_B T_{xx\alpha} / m_\alpha}$$

For isotropic pressure, we again obtain

$$v_{th\alpha} = \sqrt{(\sigma_x)_\alpha^2 + (\sigma_y)_\alpha^2 + (\sigma_z)_\alpha^2} = \sqrt{3\sigma_\alpha^2} = \sqrt{3}\sigma_\alpha = \sqrt{3k_B T_\alpha / m_\alpha}$$

Thermal velocity of a relativistic plasma:

If we define $k_B T_\alpha = (\gamma_\alpha - 1)m_\alpha c^2$, $\gamma_\alpha = (1 - v_\alpha^2 / c^2)^{-1/2}$, and $u_\alpha = \gamma_\alpha v_\alpha$, we can show that γ_α can be rewritten as

$$\gamma_\alpha = (1 + u_\alpha^2 / c^2)^{+1/2} \tag{1.0a}$$

Likewise, $k_B T_\alpha$ can be rewritten as

$$k_B T_\alpha = m_\alpha u_\alpha^2 / (\gamma_\alpha + 1) \tag{1.0b}$$

Solving the above equations (1.0a) and (1.0b), one can obtain the corresponding thermal momentum per unit mass u_α and the thermal speed v_α for a given $\sigma_\alpha = \sqrt{k_B T_\alpha / m_\alpha}$.

Exercise 1.3.

Determine the thermal speeds of the electrons with temperature 100keV.

1.4. Boltzmann Relation

Boltzmann relation is commonly used in the theoretical study of low frequency or steady state phenomena. It can be obtained from fluid equation. When the electric force is the dominant external force in the system, the fluid momentum equation of the αth species becomes

$$n_\alpha m_\alpha(\frac{\partial}{\partial t}+\mathbf{V}_\alpha\cdot\nabla)\mathbf{V}_\alpha=e_\alpha n_\alpha[-\nabla\Phi(\mathbf{x})]-\nabla p_\alpha$$

For steady state ($\partial/\partial t\approx 0$) solution with uniform background mean flow, we can choose a moving frame such that the average velocity is equal to ($\mathbf{V}_\alpha=0$). The above momentum equation becomes

$$e_\alpha n_\alpha[-\nabla\Phi(\mathbf{x})]-\nabla p_\alpha=0$$

Let each species satisfy the ideal gas law

$$p_\alpha=n_\alpha k_B T_\alpha$$

where k_B is the Boltzmann constant and the temperature T_α is measured in Kelvin. If we assume that the spatial variation of temperature is much smaller than the spatial variation of number density ($\nabla T_\alpha/T_\alpha<<\nabla n_\alpha/n_\alpha$), then

$$\nabla p_\alpha\approx k_B T_\alpha\nabla n_\alpha$$

Thus we have

$$e_\alpha n_\alpha[-\nabla\Phi(\mathbf{x})]-k_B T_\alpha\nabla n_\alpha=0$$

It can be shown that solution of the above equation is the Boltzmann relation

$$n_\alpha(\mathbf{x})=n_0\exp[-\frac{e_\alpha\Phi(\mathbf{x})}{k_B T_\alpha}]$$

where we have chosen $\Phi=0$ at $n_\alpha=n_0$. Namely, the Boltzmann relation of the electrons is

$$n_e(\mathbf{x})=n_0\exp[+\frac{e\Phi(\mathbf{x})}{k_B T_e}]$$

If there is a fixed external source of electric charges in the system, the ions might also satisfy the following Boltzmann relation

$$n_i(\mathbf{x})=n_0\exp[-\frac{e\Phi(\mathbf{x})}{k_B T_i}]$$

Note that, based on our resent studies (e.g., Tsai et al., 2009), the steady-state solution obtained from the fluid equations may not be a steady-state solution of the Vlasov equation, which will be derived in Chapter 2. Based on the steady state solution discussed later in Chapter 8, the electron temperature should be higher in the high potential region. Thus, the application of the Boltzmann relation is only good when the kinetic effect is ignored in the analysis.

