Lecture 6: Introduction to Particle Simulations

This lecture introduces the concept, the assumptions, and the limitation of different particle simulation codes.

6.1. The Tree-Code Simulation

Tree-Code simulation consider the force from the near by particles based on particle-particle interaction. But consider the average force from the distant particles in groups, which are also called the pseudo particles in the tree-code simulation. As a result, the calculation stapes will be reduced from N^2 to NM, where N is the number of particles, and M is the number of groups plus the number of nearby particles, and $N >> M$.

Let us consider the electrostatic tree-code simulation. We first divide the particle in to M groups. Let us consider a charge particle S with electric charge q_S and located at $\mathbf{x}_S$. Let $(\mathbf{F}_S)_{k,ion}$ be the effective electric force acting on the charge particle S due to the *k*th group of proton particles, which is not adjacent to the group that the charge particle S belongs to. It yields

$$(\mathbf{F}_S)_{k,ion} = \frac{eq_S}{4\pi\varepsilon_0}\sum_{i=1}^{N_k}\frac{1}{(\mathbf{x}_i-\mathbf{x}_S)\cdot(\mathbf{x}_i-\mathbf{x}_S)}$$
$$= \frac{eq_S}{4\pi\varepsilon_0}\sum_{i=1}^{N_k}\frac{1}{(\mathbf{x}_i-\mathbf{X}_k+\mathbf{X}_k-\mathbf{x}_S)\cdot(\mathbf{x}_i-\mathbf{X}_k+\mathbf{X}_k-\mathbf{x}_S)}$$
$$= \frac{eq_S}{4\pi\varepsilon_0}\sum_{i=1}^{N_k}\frac{1}{(\mathbf{x}_i-\mathbf{X}_k)\cdot(\mathbf{x}_i-\mathbf{X}_k)+2(\mathbf{x}_i-\mathbf{X}_k)\cdot\mathbf{X}_k^S+\mathbf{X}_k^S\cdot\mathbf{X}_k^S}$$
$$= \frac{eq_S}{4\pi\varepsilon_0}\frac{1}{(X_k^S)^2}\sum_{i=1}^{N_k}\frac{1}{\frac{(\mathbf{x}_i-\mathbf{X}_k)\cdot(\mathbf{x}_i-\mathbf{X}_k)}{(X_k^S)^2}+2\frac{(\mathbf{x}_i-\mathbf{X}_k)\cdot\mathbf{X}_k^S}{(X_k^S)^2}+1}$$
$$= \frac{eq_S}{4\pi\varepsilon_0}\frac{1}{(X_k^S)^2}\sum_{i=1}^{N_k}\frac{1}{1+(\varepsilon_S)_i} \tag{6.1}$$

where $\mathbf{X}_k = \langle\mathbf{x}_i\rangle_k = (\sum_{i=1}^{N_k}\mathbf{x}_i)/N_k$ is the average position of the N_k particles, $\mathbf{X}_k^S = \mathbf{X}_k - \mathbf{x}_S$ and $X_k^S = |\mathbf{X}_k^S|$. Let $\mathbf{x}_i = \mathbf{X}_k + \mathbf{d}_i$. It yields $(\varepsilon_S)_i = (2\mathbf{d}_i\cdot\mathbf{X}_k^S + \mathbf{d}_i\cdot\mathbf{d}_i)/(X_k^S)^2$. For $(\varepsilon_S)_i < 1$, the electric force given in equation (6.1) can be rewritten as

$$(\mathbf{F}_S)_{k,ion} = \frac{eq_S}{4\pi\varepsilon_0} \frac{1}{(X_k^S)^2} \sum_{i=1}^{N_k} (1 - (\varepsilon_S)_i + (\varepsilon_S)_i^2 - (\varepsilon_S)_i^3 + (\varepsilon_S)_i^4 - (\varepsilon_S)_i^5 + \ldots) \tag{6.2}$$

Let $\mathbf{d}_i = \hat{x}(\mathbf{d}_i)_x + \hat{y}(\mathbf{d}_i)_y + \hat{z}(\mathbf{d}_i)_z$ and $\mathbf{X}_k^S = \hat{x}(\mathbf{X}_k^S)_x + \hat{y}(\mathbf{X}_k^S)_y + \hat{z}(\mathbf{X}_k^S)_z$, it yields

$$\left\langle (\varepsilon_S)_i \right\rangle_k = \frac{2\left\langle \mathbf{d}_i \right\rangle_k \cdot \mathbf{X}_k^S + \left\langle \mathbf{d}_i \cdot \mathbf{d}_i \right\rangle_k}{(X_k^S)^2} = \frac{\left\langle \mathbf{d}_i \cdot \mathbf{d}_i \right\rangle_k}{(X_k^S)^2} = \frac{\left\langle (\mathbf{d}_i)_x^2 \right\rangle_k + \left\langle (\mathbf{d}_i)_y^2 \right\rangle_k + \left\langle (\mathbf{d}_i)_z^2 \right\rangle_k}{(X_k^S)^2} \tag{6.3}$$

$$\begin{aligned}
\left\langle (\varepsilon_S)_i^2 \right\rangle_k &= \frac{\left\langle (2\mathbf{d}_i \cdot \mathbf{X}_k^S + \mathbf{d}_i \cdot \mathbf{d}_i)^2 \right\rangle_k}{(X_k^S)^4} \\
&= 2\frac{\left\langle (\mathbf{d}_i)_x^2 \right\rangle_k (\mathbf{X}_k^S)_x^2 + \left\langle (\mathbf{d}_i)_y^2 \right\rangle_k (\mathbf{X}_k^S)_y^2 + \left\langle (\mathbf{d}_i)_z^2 \right\rangle_k (\mathbf{X}_k^S)_z^2}{(X_k^S)^4} \\
&\quad + 4\frac{\left\langle (\mathbf{d}_i)_x (\mathbf{d}_i)_y \right\rangle_k (\mathbf{X}_k^S)_x (\mathbf{X}_k^S)_y + \left\langle (\mathbf{d}_i)_x (\mathbf{d}_i)_z \right\rangle_k (\mathbf{X}_k^S)_x (\mathbf{X}_k^S)_z + \left\langle (\mathbf{d}_i)_y (\mathbf{d}_i)_z \right\rangle_k (\mathbf{X}_k^S)_y (\mathbf{X}_k^S)_z}{(X_k^S)^4} \\
&\quad + O[(\frac{d_i}{X_k^S})^3]
\end{aligned} \tag{6.4}$$

and so on, where by definition, $(\sum_{i=1}^{N_k} \mathbf{d}_i)/N_k = \left\langle \mathbf{d}_i \right\rangle_k = 0$, has been used to eliminate the first term in Equation (6.3). Substituting Equations (6.3) and (6.4) into Equation (6.2), it yields

$$\begin{aligned}
(\mathbf{F}_S)_{k,ion} = \frac{eq_S}{4\pi\varepsilon_0} \frac{N_k}{(X_k^S)^2} \Bigg\{ & 1 - \frac{\left\langle (\mathbf{d}_i)_x^2 \right\rangle_k + \left\langle (\mathbf{d}_i)_y^2 \right\rangle_k + \left\langle (\mathbf{d}_i)_z^2 \right\rangle_k}{(X_k^S)^2} \\
& + 2\frac{\left\langle (\mathbf{d}_i)_x^2 \right\rangle_k (\mathbf{X}_k^S)_x^2 + \left\langle (\mathbf{d}_i)_y^2 \right\rangle_k (\mathbf{X}_k^S)_y^2 + \left\langle (\mathbf{d}_i)_z^2 \right\rangle_k (\mathbf{X}_k^S)_z^2}{(X_k^S)^4} \\
& + 4\frac{\left\langle (\mathbf{d}_i)_x (\mathbf{d}_i)_y \right\rangle_k (\mathbf{X}_k^S)_x (\mathbf{X}_k^S)_y + \left\langle (\mathbf{d}_i)_x (\mathbf{d}_i)_z \right\rangle_k (\mathbf{X}_k^S)_x (\mathbf{X}_k^S)_z + \left\langle (\mathbf{d}_i)_y (\mathbf{d}_i)_z \right\rangle_k (\mathbf{X}_k^S)_y (\mathbf{X}_k^S)_z}{(X_k^S)^4} \\
& + O[(\frac{d_i}{X_k^S})^3] \Bigg\}
\end{aligned} \tag{6.5}$$

If we ignore the second and higher order terms in Equation (6.5), it yields the first order approximation:

$$(\mathbf{F}_S)_{k,ion} \approx \frac{eq_S}{4\pi\varepsilon_0} \frac{N_k}{(X_k^S)^2} \tag{6.6}$$

Equation (6.6) is commonly used in the popular tree-code simulation. If we ignore the third and higher order terms in Equation (6.5), it yields the second order approximation:

$$(\mathbf{F}_S)_{k,ion} = \frac{eq_S}{4\pi\varepsilon_0}\frac{N_k}{(X_k^S)^2}\left\{1 - \frac{\langle(\mathbf{d}_i)_x^2\rangle_k + \langle(\mathbf{d}_i)_y^2\rangle_k + \langle(\mathbf{d}_i)_z^2\rangle_k}{(X_k^S)^2}\right.$$
$$+2\frac{\langle(\mathbf{d}_i)_x^2\rangle_k(\mathbf{X}_k^S)_x^2 + \langle(\mathbf{d}_i)_y^2\rangle_k(\mathbf{X}_k^S)_y^2 + \langle(\mathbf{d}_i)_z^2\rangle_k(\mathbf{X}_k^S)_z^2}{(X_k^S)^4}$$
$$\left.+4\frac{\langle(\mathbf{d}_i)_x(\mathbf{d}_i)_y\rangle_k(\mathbf{X}_k^S)_x(\mathbf{X}_k^S)_y + \langle(\mathbf{d}_i)_x(\mathbf{d}_i)_z\rangle_k(\mathbf{X}_k^S)_x(\mathbf{X}_k^S)_z + \langle(\mathbf{d}_i)_y(\mathbf{d}_i)_z\rangle_k(\mathbf{X}_k^S)_y(\mathbf{X}_k^S)_z}{(X_k^S)^4}\right\} \quad (6.7)$$

Similarly, we can obtain $(\mathbf{F}_S)_{k,ele}$.

