Appendix D. Higher-Order Finite Differences Based on Richardson's Formula

D.1. Richardson's Formula (Maron and Lopez, 1991, Chap.7)

D.1.1. Determine $f^{(k)}(x)$*, where* k *is an even number*

Consider a tabulate function $f(x)$

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{6}f^{(3)}(x)+\frac{h^4}{24}f^{(4)}(x)+\frac{h^5}{120}f^{(5)}(x)$$
$$+\frac{h^6}{720}f^{(6)}(x)+\frac{h^7}{5040}f^{(7)}(x)+\frac{h^8}{40320}f^{(8)}(x) \tag{D.1}$$
$$+\frac{h^9}{362880}f^{(9)}(x)+\frac{h^{10}}{3628800}f^{(10)}(x)+O(h^{11}f^{(11)})$$

$$f(x-h)=f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f^{(3)}(x)+\frac{h^4}{24}f^{(4)}(x)-\frac{h^5}{120}f^{(5)}(x)$$
$$+\frac{h^6}{720}f^{(6)}(x)-\frac{h^7}{5040}f^{(7)}(x)+\frac{h^8}{40320}f^{(8)}(x) \tag{D.2}$$
$$-\frac{h^9}{362880}f^{(9)}(x)+\frac{h^{10}}{3628800}f^{(10)}(x)+O(h^{11}f^{(11)})$$

Form $\frac{(B.1)+(B.2)}{h^2}$, it yields

$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=f''(x)+\frac{h^2}{12}f^{(4)}(x)+\frac{h^4}{360}f^{(6)}(x)+\frac{h^6}{20160}f^{(8)}(x)$$
$$+\frac{h^8}{1814400}f^{(10)}(x)+O(h^{10}f^{(12)}) \tag{D.3}$$

Replacing h in equation (D.3) by $h/2$, it yields

$$\frac{f(x+\frac{h}{2})-2f(x)+f(x-\frac{h}{2})}{(h/2)^2}=f''(x)+\frac{(h/2)^2}{12}f^{(4)}(x)+\frac{(h/2)^4}{360}f^{(6)}(x)+\frac{(h/2)^6}{20160}f^{(8)}(x)$$
$$+\frac{(h/2)^8}{1814400}f^{(10)}(x)+O(h^{10}f^{(12)}) \tag{D.4}$$

Form $\frac{2^2(B.4)-(B.3)}{2^2-1}$, it yields

$$\frac{[-f(x+h)+16f(x+\frac{h}{2})-30f(x)+16f(x-\frac{h}{2})-f(x-h)]}{3h^2}$$
$$=f''(x)+0-\frac{1}{2^2}\frac{h^4}{360}f^{(6)}(x)-\frac{1+2^2}{2^4}\frac{h^6}{20160}f^{(8)}(x)-\frac{1+2^2+2^4}{2^6}\frac{h^8}{1814400}f^{(10)}(x)+O(h^{10}f^{(12)}) \tag{D.5}$$

Replacing h in equation (D.5) by $h/2$, it yields

$$\frac{-f(x+\frac{h}{2})+16f(x+\frac{h}{4})-30f(x)+16f(x-\frac{h}{4})-f(x-\frac{h}{2})}{3(h/2)^2}$$
$$=f''(x)-\frac{1}{2^2}\frac{(h/2)^4}{360}f^{(6)}(x)-\frac{1+2^2}{2^4}\frac{(h/2)^6}{20160}f^{(8)}(x)-\frac{1+2^2+2^4}{2^6}\frac{(h/2)^8}{1814400}f^{(10)}(x)+O(h^{10}f^{(12)})$$

(D.6)

Form $\frac{2^4(B.6)-(B.5)}{2^4-1}$, it yields

$$\frac{f(x+h)-80f(x+\frac{h}{2})+1024f(x+\frac{h}{4})-1890f(x)+1024f(x-\frac{h}{4})-80f(x-\frac{h}{2})+f(x-h)}{45h^2}$$
$$=f''(x)+\frac{1}{2^6}\frac{h^6}{20160}f^{(8)}(x)+\frac{1+2^2+2^4}{2^{10}}\frac{h^8}{1814400}f^{(10)}(x)+O(h^{10}f^{(12)})$$