1.5. Debye Shielding and Debye Length

Let us add a test charge particle with charge q_T and infinite mass into a plasma medium, the Poisson equation becomes

$$-\nabla^2\Phi(\mathbf{x}) = \frac{q_T\delta(\mathbf{x})}{\varepsilon_0} + \frac{e[n_i(\mathbf{x}) - n_e(\mathbf{x})]}{\varepsilon_0} \tag{1.1}$$

The electric potential due to presence of the test particle is (see Appendix A)

$$\Phi_T(r) = \frac{q_T}{4\pi\varepsilon_0 r} \tag{1.2}$$

1.5.1. Debye Shielding in the Electron Time Scale

When the time scale is much greater than the electron time scale, but is less than the ion time scale, we can assume that n_e satisfies the Boltzmann relation and n_i is still uniform, i.e.,

$$n_e(\mathbf{x}) = n_0 \exp[+\frac{e\Phi(\mathbf{x})}{k_B T_e}] \quad \text{and} \quad n_i = n_0$$

For $e\Phi(\mathbf{x}) << k_B T_e$, i.e., when the average kinetic energy is much greater than the electric potential energy, the Boltzmann relation can be approximated by

$$n_e(\mathbf{x}) \approx n_0(1 + \frac{e\Phi(\mathbf{x})}{k_B T_e}) \tag{1.3}$$

Substituting (1.3) into (1.1), the electric potential due to presence of plasma becomes

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\Phi(r) = \frac{e^2 n_0}{\varepsilon_0 k_B T_e}\Phi(r) \tag{1.4}$$

Let

$$\Phi(r) = \varphi(r)/r \tag{1.5}$$

Substituting (1.5) into (1.4), yields

$$\frac{d^2}{dr^2}\varphi(r) = \frac{e^2 n_0}{\varepsilon_0 k_B T_e}\varphi(r) = \frac{1}{\lambda_{De}^2}\varphi(r) \tag{1.6}$$

where λ_{De} is the Debye length of the plasma, which satisfies

$$\lambda_{De}^2 = \frac{\varepsilon_0 k_B T_e}{n_0 e^2}$$

Solution of Eq. (1.6) is

$$\varphi(r) = \varphi_0 \exp[-r/\lambda_{De}] \tag{1.7}$$

From (1.2), (1.5) and (1.7), the solution of Eq. (1.1) is approximately equal to

$$\Phi(r) \approx \frac{q_T}{4\pi\varepsilon_0 r}\exp[-r/\lambda_{De}] \tag{1.8}$$

As we can see, for $e\Phi << k_B T_e$, the electric potential drop exponentially when $r > \lambda_{De}$, where r is the distance measured from the test charge particle. Namely, the electric field outside the Debye sphere, which centered at the test particle, is approach to zero. This is called Debye shielding effect of the plasma.

1.5.2. Debye Shielding in the Ion Time Scale

When the time scale is much greater than the ion time scale, we can assume that both n_e and n_i satisfy the Boltzmann relations,

$$n_e(\mathbf{x}) = n_0 \exp[+\frac{e\Phi(\mathbf{x})}{k_B T_e}] \quad \text{and} \quad n_i(\mathbf{x}) = n_0 \exp[-\frac{e\Phi(\mathbf{x})}{k_B T_i}]$$

For $e\Phi(\mathbf{x}) << k_B T_\alpha$, i.e., when the average kinetic energy is much greater than the electric potential energy, the Boltzmann relations can be approximated by

$$n_i(\mathbf{x}) \approx n_0(1 - \frac{e\Phi(\mathbf{x})}{k_B T_i}) \quad \text{and} \quad n_e(\mathbf{x}) \approx n_0(1 + \frac{e\Phi(\mathbf{x})}{k_B T_e}) \tag{1.3a}$$

Substituting (1.3a) into (1.1), the electric potential due to presence of plasma becomes

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\Phi(r) = \frac{e^2 n_0}{\varepsilon_0 k_B}(\frac{1}{T_i} + \frac{1}{T_e})\Phi(r) \tag{1.4a}$$