Once we obtain the average forces due to charge particles in the non-adjacent groups, we can advance the position and velocity of the S particle by solving the following ordinary differential equations:

$$\frac{d\mathbf{x}_S}{dt} = \mathbf{v}_S$$
$$\frac{d\mathbf{v}_S}{dt} = \frac{1}{m_S}\{\sum_{\substack{j=1\\ j\neq S}}^{N_0}\frac{q_j q_S}{4\pi\varepsilon_0(\mathbf{x}_j - \mathbf{x}_S)\cdot(\mathbf{x}_j - \mathbf{x}_S)} + \sum_k[(\mathbf{F}_S)_{k,ion} + (\mathbf{F}_S)_{k,ele}]\}$$

where N_0 is the total number of particles in the adjacent groups and in the same group where the S particle belongs to.

The tree code is good to study the problems with very low plasma density at the boundaries of the simulation domain.

6.2. The Finite-Size-Particle Simulation and the Particle-in-Cell Simulation

6.2.1. Simulation Scheme

Both finite-size-particle simulation and the particle-in-cell (PIC) simulation advance the particle motion in the following way:

(1) distributing the charge density and current density to the grids

(2) solving the Maxwell's equation to advance the electric field and magnetic field on each grid

(3) interpolating the field from the grid points to the location of the particle

(4) advancing the velocity and position of a particle by solving the equations of motion

As a result, the calculation steps will be reduced from N^2 to NM, where N is the number of particles, and M is the number of grids, and $N >> M$.

6.2.2. Interpolation Scheme in the UCLA Finite-Size-Particle Simulation

The distribution (or called deposition) and interpolation can be obtained from the Taylor expansion of a function $f(x)$. In the finite-size-particle simulation, the function $f(x)$ is expanded to the nearest grid point x_0:

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + ...$$

Ignoring the second order term, it yields

$$\begin{aligned} f(x) &\approx f(x_0) + (x - x_0)\frac{f(x_1) - f(x_{-1})}{2\Delta_x} \\ &= f(x_0) + \frac{d}{2}[f(x_1) - f(x_{-1})] \end{aligned}$$

where $d = (x - x_0)/\Delta_x$. To reduce the numerical noise, finite-size-particle simulation apply a weighting function $S(x)$ to each grid to further smooth the numerical noise due to insufficient particles used in the simulation, i.e.,

$$F(x) \approx \int f(x')S(x - x')dx'$$

6.2.2. Interpolation Scheme in the classical PIC Simulation

For PIC simulation, the function $f(x)$ is expanded to the nearest half grid points $x_{1/2}$, namely, $x_0 < x < x_1$:

$$f(x) = f(x_{1/2}) + (x - x_{1/2}) f'(x_{1/2}) + \frac{1}{2}(x - x_{1/2})^2 f''(x_{1/2}) + ...$$
$$= f(x_{1/2}) + (x - x_0 - \frac{\Delta_x}{2}) f'(x_{1/2}) + \frac{1}{2}(x - x_0 - \frac{\Delta_x}{2})^2 f''(x_{1/2}) + ...$$

Let $\varepsilon = (x - x_0)/\Delta_x$. It yields

$$f(x) = f(x_{1/2}) + \Delta_x(\varepsilon - \frac{1}{2}) f'(x_{1/2}) + \frac{\Delta_x^2}{2}(\varepsilon - \frac{1}{2})^2 f''(x_{1/2}) + ...$$

Ignoring the second order term, i.e., assuming a straight line to fit the function f in the region $x_0 < x < x_1$, it yields

$$f(x) \approx \frac{f(x_1) + f(x_0)}{2} + \Delta_x(\varepsilon - \frac{1}{2}) \frac{f(x_1) - f(x_0)}{\Delta_x} = \varepsilon f(x_1) + (1 - \varepsilon) f(x_0)$$

Note that a square-particle picture is commonly used to explain the resulting equation when it was applied to the traditional PIC simulation.

6.3. Distribution of Yee Fields and Higher-Order Particle Simulation Scheme

6.3.1. Introduction to the Yee Fields

Let us consider charge density at integer grid points. That is

$$\rho_c(x = i\Delta x, y = j\Delta y, z = k\Delta z, t = n\Delta t) = (\rho_c)^n_{i,j,k}$$

If we consider the first-order finite-difference scheme in space, the charge conservation equation

$$\frac{\partial \rho_c}{\partial t} = -\nabla \cdot \mathbf{J}$$

and the Poisson equation

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0}$$

yield the electric current density and the electric field should be evaluated at half-integer, integer, integer grid points. Because, for example, the integral form of the charge continuity equation yields

$$\frac{\partial}{\partial t} \iiint_{Vol.(S)} \rho_c = -\oiint_{S(Vol.)} \mathbf{J} \cdot d\mathbf{a}$$

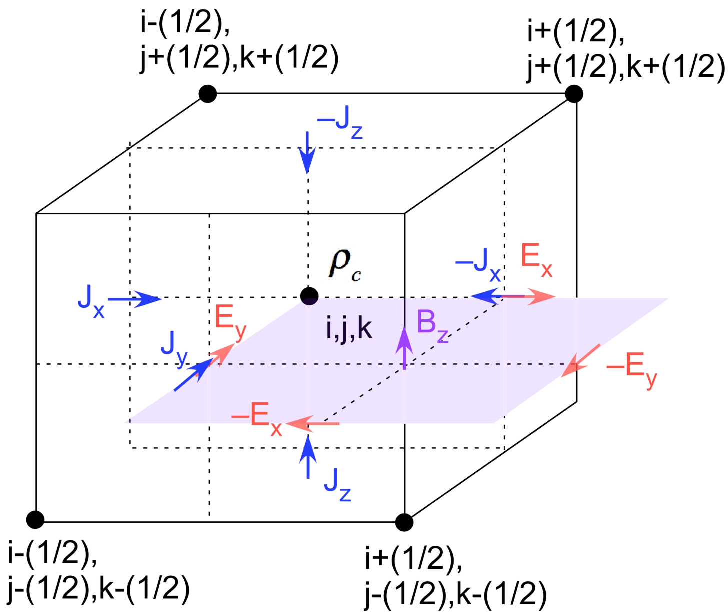

Figure 6.1. Sketches of the grids distribution of charge density, current density, electric field, and magnetic field in the particle simulation scheme.

The sketch shown in Figure 1 yields

$$\frac{(\rho_c)^{n+1}_{i,j,k} - (\rho_c)^n_{i,j,k}}{\Delta t} \Delta x\, \Delta y\, \Delta z$$
$$= -\left\{ \left[(J_x)^{n+0.5}_{i+0.5,j,k} - (J_x)^{n+0.5}_{i-0.5,j,k} \right] \Delta y\, \Delta z + \left[(J_y)^{n+0.5}_{i,j+0.5,k} - (J_y)^{n+0.5}_{i,j-0.5,k} \right] \Delta x\, \Delta z \right.$$
$$\left. + \left[(J_z)^{n+0.5}_{i,j,k+0.5} - (J_z)^{n+0.5}_{i,j,k-0.5} \right] \Delta x\, \Delta y \right\}$$

or

$$\frac{(\rho_c)^{n+1}_{i,j,k} - (\rho_c)^n_{i,j,k}}{\Delta t} = -\left\{ \frac{(J_x)^{n+0.5}_{i+0.5,j,k} - (J_x)^{n+0.5}_{i-0.5,j,k}}{\Delta x} + \frac{(J_y)^{n+0.5}_{i,j+0.5,k} - (J_y)^{n+0.5}_{i,j-0.5,k}}{\Delta y} + \frac{(J_z)^{n+0.5}_{i,j,k+0.5} - (J_z)^{n+0.5}_{i,j,k-0.5}}{\Delta z} \right\}$$

Likewise, the Poisson equation can be written in the following first-order finite difference form

$$\frac{(E_x)^n_{i+0.5,j,k}-(E_x)^n_{i-0.5,j,k}}{\Delta x}+\frac{(E_y)^n_{i,j+0.5,k}-(E_y)^n_{i,j-0.5,k}}{\Delta y}+\frac{(E_z)^n_{i,j,k+0.5}-(E_z)^n_{i,j,k-0.5}}{\Delta z}=\frac{(\rho_c)^n_{i,j,k}}{\varepsilon_0}$$

Thus, the electric field and the electric current density should be evaluated at half-integer, integer, integer grid points. Namely,

$$\begin{bmatrix} (J_x)^H_{H,N,N} & (J_y)^H_{N,H,N} & (J_z)^H_{N,N,H} \end{bmatrix}$$

$$\begin{bmatrix} (E_x)^N_{H,N,N} & (E_y)^N_{N,H,N} & (E_z)^N_{N,N,H} \end{bmatrix}$$

where the N and H in the superscripts/subscripts denote the integer and the half-integer time steps/grid points, respectively. Likewise, the Faraday's law and the Ampere's law yield the magnetic field should be evaluated at integer, half-integer, half-integer grid points. Because, for example, the integral form of the Faraday's law yields

$$\frac{\partial}{\partial t}\iint_{S(C)}\mathbf{B}\cdot d\mathbf{a}=-\oint_{C(S)}\mathbf{E}\cdot d\mathbf{l}$$

The sketch shown in Figure 1 yields

$$\frac{(B_z)^{n+0.5}_{i+0.5,j-0.5,k}-(B_z)^{n-0.5}_{i+0.5,j-0.5,k}}{\Delta t}\Delta x\,\Delta y$$
$$=\left\{\left[(E_x)^n_{i+0.5,j,k}-(E_x)^n_{i+0.5,j-1,k}\right]\Delta x+\left[(E_y)^n_{i,j-0.5,k}-(E_y)^n_{i+1,j-0.5,k}\right]\Delta y\right\}$$

or

$$\frac{(B_z)^{n+0.5}_{i+0.5,j-0.5,k}-(B_z)^{n-0.5}_{i+0.5,j-0.5,k}}{\Delta t}=\frac{(E_x)^n_{i+0.5,j,k}-(E_x)^n_{i+0.5,j-1,k}}{\Delta y}-\frac{(E_y)^n_{i+1,j-0.5,k}-(E_y)^n_{i,j-0.5,k}}{\Delta x}$$

Thus, the magnetic field should be evaluated at integer, half-integer, half-integer grid points and half-integer time steps. That is,

$$\begin{bmatrix} (B_x)^H_{N,H,H} & (B_y)^H_{H,N,H} & (B_z)^H_{H,H,N} \end{bmatrix}$$

6.3.2. Lower-Order Particle Simulation Scheme

We can advance particle position $\mathbf{x}_s$, particle momentum per unit mass $\mathbf{u}_s$, and the $\mathbf{E}$, $\mathbf{B}$ fields based on the second-order leapfrog integration scheme.