(D.7)

Replacing h in equation (D.7) by $h/2$, it yields

$$\frac{f(x+\frac{h}{2})-80f(x+\frac{h}{4})+1024f(x+\frac{h}{8})-1890f(x)+1024f(x-\frac{h}{8})-80f(x-\frac{h}{4})+f(x-\frac{h}{2})}{45(h/2)^2}$$
$$=f''(x)+\frac{1}{2^6}\frac{(h/2)^6}{20160}f^{(8)}(x)+\frac{1+2^2+2^4}{2^{10}}\frac{(h/2)^8}{1814400}f^{(10)}(x)+O(h^{10}f^{(12)})$$

(D.8)

Form $\frac{2^6(B.8)-(B.7)}{2^6-1}$, it yields

$$\frac{1}{2835h^2}[-f(x+h)+336f(x+\frac{h}{2})-21504f(x+\frac{h}{4})+256\cdot 1024f(x+\frac{h}{8})-255\cdot 1890f(x)$$
$$-f(x+h)+336f(x-\frac{h}{2})-21504f(x-\frac{h}{4})+256\cdot 1024f(x-\frac{h}{8})]$$
$$=f''(x)-\frac{1}{2^{12}}\frac{h^8}{1814400}f^{(10)}(x)+O(h^{10}f^{(12)})$$

(D.9)

We have obtained finite difference expressions of $f''(x)$.

Likewise, we can obtain finite difference expressions of $f^{(4)}(x)$ from $\frac{[(B.3)-(B.4)]}{h^2/2^4}$.

D.1.2. Determine $f^{(k)}(x)$*, where* k *is an odd number*

Consider a tabulate function $f(x)$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f^{(3)}(x) + \frac{h^4}{24}f^{(4)}(x) + \frac{h^5}{120}f^{(5)}(x) + \frac{h^6}{720}f^{(6)}(x) + \frac{h^7}{5040}f^{(7)}(x) + \frac{h^8}{40320}f^{(8)}(x) + \frac{h^9}{362880}f^{(9)}(x) + \frac{h^{10}}{3628800}f^{(10)}(x) + O(h^{11}f^{(11)}) \tag{D.1}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f^{(3)}(x) + \frac{h^4}{24}f^{(4)}(x) - \frac{h^5}{120}f^{(5)}(x) + \frac{h^6}{720}f^{(6)}(x) - \frac{h^7}{5040}f^{(7)}(x) + \frac{h^8}{40320}f^{(8)}(x) - \frac{h^9}{362880}f^{(9)}(x) + \frac{h^{10}}{3628800}f^{(10)}(x) + O(h^{11}f^{(11)}) \tag{D.2}$$

Form $\frac{(B.1)-(B.2)}{2h}$, it yields

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + \frac{h^2}{6}f^{(3)}(x) + \frac{h^4}{120}f^{(5)}(x) + \frac{h^6}{5040}f^{(7)}(x) + \frac{h^8}{362880}f^{(9)}(x) + O(h^{10}f^{(11)}) \tag{D.3a}$$

Replacing h in equation (D.3*a*) by $h/2$, it yields

$$\frac{1}{2(h/2)}[f(x+\frac{h}{2}) - f(x-\frac{h}{2})] = f'(x) + \frac{(h/2)^2}{6}f^{(3)}(x) + \frac{(h/2)^4}{120}f^{(5)}(x) + \frac{(h/2)^6}{5040}f^{(7)}(x) + \frac{(h/2)^8}{362880}f^{(9)}(x) + O(h^{10}f^{(11)}) \tag{D.4a}$$

Form $\frac{2^2(B.4a)-(B.3a)}{2^2-1}$, it yields

$$\frac{[-f(x+h) + 8f(x+\frac{h}{2}) - 8f(x-\frac{h}{2}) + f(x-h)]}{6h} = f'(x) - \frac{1}{2^2}\frac{h^4}{120}f^{(5)}(x) - \frac{1+2^2}{2^4}\frac{h^6}{5040}f^{(7)}(x) - \frac{1+2^2+2^4}{2^6}\frac{h^8}{362880}f^{(9)}(x) + O(h^{10}f^{(11)}) \tag{D.5a}$$