Let

$$\Phi(r) = \varphi(r)/r \tag{1.5a}$$

Substituting (1.5a) into (1.4a), yields

$$\frac{d^2}{dr^2}\varphi(r) = \frac{e^2 n_0}{\varepsilon_0 k_B}(\frac{1}{T_i} + \frac{1}{T_e})\varphi(r) = \frac{1}{\lambda_D^2}\varphi(r) \tag{1.6a}$$

where λ_D is the Debye length of the plasma, which satisfies

$$\lambda_D^{-2} = \lambda_{Di}^{-2} + \lambda_{De}^{-2} = \frac{n_0 e^2}{\varepsilon_0 k_B T_i} + \frac{n_0 e^2}{\varepsilon_0 k_B T_e}$$

Solution of Eq. (1.6a) is

$$\varphi(r) = \varphi_0 \exp[-r/\lambda_D] \tag{1.7a}$$

From (1.2), (1.5a) and (1.7a), the solution of Eq. (1.1) is approximately equal to

$$\Phi(r) \approx \frac{q_T}{4\pi\varepsilon_0 r}\exp[-r/\lambda_D] \tag{1.8a}$$

For $T_e = T_i$, the Debye length becomes

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T_e}{2n_0 e^2}} \qquad \text{in the SI units} \tag{1.9}$$

or

$$\lambda_D = \sqrt{\frac{k_B T_e}{8\pi n_0 e^2}} \qquad \text{in the Gaussian units} \tag{1.10}$$

Again, for $e\Phi << k_B T_e$, the electric potential drop exponentially when $r > \lambda_D$, where r is the distance measured from the test charge particle. Namely, the electric field outside the Debye sphere, which centered at the test particle, is approach to zero. This is called Debye shielding effect of the plasma.

1.6. Plasma Parameter

To minimize the chance of recombination, the electric potential between two charged particle must satisfies

$$|e\Phi| \approx \frac{e^2}{4\pi\varepsilon_0|\mathbf{r}_1 - \mathbf{r}_2|} \approx \frac{e^2 n^{1/3}}{4\pi\varepsilon_0} << k_B T$$

or

$$n^{-2/3} << \frac{4\pi\varepsilon_0 k_B T}{ne^2}$$

or

$$(\frac{4\pi\varepsilon_0 k_B T}{ne^2})^{3/2} n >> 1$$

Ignoring the leading constant, we have the plasma parameter

$$\Lambda \equiv n\lambda_D^3 >> 1 \tag{1.11}$$

Thus, we can talk about all the statistic variables, including number density, temperature, bulk velocity (or average velocity), … etc., if we consider charge particles with a spatial dimension equal to or greater than the Debye length.

1.7. Plasma Frequency

Let us consider a uniform plasma medium, which consists of proton ions and electrons. Since $m_i >> m_e$, the plasma oscillation frequency can be obtained by assuming a virtual displacement of a sheet of electrons without perturbing ions. Let use consider a sheet of electrons with sheet size $L_y \times L_z$ and thickness W_x, where $0 < W_x \le \lambda_D$. Let the sheet of electrons make a virtual displacement $\delta_{e0}\mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector along the x direction and $\delta_{e0} << W_x$. The displacement of the electron sheet $\delta_e(t)\mathbf{e}_x$ can result in a positive charge layer with surface charge density $en_e\delta_e(t)$ and a negative charge layer with surface charge density $-en_e\delta_e(t)$ on two sides of the electron sheet. The electric field between the two charged layers is

$$\mathbf{E} = \mathbf{e}_x E_x \approx \mathbf{e}_x(en_e\delta_e/\varepsilon_0) \tag{1.12}$$