Step	Initial condition: $n=1$. Given $\mathbf{x}_S^{n-1}, \mathbf{E}^{n-1}, \mathbf{u}_S^{n-0.5}, \mathbf{B}^{n-0.5}$, we can obtain
1	$\mathbf{x}_S^{\ n} = \mathbf{x}_S^{\ n-1} + \Delta t\, \mathbf{u}_S^{\ n-0.5} / \sqrt{1+(u_S^{\ n-0.5}/c)^2}$
2	$\mathbf{x}_S^{\ n-0.5} = (\mathbf{x}_S^{\ n} + \mathbf{x}_S^{\ n-1})/2$
3	Determine $\mathbf{J}^{n-0.5}$ from $\mathbf{v}_S^{\ n-0.5}$ and $\mathbf{x}_S^{\ n-0.5}$ (see Tables 6.1*a*, 6.4*a*)
4	$\mathbf{E}^n = \mathbf{E}^{n-1} + c^2 \Delta t (\nabla \times \mathbf{B} - \mu_0 \mathbf{J})^{n-0.5}$
5	$\mathbf{B}^{n+0.5} = \mathbf{B}^{n-0.5} - \Delta t (\nabla \times \mathbf{E})^n$
6	Determine $\mathbf{E}^n(\mathbf{x}_S^n)$ from $\mathbf{E}^n$ and $\mathbf{x}_S^{\ n}$ (see Tables 6.1, 6.4)
7	$(\mathbf{u}_S^{\ temp})^n = \mathbf{u}_S^{\ n-0.5} + \dfrac{\Delta t}{2} \dfrac{q_S}{m_S} \mathbf{E}^n(\mathbf{x}_S^{\ n})$
8	Solving the following equation to obtain $\mathbf{u}_S^{\ n+0.5}$ $\dfrac{\mathbf{u}_S^{\ n+0.5} - \mathbf{u}_S^{\ n-0.5}}{\Delta t} = \dfrac{q_S}{m_S}[\mathbf{E}^n(\mathbf{x}_S^{\ n}) + \dfrac{\mathbf{u}_S^{\ n+0.5} + \mathbf{u}_S^{\ n-0.5}}{2\sqrt{1+[(u_S^{\ temp})^n/c]^2}} \times \dfrac{\mathbf{B}^{n+0.5}(\mathbf{x}_S^{\ n}) + \mathbf{B}^{n-0.5}(\mathbf{x}_S^{\ n})}{2}]$
9	Advance the time step ($n \leftarrow n+1$) and repeat the main loop (go to Step 1)

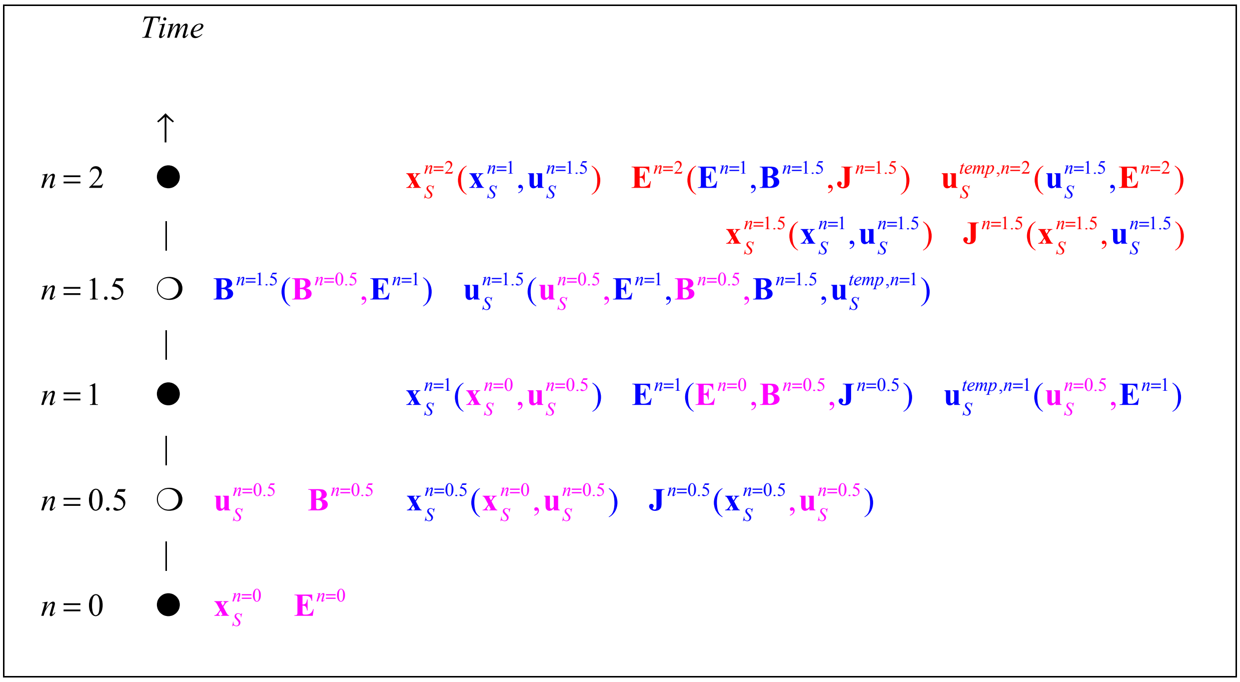

Figure 6.2. The second-order leapfrog simulation scheme

6.3.3. Higher-Order Finite Difference Scheme for the Yee Fields

For fifth-order finite difference scheme in space, we have (Appendix B in this book)

$$f_0^{(1)} = \frac{1}{h}[\frac{75}{64}(f_{+0.5} - f_{-0.5}) - \frac{25}{64\cdot 6}(f_{+1.5} - f_{-1.5}) + \frac{3}{64\cdot 10}(f_{+2.5} - f_{-2.5})] + O(h^6 f_0^{(7)})$$

Thus, we can determine the time derivative of the magnetic field by the following spatial fifth-order finite difference scheme in space

$$BXP^n_{i,j+0.5,k+0.5} = (\frac{\partial B_x}{\partial t})^n_{i,j+0.5,k+0.5} = -[(\nabla\times\mathbf{E})_x]^n_{i,j+0.5,k+0.5}$$

$$= \frac{1}{\Delta z}\{\frac{75}{64}[(E_y)^n_{i,j+0.5,k+1} - (E_y)^n_{i,j+0.5,k}] - \frac{25}{64\cdot 6}[(E_y)^n_{i,j+0.5,k+2} - (E_y)^n_{i,j+0.5,k-1}]$$
$$+ \frac{3}{64\cdot 10}[(E_y)^n_{i,j+0.5,k+3} - (E_y)^n_{i,j+0.5,k-2}]\}$$
$$- \frac{1}{\Delta y}\{\frac{75}{64}[(E_z)^n_{i,j+1,k+0.5} - (E_z)^n_{i,j,k+0.5}] - \frac{25}{64\cdot 6}[(E_z)^n_{i,j+2,k+0.5} - (E_z)^n_{i,j-1,k+0.5}]$$
$$+ \frac{3}{64\cdot 10}[(E_z)^n_{i,j+3,k+0.5} - (E_z)^n_{i,j-2,k+0.5}]\}$$

$$BYP^n_{i+0.5,j,k+0.5} = (\frac{\partial B_y}{\partial t})^n_{i+0.5,j,k+0.5} = -[(\nabla\times\mathbf{E})_y]^n_{i,j+0.5,k+0.5}$$

$$= \frac{1}{\Delta x}\{\frac{75}{64}[(E_z)^n_{i+1,j,k+0.5} - (E_z)^n_{i,j,k+0.5}] - \frac{25}{64\cdot 6}[(E_z)^n_{i+2,j,k+0.5} - (E_z)^n_{i-1,j,k+0.5}]$$
$$+ \frac{3}{64\cdot 10}[(E_z)^n_{i+3,j,k+0.5} - (E_z)^n_{i-2,j,k+0.5}]\}$$
$$- \frac{1}{\Delta z}\{\frac{75}{64}[(E_x)^n_{i+0.5,j,k+1} - (E_x)^n_{i+0.5,j,k}] - \frac{25}{64\cdot 6}[(E_x)^n_{i+0.5,j,k+2} - (E_x)^n_{i+0.5,j,k-1}]$$
$$+ \frac{3}{64\cdot 10}[(E_x)^n_{i+0.5,j,k+3} - (E_x)^n_{i+0.5,j,k-2}]\}$$

$$BZP^n_{i+0.5,j+0.5,k} = (\frac{\partial B_z}{\partial t})^n_{i+0.5,j+0.5,k} = -[(\nabla\times\mathbf{E})_z]^n_{i+0.5,j+0.5,k}$$

$$= \frac{1}{\Delta y}\{\frac{75}{64}[(E_x)^n_{i+0.5,j+1,k} - (E_x)^n_{i+0.5,j,k}] - \frac{25}{64\cdot 6}[(E_x)^n_{i+0.5,j+2,k} - (E_x)^n_{i+0.5,j-1,k}]$$
$$+ \frac{3}{64\cdot 10}[(E_x)^n_{i+0.5,j+3,k} - (E_x)^n_{i+0.5,j-2,k}]\}$$
$$- \frac{1}{\Delta x}\{\frac{75}{64}[(E_y)^n_{i+1,j+0.5,k} - (E_y)^n_{i,j+0.5,k}] - \frac{25}{64\cdot 6}[(E_y)^n_{i+2,j+0.5,k} - (E_y)^n_{i-1,j+0.5,k}]$$
$$+ \frac{3}{64\cdot 10}[(E_y)^n_{i+3,j+0.5,k} - (E_y)^n_{i-2,j+0.5,k}]\}$$

Likewise, we can determine the time derivative of the electric field by the following spatial fifth-order finite difference scheme in space

$$\frac{1}{c^2}EXP_{i+0.5,j,k}^{n+0.5}=\frac{1}{c^2}(\frac{\partial E_x}{\partial t})_{i+0.5,j,k}^{n+0.5}=[(\nabla\times\mathbf{B})_x]_{i+0.5,j,k}^{n+0.5}-\mu_0(J_x)_{i+0.5,j,k}^{n+0.5}$$

$$=\frac{1}{\Delta y}\{\frac{75}{64}[(B_z)_{i+0.5,j+0.5,k}^{n+0.5}-(B_z)_{i+0.5,j-0.5,k}^{n+0.5}]-\frac{25}{64\cdot 6}[(B_z)_{i+0.5,j+1.5,k}^{n+0.5}-(B_z)_{i+0.5,j-1.5,k}^{n+0.5}]$$
$$+\frac{3}{64\cdot 10}[(B_z)_{i+0.5,j+2.5,k}^{n+0.5}-(B_z)_{i+0.5,j-2.5,k}^{n+0.5}]\}$$
$$-\frac{1}{\Delta z}\{\frac{75}{64}[(B_y)_{i+0.5,j,k+0.5}^{n+0.5}-(B_y)_{i+0.5,j,k-0.5}^{n+0.5}]-\frac{25}{64\cdot 6}[(B_y)_{i+0.5,j,k+1.5}^{n+0.5}-(B_y)_{i+0.5,j,k-1.5}^{n+0.5}]$$
$$+\frac{3}{64\cdot 10}[(B_y)_{i+0.5,j,k+2.5}^{n+0.5}-(B_y)_{i+0.5,j,k-2.5}^{n+0.5}]\}-\mu_0(J_x)_{i+0.5,j,k}^{n+0.5}$$

$$\frac{1}{c^2}EYP_{i,j+0.5,k}^{n+0.5}=\frac{1}{c^2}(\frac{\partial E_y}{\partial t})_{i,j+0.5,k}^{n+0.5}=[(\nabla\times\mathbf{B})_y]_{i,j+0.5,k}^{n+0.5}-\mu_0(J_y)_{i,j+0.5,k}^{n+0.5}$$

$$=\frac{1}{\Delta z}\{\frac{75}{64}[(B_x)_{i,j+0.5,k+0.5}^{n+0.5}-(B_x)_{i,j+0.5,k-0.5}^{n+0.5}]-\frac{25}{64\cdot 6}[(B_x)_{i,j+0.5,k+1.5}^{n+0.5}-(B_x)_{i,j+0.5,k-1.5}^{n+0.5}]$$
$$+\frac{3}{64\cdot 10}[(B_x)_{i,j+0.5,k+2.5}^{n+0.5}-(B_x)_{i,j+0.5,k-2.5}^{n+0.5}]\}$$
$$-\frac{1}{\Delta x}\{\frac{75}{64}[(B_z)_{i+0.5,j+0.5,k}^{n+0.5}-(B_z)_{i-0.5,j+0.5,k}^{n+0.5}]-\frac{25}{64\cdot 6}[(B_z)_{i+1.5,j+0.5,k}^{n+0.5}-(B_z)_{i-1.5,j+0.5,k}^{n+0.5}]$$
$$+\frac{3}{64\cdot 10}[(B_z)_{i+2.5,j+0.5,k}^{n+0.5}-(B_z)_{i-2.5,j+0.5,k}^{n+0.5}]\}-\mu_0(J_y)_{i,j+0.5,k}^{n+0.5}$$

$$\frac{1}{c^2}EZP_{i,j,k+0.5}^{n+0.5}=\frac{1}{c^2}(\frac{\partial E_z}{\partial t})_{i,j,k+0.5}^{n+0.5}=[(\nabla\times\mathbf{B})_z]_{i,j,k+0.5}^{n+0.5}-\mu_0(J_z)_{i,j,k+0.5}^{n+0.5}$$

$$=+\frac{1}{\Delta x}\{\frac{75}{64}[(B_y)_{i+0.5,j,k+0.5}^{n+0.5}-(B_y)_{i-0.5,j,k+0.5}^{n+0.5}]-\frac{25}{64\cdot 6}[(B_y)_{i+1.5,j,k+0.5}^{n+0.5}-(B_y)_{i-1.5,j,k+0.5}^{n+0.5}]$$
$$+\frac{3}{64\cdot 10}[(B_y)_{i+2.5,j,k+0.5}^{n+0.5}-(B_y)_{i-2.5,j,k+0.5}^{n+0.5}]\}$$
$$-\frac{1}{\Delta y}\{\frac{75}{64}[(B_x)_{i,j+0.5,k+0.5}^{n+0.5}-(B_x)_{i,j-0.5,k+0.5}^{n+0.5}]-\frac{25}{64\cdot 6}[(B_x)_{i,j+1.5,k+0.5}^{n+0.5}-(B_x)_{i,j-1.5,k+0.5}^{n+0.5}]$$
$$+\frac{3}{64\cdot 10}[(B_x)_{i,j+2.5,k+0.5}^{n+0.5}-(B_x)_{i,j-2.5,k+0.5}^{n+0.5}]\}-\mu_0(J_z)_{i,j,k+0.5}^{n+0.5}$$

Time integration based on the fourth-order open formula at half time steps (Appendix C in this book) yields

$$y^{n+1}=y^n+h[\frac{26}{24}f^{n+0.5}-\frac{5}{24}f^{n-0.5}+\frac{4}{24}f^{n-1.5}-\frac{1}{24}f^{n-2.5}]+O(h^5f^{(4)})$$

Thus, we can advance the magnetic field and electric field by the following fourth-order finite difference scheme

$$(B_x)^{n+0.5}_{i,j+0.5,k+0.5} = (B_x)^{n-0.5}_{i,j+0.5,k+0.5} + \Delta t(\frac{26}{24} BXP^{n}_{i,j+0.5,k+0.5} - \frac{5}{24} BXP^{n-1}_{i,j+0.5,k+0.5} + \frac{4}{24} BXP^{n-2}_{i,j+0.5,k+0.5} - \frac{1}{24} BXP^{n-3}_{i,j+0.5,k+0.5})$$

$$(B_y)^{n+0.5}_{i+0.5,j,k+0.5} = (B_y)^{n-0.5}_{i+0.5,j,k+0.5} + \Delta t(\frac{26}{24} BYP^{n}_{i+0.5,j,k+0.5} - \frac{5}{24} BYP^{n-1}_{i+0.5,j,k+0.5} + \frac{4}{24} BYP^{n-2}_{i+0.5,j,k+0.5} - \frac{1}{24} BYP^{n-3}_{i+0.5,j,k+0.5})$$

$$(B_z)^{n+0.5}_{i+0.5,j+0.5,k} = (B_z)^{n-0.5}_{i+0.5,j+0.5,k} + \Delta t(\frac{26}{24} BZP^{n}_{i+0.5,j+0.5,k} - \frac{5}{24} BZP^{n-1}_{i+0.5,j+0.5,k} + \frac{4}{24} BZP^{n-2}_{i+0.5,j+0.5,k} - \frac{1}{24} BZP^{n-3}_{i+0.5,j+0.5,k})$$

$$(E_x)^{n+1}_{i+0.5,j,k} = (E_x)^{n}_{i+0.5,j,k} + \Delta t(\frac{26}{24} EXP^{n+0.5}_{i+0.5,j,k} - \frac{5}{24} EXP^{n-0.5}_{i+0.5,j,k} + \frac{4}{24} EXP^{n-1.5}_{i+0.5,j,k} - \frac{1}{24} EXP^{n-2.5}_{i+0.5,j,k})$$

$$(E_y)^{n+1}_{i,j+0.5,k} = (E_y)^{n}_{i,j+0.5,k} + \Delta t(\frac{26}{24} EYP^{n+0.5}_{i,j+0.5,k} - \frac{5}{24} EYP^{n-0.5}_{i,j+0.5,k} + \frac{4}{24} EYP^{n-1.5}_{i,j+0.5,k} - \frac{1}{24} EYP^{n-2.5}_{i,j+0.5,k})$$

$$(E_z)^{n+1}_{i,j,k+0.5} = (E_z)^{n}_{i,j,k+0.5} + \Delta t(\frac{26}{24} EZP^{n+0.5}_{i,j,k+0.5} - \frac{5}{24} EZP^{n-0.5}_{i,j,k+0.5} + \frac{4}{24} EZP^{n-1.5}_{i,j,k+0.5} - \frac{1}{24} EZP^{n-2.5}_{i,j,k+0.5})$$

6.3.4. Higher-Order Particle Simulation Scheme

We use the second-order leapfrog time integration scheme to obtain the time integrations in the first four steps. Then, for $n > 4$, we use the following fourth-order open formula to advance particle position and the $\mathbf{E}$, $\mathbf{B}$ fields.