The equation of motion of the electron sheet becomes

$$M\mathbf{a} = [(W_xL_yL_z)n_e m_e]\ddot{\delta}_e\mathbf{e}_x = Q\mathbf{E} = [-(W_xL_yL_z)en_e][\mathbf{e}_x(en_e\delta_e/\varepsilon_0)]$$

or

$$\ddot{\delta}_e = -(e^2n_e/m_e\varepsilon_0)\delta_e \tag{1.13}$$

Let $\omega_{pe}^2 = e^2n_e/m_e\varepsilon_0$. Solution of (1.13) can be written in the following form

$$\delta_e = A\cos(\omega_{pe}t + \phi)$$

For $\delta_e(t=0) = \delta_{e0}$, we have

$$\delta_e = \delta_{e0}\cos(\omega_{pe}t)$$

where

$$\omega_{pe} = (e^2n_e/m_e\varepsilon_0)^{1/2} \tag{1.14}$$

is called the plasma frequency of electrons.

Let us consider a plasma oscillation, which is initiated by a virtual displacement of a sheet of electrons and a virtual displacement of a sheet of ions. Both of them are characterized by a sheet size $L_y \times L_z$ and a thickness W_x. Let the sheet of electrons make a virtual displacement $\delta_{e0}\mathbf{e}_x$, and the sheet of ions make a virtual displacement $-\delta_{i0}\mathbf{e}_x$ and the displacements satisfy $0 < \delta_{e0} << W_x$ and $0 < \delta_{i0} << W_x$. The displacement of the electron sheet $\delta_e(t)\mathbf{e}_x$ and the displacement of the ion sheet $-\delta_i(t)\mathbf{e}_x$ can result in a positive charge layer with surface charge density $en_0(\delta_e + \delta_i)$ and a negative charge layer with surface charge density $-en_0(\delta_e + \delta_i)$, where we have assumed that $n_e \approx n_i \approx n_0$ in the background equilibrium state. The electric field between the two charged layers is

$$\mathbf{E} = \mathbf{e}_x E_x \approx \mathbf{e}_x [en_0(\delta_e + \delta_i)/\varepsilon_0] \tag{1.15}$$

It can be shown that the equation of motion of the electron sheet is

$$\ddot{\delta}_e \mathbf{e}_x = -(e^2 n_0 / m_e \varepsilon_0)(\delta_e + \delta_i)\mathbf{e}_x \tag{1.16}$$

The equation of motion of the ion sheet is

$$-\ddot{\delta}_i \mathbf{e}_x = (e^2 n_0 / m_e \varepsilon_0)(\delta_e + \delta_i)\mathbf{e}_x \tag{1.17}$$

Solving (1.16) and (1.17), yields

$$\frac{d^2}{dt^2}(\delta_e + \delta_i) = -(\frac{e^2 n_0}{m_e \varepsilon_0} + \frac{e^2 n_0}{m_i \varepsilon_0})(\delta_e + \delta_i) = -(\omega_{pe}^2 + \omega_{pi}^2)(\delta_e + \delta_i) = -\omega_p^2(\delta_e + \delta_i) \tag{1.18}$$

Let $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$. Solution of (1.18) can be written in the following form

$$\delta = \delta_e + \delta_i = A\cos(\omega_p t + \phi) = (\delta_{e0} + \delta_{i0})\cos(\omega_p t)$$

In summary, the plasma frequency of the αth species is defined as

$$\omega_{p\alpha} = \sqrt{\frac{ne^2}{\varepsilon_0 m_\alpha}} \tag{1.19}$$

Total plasma frequency of a two-component plasma can be defined as

$$\omega_p = (\omega_{pe}^2 + \omega_{pi}^2)^{1/2} \approx \omega_{pe} \tag{1.20}$$

1.8. Gyro Frequency and Gyro Radius (or Larmor Radius)

Equations of motion of a single charged ion with absence of electric filed and gravitational field are

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \tag{1.21}$$

$$\frac{d\mathbf{v}}{dt} = \frac{e}{m_i}\mathbf{v} \times \mathbf{B} \tag{1.22}$$

Exercise 1.4.