Step	Initial condition: $n = 4$. Given $\mathbf{x}_S^{\,n-1}, \mathbf{E}^{n-1}, \mathbf{u}_S^{\,n-0.5}, \mathbf{B}^{n-0.5}$, we can obtain
1	$\mathbf{v}_S^{\,n-0.5} = \mathbf{u}_S^{\,n-0.5} / \sqrt{1 + (u_S^{\,n-0.5}/c)^2}$
2	$\mathbf{x}_S^{\,n} = \mathbf{x}_S^{\,n-1} + \Delta t[\frac{26}{24}\mathbf{v}_S^{\,n-0.5} - \frac{5}{24}\mathbf{v}_S^{\,n-1.5} + \frac{4}{24}\mathbf{v}_S^{\,n-2.5} - \frac{1}{24}\mathbf{v}_S^{\,n-3.5}]$
3	$\mathbf{x}_S^{\,n-0.5} = (\mathbf{x}_S^{\,n} + \mathbf{x}_S^{\,n-1})/2$
4	Determine $\mathbf{J}^{n-0.5}$ from $\mathbf{v}_S^{\,n-0.5}$ and $\mathbf{x}_S^{\,n-0.5}$ (see Tables 6.1*a*~6.6*a*)
5	$\mathbf{E}^n = \mathbf{E}^{n-1} + c^2\Delta t[\frac{26}{24}(\nabla\times\mathbf{B} - \mu_0\mathbf{J})^{n-0.5} - \frac{5}{24}(\nabla\times\mathbf{B} - \mu_0\mathbf{J})^{n-1.5}$ $+ \frac{4}{24}(\nabla\times\mathbf{B} - \mu_0\mathbf{J})^{n-2.5} - \frac{1}{24}(\nabla\times\mathbf{B} - \mu_0\mathbf{J})^{n-3.5}]$
6	$\mathbf{B}^{n+0.5} = \mathbf{B}^{n-0.5} - \Delta t[\frac{26}{24}(\nabla\times\mathbf{E})^n - \frac{5}{24}(\nabla\times\mathbf{E})^{n-1} + \frac{4}{24}(\nabla\times\mathbf{E})^{n-2} - \frac{1}{24}(\nabla\times\mathbf{E})^{n-3}]$
7	Determine $\mathbf{E}^n(\mathbf{x}_S^n)$ from $\mathbf{E}^n$ and $\mathbf{x}_S^{\,n}$ (see Tables 6.1~6.6)
8	$(\mathbf{u}_S^{\,temp})^n = \mathbf{u}_S^{\,n-0.5} + \frac{\Delta t}{2}\frac{q_S}{m_S}\mathbf{E}^n(\mathbf{x}_S^{\,n})$
9	Solving the following equation to obtain $\mathbf{u}_S^{\,n+0.5}$ $\frac{\mathbf{u}_S^{\,n+0.5} - \mathbf{u}_S^{\,n-0.5}}{\Delta t} = \frac{q_S}{m_S}[\mathbf{E}^n(\mathbf{x}_S^{\,n}) + \frac{\mathbf{u}_S^{\,n+0.5} + \mathbf{u}_S^{\,n-0.5}}{2\sqrt{1 + [(u_S^{\,temp})^n/c]^2}} \times \frac{\mathbf{B}^{n+0.5}(\mathbf{x}_S^{\,n}) + \mathbf{B}^{n-0.5}(\mathbf{x}_S^{\,n})}{2}]$
10	Advance the time step ($n \leftarrow n+1$) and repeat the main loop (go to Step 1)

6.3.5. Higher-Order Deposition and Interpolation

A summary of the floor and ceiling functions

https://en.wikipedia.org/wiki/Floor_and_ceiling_functions

In mathematics and computer science, the **floor** and **ceiling** functions map a real number to the largest previous or the smallest following integer, respectively. More precisely, floor(x) = $\lfloor x \rfloor$ is the largest integer less than or equal to x and ceiling(x) = $\lceil x \rceil$ is the smallest integer greater than or equal to x. In the following formulas, x is real numbers, m, and n are integers, and Z is the set of integers (positive, negative, and zero). Floor and ceiling may be defined by the set equations

$$\lfloor x \rfloor = \max\{m \in Z \mid m \le x\},$$

$$\lceil x \rceil = \min\{n \in Z \mid n \ge x\}$$

The **fractional part** is denoted by ε_x for real x and defined by the formula

$$\varepsilon_x = x - \lfloor x \rfloor$$

For all x, we have $0 \le \varepsilon_x < 1$.

Examples

x	**Int(x)**	**Floor(x)**	**Ceiling(x)**	**Fractional part** ε_x
2	2	2	2	0
2.4	2	2	3	0.4
2.9	2	2	3	0.9
−2.7	−2	−3	−2	0.3
−2	−2	−2	−2	0

Let us consider the field variable f at integer grid points $\{(x_L, y_L, z_L), ..., (x_R, y_R, z_R)\}$

For C++, let

$$i = \text{int}(\frac{x_S - x_L}{\Delta x}) \qquad j = \text{int}(\frac{y_S - y_L}{\Delta y}) \qquad k = \text{int}(\frac{z_S - z_L}{\Delta z})$$

$$\varepsilon_x = (x_S - x_i) / \Delta x \quad \varepsilon_y = (y_S - y_j) / \Delta y \quad \varepsilon_z = (z_S - z_k) / \Delta z$$

For FORTRAN, let

$$i = 1 + \text{int}(\frac{x_S - x_L}{\Delta x}) \quad j = 1 + \text{int}(\frac{y_S - y_L}{\Delta y}) \quad k = 1 + \text{int}(\frac{z_S - z_L}{\Delta z})$$

$$\varepsilon_x = (x_S - x_i) / \Delta x \quad \varepsilon_y = (y_S - y_j) / \Delta y \quad \varepsilon_z = (z_S - z_k) / \Delta z$$

where $\mathbf{x}_L = (x_L, y_L, z_L)$ is the location where the first element of the field array f is specified. Let $f(\mathbf{x}_S)$ be the field at the particle's position $\mathbf{x}_S = (x_S, y_S, z_S)$. If the grid size is uniform, we can determine $f(\mathbf{x}_S)$ based on the interpolation of the fields at the nearby integer gird points as listed in Tables 1~3. Likewise, if the grid size is uniform, we can deposition the information at $f(\mathbf{x}_S)$ to the nearby integer gird points as listed in Tables 1*a*~3*a*.

Table 6.1. The first-order interpolation

1-D	$f(x_S) = a1_1 f_i + a1_2 f_{i+1} = \sum_{m=1}^{2} a1_m f_{i+(m-1)}$
2-D	$f(x_S, y_S) = b1_1(a1_1 f_{i,j} + a1_2 f_{i+1,j}) + b1_2(a1_1 f_{i,j+1} + a1_2 f_{i+1,j+1})$ $= \sum_{my=1}^{2} b1_{my}(\sum_{mx=1}^{2} a1_{mx} f_{i+(mx-1),j+(my-1)})$
3-D	$f(x_S, y_S, z_S) = c1_1[b1_1(a1_1 f_{i,j,k} + a1_2 f_{i+1,j,k}) + b1_2(a1_1 f_{i,j+1,k} + a1_2 f_{i+1,j+1,k})]$ $+ c1_2[b1_1(a1_1 f_{i,j,k+1} + a1_2 f_{i+1,j,k+1}) + b1_2(a1_1 f_{i,j+1,k+1} + a1_2 f_{i+1,j+1,k+1})]$ $= \sum_{mz=1}^{2} c1_{mz}[\sum_{my=1}^{2} b1_{my}(\sum_{mx=1}^{2} a1_{mx} f_{i+(mx-1),j+(my-1),k(mz-1)})]$

Table 6.1*a*. The first-order deposition

1-D	$f_{i+(m-1)} = f_{i+(m-1)} + a1_m f(x_S)$ for $m = 1$ to 2
2-D	$f_{i+(mx-1),j+(my-1)} = f_{i+(mx-1),j+(my-1)} + a1_{mx} b1_{my} f(x_S, y_S)$ for $mx, my = 1$ to 2
3-D	$f_{i+(mx-1),j+(my-1),k(mz-1)} = f_{i+(mx-1),j+(my-1),k(mz-1)} + a1_{mx} b1_{my} c1_{mz} f(x_S, y_S, z_S)$ for $mx, my, mz = 1$ to 2

where

$a1_1 = \frac{(x_S - x_{i+1})}{(x_i - x_{i+1})} = \frac{\varepsilon_x - 1}{-1} = 1 - \varepsilon_x$, and $a1_2 = \frac{(x_S - x_i)}{(x_{i+1} - x_i)} = \varepsilon_x$

Likewise,

$b1_1 = 1 - \varepsilon_y \quad c1_1 = 1 - \varepsilon_z$

$b1_2 = \varepsilon_y \quad c1_2 = \varepsilon_z$

Table 6.2. The third-order interpolation

1-D	$f(x_S) = a3_1 f_{i-1} + a3_2 f_i + a3_3 f_{i+1} + a3_4 f_{i+2} = \sum_{m=1}^{4} a3_m f_{i+(m-2)}$
2-D	$f(x_S, y_S) = \sum_{my=1}^{4} b3_{my} (\sum_{mx=1}^{4} a3_{mx} f_{i+(mx-2),j+(my-2)})$
3-D	$f(x_S, y_S, z_S) = \sum_{mz=1}^{4} c3_{mz} [\sum_{my=1}^{4} b3_{my} (\sum_{mx=1}^{4} a3_{mx} f_{i+(mx-2),j+(my-2),k(mz-2)})]$

Table 6.2*a*. The third-order deposition

1-D	$f_{i+(m-2)} = f_{i+(m-2)} + a3_m f(x_S)$ for $m = 1$ to 4
2-D	$f_{i+(mx-2),j+(my-2)} = f_{i+(mx-2),j+(my-2)} + a3_{mx} b3_{my} f(x_S, y_S)$ for $mx, my = 1$ to 4
3-D	$f_{i+(mx-2),j+(my-2),k(mz-2)} = f_{i+(mx-2),j+(my-2),k(mz-2)} + a3_{mx} b3_{my} c3_{mz} f(x_S, y_S, z_S)$ for $mx, my, mz = 1$ to 4

where

$$a3_1 = \frac{(x_S - x_{i-1})(x_S - x_i)(x_S - x_{i+1})(x_S - x_{i+2})}{(x_{i-1} - x_{i-1})(x_{i-1} - x_i)(x_{i-1} - x_{i+1})(x_{i-1} - x_{i+2})} = \frac{(\varepsilon_x + 1)\varepsilon_x(\varepsilon_x - 1)(\varepsilon_x - 2)}{0 \cdot -1 \cdot -2 \cdot -3}$$
$$= -\frac{1}{6}\varepsilon_x(\varepsilon_x - 1)(\varepsilon_x - 2)$$

$$a3_2 = \frac{(x_S - x_{i-1})(x_S - x_i)(x_S - x_{i+1})(x_S - x_{i+2})}{(x_i - x_{i-1})(x_i - x_i)(x_i - x_{i+1})(x_i - x_{i+2})} = \frac{(\varepsilon_x + 1)\varepsilon_x(\varepsilon_x - 1)(\varepsilon_x - 2)}{1 \cdot 0 \cdot -1 \cdot -2}$$
$$= \frac{1}{2}(\varepsilon_x + 1)(\varepsilon_x - 1)(\varepsilon_x - 2)$$

$$a3_3 = \frac{(x_S - x_{i-1})(x_S - x_i)(x_S - x_{i+1})(x_S - x_{i-2})}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)(x_{i+1} - x_{i+1})(x_{i+1} - x_{i+2})} = \frac{(\varepsilon_x + 1)\varepsilon_x(\varepsilon_x - 1)(\varepsilon_x - 2)}{2 \cdot 1 \cdot 0 \cdot -1}$$
$$= -\frac{1}{2}(\varepsilon_x + 1)\varepsilon_x(\varepsilon_x - 2)$$

$$a3_4 = \frac{(x_S - x_{i-1})(x_S - x_i)(x_S - x_{i+1})(x_S - x_{i+2})}{(x_{i+2} - x_{i-1})(x_{i+2} - x_i)(x_{i+2} - x_{i+1})(x_{i+2} - x_{i+2})} = \frac{(\varepsilon_x + 1)\varepsilon_x(\varepsilon_x - 1)(\varepsilon_x - 2)}{3 \cdot 2 \cdot 1 \cdot 0}$$
$$= \frac{1}{6}(\varepsilon_x + 1)\varepsilon_x(\varepsilon_x - 1)$$

Likewise,

$$b3_1 = -\frac{1}{6}\varepsilon_y(\varepsilon_y - 1)(\varepsilon_y - 2) \qquad c3_1 = -\frac{1}{6}\varepsilon_z(\varepsilon_z - 1)(\varepsilon_z - 2)$$

$$b3_2 = +\frac{1}{2}(\varepsilon_y + 1)(\varepsilon_y - 1)(\varepsilon_y - 2) \qquad c3_2 = +\frac{1}{2}(\varepsilon_z + 1)(\varepsilon_z - 1)(\varepsilon_z - 2)$$

$$b3_3 = -\frac{1}{2}(\varepsilon_y + 1)\varepsilon_y(\varepsilon_y - 2) \qquad c3_2 = -\frac{1}{2}(\varepsilon_z + 1)\varepsilon_z(\varepsilon_z - 2)$$

$$b3_4 = +\frac{1}{6}(\varepsilon_y + 1)\varepsilon_y(\varepsilon_y - 1) \qquad c3_2 = +\frac{1}{6}(\varepsilon_z + 1)\varepsilon_z(\varepsilon_z - 1)$$

Table 6.3. The fifth-order interpolation

1-D	$f(x_S) = a5_1 f_{i-2} + a5_2 f_{i-1} + a5_3 f_i + a5_4 f_{i+1} + a5_5 f_{i+2} + a5_6 f_{i+3} = \sum_{m=1}^{6} a5_m f_{i+(m-3)}$
2-D	$f(x_S, y_S) = \sum_{my=1}^{6} b5_{my} (\sum_{mx=1}^{6} a5_{mx} f_{i+(mx-3),j+(my-3)})$
3-D	$f(x_S, y_S, z_S) = \sum_{mz=1}^{6} c5_{mz} [\sum_{my=1}^{6} b5_{my} (\sum_{mx=1}^{6} a5_{mx} f_{i+(mx-3),j+(my-3),k(mz-3)})]$

Table 6.3*a*. The fifth-order deposition

1-D	$f_{i+(m-3)} = f_{i+(m-3)} + a5_m f(x_S)$ for $m = 1$ to 6
2-D	$f_{i+(mx-3),j+(my-3)} = f_{i+(mx-3),j+(my-3)} + a5_{mx} b5_{my} f(x_S, y_S)$ for $mx, my = 1$ to 6
3-D	$f_{i+(mx-3),j+(my-3),k(mz-3)} = f_{i+(mx-3),j+(my-3),k(mz-3)} + a5_{mx} b5_{my} c5_{mz} f(x_S, y_S, z_S)$ for $mx, my, mz = 1$ to 6

where

$$a5_1 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-1\cdot-2\cdot-3\cdot-4\cdot-5} = \frac{(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-120}$$

$$a5_2 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+1\cdot-1\cdot-2\cdot-3\cdot-4} = \frac{(\varepsilon_x+2)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+24}$$

$$a5_3 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+2\cdot+1\cdot-1\cdot-2\cdot-3} = \frac{(\varepsilon_x+2)(\varepsilon_x+1)(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-12}$$

$$a5_4 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+3\cdot+2\cdot+1\cdot-1\cdot-2} = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-2)(\varepsilon_x-3)}{+12}$$

$$a5_5 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+4\cdot+3\cdot+2\cdot+1\cdot-1} = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-3)}{-24}$$

$$a5_6 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+5\cdot+4\cdot+3\cdot+2\cdot+1} = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{+120}$$

Likewise,

$$b5_1 = \frac{(\varepsilon_y+1)\varepsilon_y(\varepsilon_y-1)(\varepsilon_y-2)(\varepsilon_y-3)}{-120} \qquad c5_1 = \frac{(\varepsilon_z+1)\varepsilon_z(\varepsilon_z-1)(\varepsilon_z-2)(\varepsilon_z-3)}{-120}$$

$$b5_2 = \frac{(\varepsilon_y+2)\varepsilon_y(\varepsilon_y-1)(\varepsilon_y-2)(\varepsilon_y-3)}{+24} \qquad c5_2 = \frac{(\varepsilon_z+2)\varepsilon_z(\varepsilon_z-1)(\varepsilon_z-2)(\varepsilon_z-3)}{+24}$$

$$b5_3 = \frac{(\varepsilon_y+2)(\varepsilon_y+1)(\varepsilon_y-1)(\varepsilon_y-2)(\varepsilon_y-3)}{-12} \qquad c5_3 = \frac{(\varepsilon_z+2)(\varepsilon_z+1)(\varepsilon_z-1)(\varepsilon_z-2)(\varepsilon_z-3)}{-12}$$

$$b5_4 = \frac{(\varepsilon_y+2)(\varepsilon_y+1)\varepsilon_y(\varepsilon_y-2)(\varepsilon_y-3)}{+12} \qquad c5_4 = \frac{(\varepsilon_z+2)(\varepsilon_z+1)\varepsilon_z(\varepsilon_z-2)(\varepsilon_z-3)}{+12}$$

$$b5_5 = \frac{(\varepsilon_y+2)(\varepsilon_y+1)\varepsilon_y(\varepsilon_y-1)(\varepsilon_y-3)}{-24} \qquad c5_5 = \frac{(\varepsilon_z+2)(\varepsilon_z+1)\varepsilon_z(\varepsilon_z-1)(\varepsilon_z-3)}{-24}$$

$$b5_6 = \frac{(\varepsilon_y+2)(\varepsilon_y+1)\varepsilon_y(\varepsilon_y-1)(\varepsilon_y-2)}{+120} \qquad c5_6 = \frac{(\varepsilon_z+2)(\varepsilon_z+1)\varepsilon_z(\varepsilon_z-1)(\varepsilon_z-2)}{+120}$$

Let us consider the field variable f at integer grid points $\{(x_L, y_L, z_L),...,(x_R, y_R, z_R)\}$

For C++, let

$$i = \mathrm{int}(\frac{x_S - x_L}{\Delta x} + 0.5) \quad j = \mathrm{int}(\frac{y_S - y_L}{\Delta y} + 0.5) \quad k = \mathrm{int}(\frac{z_S - z_L}{\Delta z} + 0.5)$$
$$\varepsilon_x = (x_S - x_i) / \Delta x \qquad \varepsilon_y = (y_S - y_j) / \Delta y \qquad \varepsilon_z = (z_S - z_k) / \Delta z$$

For FORTRAN, let

$$i = 1 + \mathrm{int}(\frac{x_S - x_L}{\Delta x} + 0.5) \quad j = 1 + \mathrm{int}(\frac{y_S - y_L}{\Delta y} + 0.5) \quad k = 1 + \mathrm{int}(\frac{z_S - z_L}{\Delta z} + 0.5)$$
$$\varepsilon_x = (x_S - x_i) / \Delta x \qquad \varepsilon_y = (y_S - y_j) / \Delta y \qquad \varepsilon_z = (z_S - z_k) / \Delta z$$

where $\mathbf{x}_L = (x_L, y_L, z_L)$ is the location where the first element of the field array f is specified. Let $f(\mathbf{x}_S)$ be the field at the particle's position $\mathbf{x}_S = (x_S, y_S, z_S)$. If the grid size is uniform, we can determine $f(\mathbf{x}_S)$ based on the interpolation of the fields at the nearest integer grid points as listed in Tables 6.4~6.6, or we can deposition the information at $f(\mathbf{x}_S)$ to the nearby integer gird points as listed in Tables 6.4*a*~6.6*a*.