Solve equations (1.21) and (1.22) for uniform magnetic field $\mathbf{B} = \hat{z}B$ initial velocity $\mathbf{v}(t=0) = \hat{x}v_0 \sin\phi_0 + \hat{y}v_0 \cos\phi_0$ and initial location at $\mathbf{x}(t=0) = 0$. Express your solution in terms of *gyro frequency* $\Omega_{ci} = eB/m_i$ and *gyro radius* (or Larmor radius) $r_{Li} = v_0 / \Omega_{ci}$.

Note that Nicholson (1983) defines $\Omega_{c\alpha} = e_\alpha B / cm_\alpha$ in the Gaussian units, so that there is a sign in his definition of gyro frequency. Namely, $\Omega_{ce} = -eB / cm_e$ in Nicholson's textbook (Nicholson, 1983). The negative sign denotes the gyro motion is right-handed with respect to the background magnetic field. A positive sign denotes the gyro motion is left-handed with respect to the background magnetic field. However, gyro frequency discussed in this book will always be positive. Namely, we shall *not* include the sign in *our* definition of gyro frequency.

1.9. Collisions

Two types of collisions have been discussed in section 1.6 of [Nicholson (1983)]. The collision frequency due to one large-angle collision is

$$\nu_L = \frac{n_0 e^4}{4\pi\varepsilon_0^2 m_e^2 v_0^3} = \frac{n_0^2 e^4}{4\pi\varepsilon_0^2 m_e^2 v_0^3}\frac{1}{n_0} \approx \frac{\omega_{pe}}{4\pi n_0 \lambda_{De}^3} \approx \frac{\omega_{pe}}{4\pi\Lambda_e} << \omega_{pe} \qquad (1.23)$$

The collision frequency due to many small-angle collisions is

$$\nu_c = \frac{2n_0 e^4 \ln\Lambda}{4\pi\varepsilon_0^2 m_e^2 v_0^3} \approx \frac{2\ln\Lambda\omega_{pe}}{4\pi n_0 \lambda_{De}^3} \approx \frac{2\ln\Lambda\omega_{pe}}{4\pi\Lambda_e} = 2\ln\Lambda\nu_L > \nu_L \qquad (1.24)$$

and

$$\nu_c = \frac{\ln\Lambda}{2\pi\Lambda_e}\omega_{pe} << \omega_{pe} \qquad (1.25)$$

where the collision frequency is defined by the inverse of a time interval in which more than half of the particles will change their moving directions by more than 45 degrees. From equations (1.23) and (1.25), we can conclude that *many small-angle collisions* are more efficient than *one large-angle collision*. Equations (1.23) ~ (1.25) yield $\nu_L < \nu_c << \omega_{pe}$. Thus, we can ignore particle-particle collisions in compare with wave-particle interactions, if the wave has a frequency near the plasma frequency ω_{pe}.

Let $\nu_{\alpha\beta}$ denotes collision frequency that the αth species is scattered by the βth species. Scientists have shown that

$$\nu_{ee} \approx \nu_{ei}$$

$$\nu_{ii} \approx \sqrt{m_e / m_i}\,\nu_{ee} < \nu_{ee}$$

$$\nu_{ie} \approx (m_e / m_i)\nu_{ee} << \nu_{ee}$$

These results indicate that (1) it takes longer time to thermalize two ion beams than to thermalize two electron beams; (2) a cold ion beam can hardly be heated by warm electrons; (3) fast electron heating can occur when there are two electron beams or when there is at least one ion beam that moves at different speed with respect to the beam electrons.

References

Chen, F. F. (1984), *Introduction to Plasma Physics and Controlled Fusion, Volume 1: Plasma Physics*, 2nd edition, Plenum Press, New York.

Krall, N. A., and A. W. Trivelpiece (1973), *Principles of Plasma Physics*, McGraw-Hill Book Company, New York.

Nicholson, D. R. (1983), *Introduction to Plasma Theory*, John Wiley & Sons, New York.

Tsai, T. C., L. H. Lyu, J. K. Chao, M. Q. Chen, and W. H. Tsai (2009), A theoretical and simulation study of the contact discontinuities based on a Vlasov simulation code, *J. Geophys. Res., 114*, A12103, doi:10.1029/2009JA014121.