Let us consider the field variable f at half-integer grid points

$\{(x_{L+0.5}, y_{L+0.5}, z_{L+0.5}),...,(x_{R+0.5}, y_{R+0.5}, z_{R+0.5})\}$

For C++, let

$$i = \mathrm{int}(\frac{x_S - x_{LH}}{\Delta x} + 0.5) \quad j = \mathrm{int}(\frac{y_S - y_{LH}}{\Delta y} + 0.5) \quad k = \mathrm{int}(\frac{z_S - z_{LH}}{\Delta z} + 0.5)$$
$$\varepsilon_x = (x_S - x_i) / \Delta x \qquad \varepsilon_y = (y_S - y_j) / \Delta y \qquad \varepsilon_z = (z_S - z_k) / \Delta z$$

For FORTRAN, let

$$i = 1 + \mathrm{int}(\frac{x_S - x_{LH}}{\Delta x} + 0.5) \quad j = 1 + \mathrm{int}(\frac{y_S - y_{LH}}{\Delta y} + 0.5) \quad k = 1 + \mathrm{int}(\frac{z_S - z_{LH}}{\Delta z} + 0.5)$$
$$\varepsilon_x = (x_S - x_i) / \Delta x \qquad \varepsilon_y = (y_S - y_j) / \Delta y \qquad \varepsilon_z = (z_S - z_k) / \Delta z$$

where $\mathbf{x}_{LH} = (x_{LH}, y_{LH}, z_{LH}) = (x_{L+0.5}, y_{L+0.5}, z_{L+0.5})$ is the location where the first element of the field array f is specified. Let $f(\mathbf{x}_S)$ be the field at the particle's position $\mathbf{x}_S = (x_S, y_S, z_S)$. If the grid size is uniform, we can determine $f(\mathbf{x}_S)$ based on the interpolation of the fields at the nearest by half-integer grid points as listed in Tables 6.4~6.6 or we can deposition the information at $f(\mathbf{x}_S)$ to the nearby half-integer gird points as listed in Tables 6.4*a*~6.6*a*.

Table 6.4. The second-order interpolation

1-D	$f(x_S)=a2_1 f_{i-1}+a2_2 f_i+a2_3 f_{i+1}=\sum_{m=1}^{3} a2_m f_{i+(m-2)}$
2-D	$f(x_S,y_S)=\sum_{my=1}^{3} b2_{my}(\sum_{mx=1}^{3} a2_{mx} f_{i+(mx-2),j+(my-2)})$
3-D	$f(x_S,y_S,z_S)=\sum_{mz=1}^{3} c2_{mz}[\sum_{my=1}^{3} b2_{my}(\sum_{mx=1}^{3} a2_{mx} f_{i+(mx-2),j+(my-2),k(mz-2)})]$

Table 6.4*a*. The second-order deposition

1-D	$f_{i+(m-2)}=f_{i+(m-2)}+a2_m f(x_S)$ for $m=1$ to 3
2-D	$f_{i+(mx-2),j+(my-2)}=f_{i+(mx-2),j+(my-2)}+a2_{mx}b2_{my}f(x_S,y_S)$ for $mx,my=1$ to 3
3-D	$f_{i+(mx-2),j+(my-2),k(mz-2)}=f_{i+(mx-2),j+(my-2),k(mz-2)}+a2_{mx}b2_{my}c2_{mz}f(x_S,y_S,z_S)$ for $mx,my,mz=1$ to 3

where

$$a2_1=\frac{(x_S-x_{i-1})(x_S-x_i)(x_S-x_{i+1})}{(x_{i-1}-x_{i-1})(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}=\frac{(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)}{0\cdot -1\cdot -2}=\frac{1}{2}\varepsilon_x(\varepsilon_x-1)$$

$$a2_2=\frac{(x_S-x_{i-1})(x_S-x_i)(x_S-x_{i+1})}{(x_i-x_{i-1})(x_i-x_i)(x_i-x_{i+1})}=\frac{(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)}{1\cdot 0\cdot -1}=\frac{1}{-1}(\varepsilon_x+1)(\varepsilon_x-1)=1-\varepsilon_x^2$$

$$a2_3=\frac{(x_S-x_{i-1})(x_S-x_i)(x_S-x_{i+1})}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)(x_{i+1}-x_{i+1})}=\frac{(\varepsilon_x+1)\varepsilon_x(1-\varepsilon_x)}{2\cdot 1\cdot 0}=\frac{1}{2}(\varepsilon_x+1)\varepsilon_x$$

Likewise,

$$b2_1=\frac{1}{2}\varepsilon_y(\varepsilon_y-1) \qquad c2_1=\frac{1}{2}\varepsilon_z(\varepsilon_z-1)$$

$$b2_2=1-\varepsilon_y^2 \qquad c2_2=1-\varepsilon_z^2$$

$$b2_3=\frac{1}{2}(\varepsilon_y+1)\varepsilon_y \qquad c2_3=\frac{1}{2}(\varepsilon_z+1)\varepsilon_z$$

Table 6.5. The fourth-order interpolation

1-D	$f(x_S) = a4_1 f_{i-2} + a4_2 f_{i-1} + a4_3 f_i + a4_4 f_{i+1} + a4_5 f_{i+2} = \sum_{m=1}^{5} a4_m f_{i+(m-3)}$
2-D	$f(x_S, y_S) = \sum_{my=1}^{5} b4_{my} (\sum_{mx=1}^{5} a4_{mx} f_{i+(mx-3),j+(my-3)})$
3-D	$f(x_S, y_S, z_S) = \sum_{mz=1}^{5} c4_{mz} [\sum_{my=1}^{5} b4_{my} (\sum_{mx=1}^{5} a4_{mx} f_{i+(mx-3),j+(my-3),k(mz-3)})]$

Table 6.5*a*. The fourth-order deposition

1-D	$f_{i+(m-3)} = f_{i+(m-3)} + a4_m f(x_S)$ for $m = 1$ to 5
2-D	$f_{i+(mx-3),j+(my-3)} = f_{i+(mx-3),j+(my-3)} + a4_{mx} b4_{my} f(x_S, y_S)$ for $mx, my = 1$ to 5
3-D	$f_{i+(mx-3),j+(my-3),k(mz-3)} = f_{i+(mx-3),j+(my-3),k(mz-3)} + a4_{mx} b4_{my} c4_{mz} f(x_S, y_S, z_S)$ for $mx, my, mz = 1$ to 5

where

$$a4_1 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{-1\cdot-2\cdot-3\cdot-4} = \frac{(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{24}$$

$$a4_2 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{+1\cdot-1\cdot-2\cdot-3} = \frac{(\varepsilon_x+2)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{-6}$$

$$a4_3 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{+2\cdot+1\cdot-1\cdot-2} = \frac{(\varepsilon_x+2)(\varepsilon_x+1)(\varepsilon_x-1)(\varepsilon_x-2)}{4}$$

$$a4_4 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{+3\cdot+2\cdot+1\cdot-1} = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-2)}{-6}$$

$$a4_5 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{+4\cdot+3\cdot+2\cdot+1} = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)}{24}$$

Likewise,

$$b4_1 = \frac{(\varepsilon_y + 1)\varepsilon_y(\varepsilon_y - 1)(\varepsilon_y - 2)}{24} \qquad c4_1 = \frac{(\varepsilon_z + 1)\varepsilon_z(\varepsilon_z - 1)(\varepsilon_z - 2)}{24}$$

$$b4_2 = \frac{(\varepsilon_y + 2)\varepsilon_y(\varepsilon_y - 1)(\varepsilon_y - 2)}{-6} \qquad c4_2 = \frac{(\varepsilon_z + 2)\varepsilon_z(\varepsilon_z - 1)(\varepsilon_z - 2)}{-6}$$

$$b4_3 = \frac{(\varepsilon_y + 2)(\varepsilon_y + 1)(\varepsilon_y - 1)(\varepsilon_y - 2)}{4} \qquad c4_3 = \frac{(\varepsilon_z + 2)(\varepsilon_z + 1)(\varepsilon_z - 1)(\varepsilon_z - 2)}{4}$$

$$b4_4 = \frac{(\varepsilon_y + 2)(\varepsilon_y + 1)\varepsilon_y(\varepsilon_y - 2)}{-6} \qquad c4_4 = \frac{(\varepsilon_z + 2)(\varepsilon_z + 1)\varepsilon_z(\varepsilon_z - 2)}{-6}$$

$$b4_5 = \frac{(\varepsilon_y + 2)(\varepsilon_y + 1)\varepsilon_y(\varepsilon_y - 1)}{24} \qquad c4_5 = \frac{(\varepsilon_z + 2)(\varepsilon_z + 1)\varepsilon_z(\varepsilon_z - 1)}{24}$$

Table 6.6. The sixth-order interpolation

1-D	$f(x_S) = a6_1 f_{i-3} + a6_2 f_{i-2} + a6_3 f_{i-1} + a6_4 f_i + a6_5 f_{i+1} + a6_6 f_{i+2} + a6_7 f_{i+3}$ $= \sum_{m=1}^{7} a6_m f_{i+(m-4)}$
2-D	$f(x_S, y_S) = \sum_{my=1}^{7} b6_{my} (\sum_{mx=1}^{7} a6_{mx} f_{i+(mx-4),j+(my-4)})$
3-D	$f(x_S, y_S, z_S) = \sum_{mz=1}^{7} c6_{mz} [\sum_{my=1}^{7} b6_{my} (\sum_{mx=1}^{7} a6_{mx} f_{i+(mx-4),j+(my-4),k(mz-4)})]$

Table 6.6*a*. The sixth-order deposition

1-D	$f_{i+(m-4)} = f_{i+(m-4)} + a6_m f(x_S)$ for $m = 1$ to 7
2-D	$f_{i+(mx-4),j+(my-4)} = f_{i+(mx-4),j+(my-4)} + a6_{mx} b6_{my} f(x_S, y_S)$ for $mx, my = 1$ to 7
3-D	$f_{i+(mx-4),j+(my-4),k(mz-4)} = f_{i+(mx-4),j+(my-4),k(mz-4)} + a6_{mx} b6_{my} c6_{mz} f(x_S, y_S, z_S)$ for $mx, my, mz = 1$ to 7

where

$$a6_1 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-1\cdot -2\cdot -3\cdot -4\cdot -5\cdot -6} = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+720}$$

$$a6_2 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+1\cdot -1\cdot -2\cdot -3\cdot -4\cdot -5} = \frac{(\varepsilon_x+3)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-120}$$

$$a6_3 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+2\cdot +1\cdot -1\cdot -2\cdot -3\cdot -4} = \frac{(\varepsilon_x+3)(\varepsilon_x+2)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+48}$$

$$a6_4 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+3\cdot +2\cdot +1\cdot -1\cdot -2\cdot -3} = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-36}$$

$$a6_5 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+4\cdot +3\cdot +2\cdot +1\cdot -1\cdot -2} = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-2)(\varepsilon_x-3)}{+48}$$

$$a6_6 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+5\cdot +4\cdot +3\cdot +2\cdot +1\cdot -1} = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-3)}{-120}$$

$$a6_7 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+6\cdot +5\cdot +4\cdot +3\cdot +2\cdot +1} = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{+720}$$

Likewise,

$$b6_1 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+720} \qquad c6_1 = \frac{(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+720}$$

$$b6_2 = \frac{(\varepsilon_x+3)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-120} \qquad c6_2 = \frac{(\varepsilon_x+3)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-120}$$

$$b6_3 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+48} \qquad c6_3 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{+48}$$

$$b6_4 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-36} \qquad c6_4 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)(\varepsilon_x-1)(\varepsilon_x-2)(\varepsilon_x-3)}{-36}$$

$$b6_5 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-2)(\varepsilon_x-3)}{+48} \qquad c6_5 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-2)(\varepsilon_x-3)}{+48}$$

$$b6_6 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-3)}{-120} \qquad c6_6 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-3)}{-120}$$

$$b6_7 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{+720} \qquad c6_7 = \frac{(\varepsilon_x+3)(\varepsilon_x+2)(\varepsilon_x+1)\varepsilon_x(\varepsilon_x-1)(\varepsilon_x-2)}{+720}$$

For mixing integer and half-integer fields, we can use respectively a combination of Tables 6.1~6.3 and Tables 6.4~6.6 for interpolation and use respectively a combination of Tables 6.1*a*~6.3a and Tables 6.4*a*~6.6*a* for deposition. For instance, we can deposit the electric current carried by a charge particle with charge q_S, velocity $\mathbf{v}_S$, and position $\mathbf{x}_S$ to the electric current density field [$(J_x)^H_{H,N,N}$ $(J_y)^H_{N,H,N}$ $(J_z)^H_{N,N,H}$] based on the following deposition schemes

$$(J_x)_{i+(mx-4),j+(my-3),k(mz-3)} = (J_x)_{i+(mx-4),j+(my-3),k(mz-3)} + a6_{mx} b5_{my} c5_{mz} [q_S(\mathbf{v}_S)_x]$$
$$\text{for} \quad mx = 1 \text{ to } 7, \quad my, mz = 1 \text{ to } 6$$
$$(J_y)_{i+(mx-3),j+(my-4),k(mz-3)} = (J_y)_{i+(mx-3),j+(my-4),k(mz-3)} + a5_{mx} b6_{my} c5_{mz} [q_S(\mathbf{v}_S)_y]$$
$$\text{for} \quad my = 1 \text{ to } 7, \quad mx, mz = 1 \text{ to } 6$$
$$(J_z)_{i+(mx-3),j+(my-3),k(mz-4)} = (J_z)_{i+(mx-3),j+(my-3),k(mz-4)} + a5_{mx} b5_{my} c6_{mz} [q_S(\mathbf{v}_S)_z]$$
$$\text{for} \quad mz = 1 \text{ to } 7, \quad mx, my = 1 \text{ to } 6$$

Likewise, we can find the electric field [$(E_x)^N_{H,N,N}$ $(E_y)^N_{N,H,N}$ $(E_z)^N_{N,N,H}$] and magnetic filed [$(B_x)^H_{N,H,H}$ $(B_y)^H_{H,N,H}$ $(B_z)^H_{H,H,N}$] at $\mathbf{x}_S$ based on the following interpolation schemes

$$E_x^N(x_S, y_S, z_S) = \sum_{mz=1}^{6} c5_{mz} [\sum_{my=1}^{6} b5_{my} (\sum_{mx=1}^{7} a6_{mx} (E_x^N)_{i+(mx-4),j+(my-3),k(mz-3)})]$$
$$E_y^N(x_S, y_S, z_S) = \sum_{mz=1}^{6} c5_{mz} [\sum_{my=1}^{7} b6_{my} (\sum_{mx=1}^{6} a5_{mx} (E_y^N)_{i+(mx-3),j+(my-4),k(mz-3)})]$$
$$E_z^N(x_S, y_S, z_S) = \sum_{mz=1}^{7} c6_{mz} [\sum_{my=1}^{6} b5_{my} (\sum_{mx=1}^{6} a5_{mx} (E_z^N)_{i+(mx-3),j+(my-3),k(mz-4)})]$$

$$B_x^H(x_S, y_S, z_S) = \sum_{mz=1}^{7} c6_{mz} [\sum_{my=1}^{7} b6_{my} (\sum_{mx=1}^{6} a5_{mx} (B_x^H)_{i+(mx-3),j+(my-4),k(mz-4)})]$$
$$B_y^H(x_S, y_S, z_S) = \sum_{mz=1}^{7} c6_{mz} [\sum_{my=1}^{6} b5_{my} (\sum_{mx=1}^{7} a6_{mx} (B_y^H)_{i+(mx-4),j+(my-3),k(mz-4)})]$$
$$B_z^H(x_S, y_S, z_S) = \sum_{mz=1}^{6} c5_{mz} [\sum_{my=1}^{7} b6_{my} (\sum_{mx=1}^{7} a6_{mx} (B_z^H)_{i+(mx-4),j+(my-4),k(mz-3)})]$$

Finally, it is shown in Figure 6.3 that the accumulated numerical errors will be reduced significantly in the simulations with higher order-deposition and interpolation schemes.

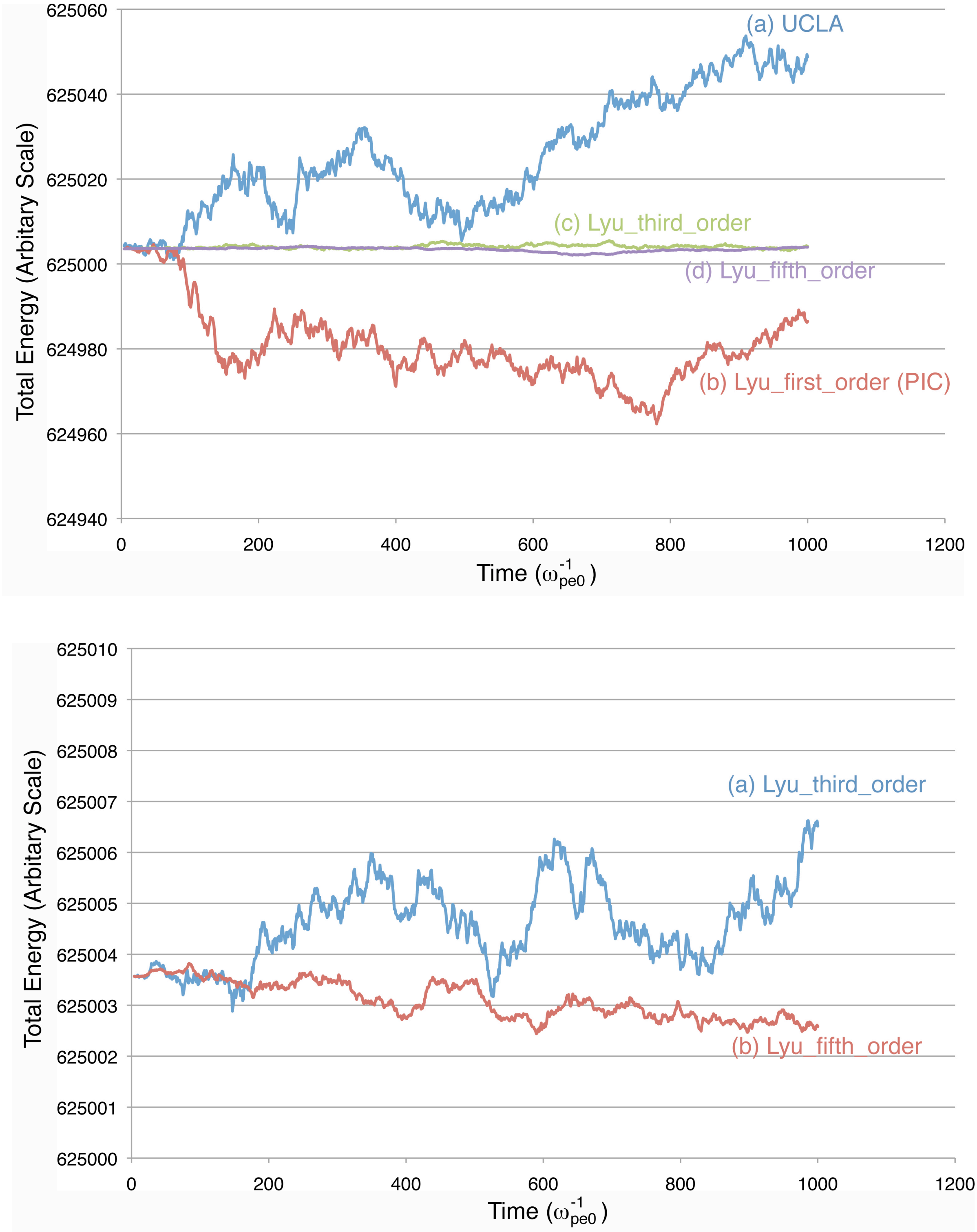

Figure 6.3. Differences in the accumulated numerical errors in the particle simulations with interpolations and depositions at different orders.