Replacing h in equation (D.5*a*) by $h/2$, it yields

$$\frac{[-f(x+\frac{h}{2})+8f(x+\frac{h}{4})-8f(x-\frac{h}{4})+f(x-\frac{h}{2})]}{6(h/2)}$$
$$= f'(x)-\frac{1}{2^2}\frac{(h/2)^4}{120}f^{(5)}(x)-\frac{1+2^2}{2^4}\frac{(h/2)^6}{5040}f^{(7)}(x)-\frac{1+2^2+2^4}{2^6}\frac{(h/2)^8}{362880}f^{(9)}(x)+O(h^{10}f^{(11)}) \quad \text{(D.6a)}$$

Form $\frac{2^4(B.6a)-(B.5a)}{2^4-1}$, it yields

$$\frac{[f(x+h)-40f(x+\frac{h}{2})+256f(x+\frac{h}{4})-256f(x+\frac{h}{4})+40f(x-\frac{h}{2})-f(x-h)]}{90h} \quad \text{(D.7a)}$$
$$= f'(x)+\frac{1}{2^4}\frac{h^6}{5040}f^{(7)}(x)+\frac{1+2^2+2^4}{2^{10}}\frac{h^8}{362880}f^{(9)}(x)+O(h^{10}f^{(11)})$$

Replacing h in equation (D.7a) by $h/2$, it yields

$$\frac{[f(x+\frac{h}{2})-40f(x+\frac{h}{4})+256f(x+\frac{h}{8})-256f(x+\frac{h}{8})+40f(x-\frac{h}{4})-f(x-\frac{h}{2})]}{90(h/2)}$$
$$= f'(x)+\frac{1}{2^4}\frac{(h/2)^6}{5040}f^{(7)}(x)+\frac{1+2^2+2^4}{2^{10}}\frac{(h/2)^8}{362880}f^{(9)}(x)+O(h^{10}f^{(11)}) \quad \text{(D.8a)}$$

Form $\frac{2^6(B.8a)-(B.7a)}{2^6-1}$, it yields

$$\frac{1}{5670h}[-f(x+h)+168f(x+\frac{h}{2})-5376f(x+\frac{h}{4})+32768f(x+\frac{h}{8})$$
$$+f(x-h)-168f(x-\frac{h}{2})+5376f(x-\frac{h}{4})-32768f(x-\frac{h}{8})] \quad \text{(D.9a)}$$
$$= f'(x)-\frac{1}{2^{12}}\frac{h^8}{362880}f^{(9)}(x)+O(h^{10}f^{(11)})$$

We have obtained finite difference expressions of $f'(x)$.

Likewise, we can obtain finite difference expressions of $f^{(3)}(x)$ from $\frac{[(B.3)-(B.4)]}{h^2/2^3}$.

D.2. Summary of Richardson's Formula

The first order finite difference $O(h^2)$

$$f_i' = \frac{1}{2h}(f_{i+1} - f_{i-1}) + O(h^2 f^{(3)})$$

$$f_i'' = \frac{1}{h^2}(f_{i+1} - 2f_i + f_{i-1}) + O(h^2 f^{(4)})$$

$$f_i^{(3)} = \frac{1}{2h^3}(f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}) + O(h^2 f^{(5)})$$

$$f_i^{(4)} = \frac{1}{h^4}(f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}) + O(h^2 f^{(6)})$$

The third order finite difference $O(h^4)$

$$f_i' = \frac{1}{2h}[-\frac{1}{6}(f_{i+2} - f_{i-2}) + \frac{4}{3}(f_{i+1} - f_{i-1})] + O(h^4 f^{(5)})$$

$$f_i'' = \frac{1}{h^2}[-\frac{1}{12}(f_{i+2} - 2f_i + f_{i-2}) + \frac{4}{3}(f_{i+1} - 2f_i + f_{i-1})] + O(h^4 f^{(6)})$$

$$f_i^{(3)} = \frac{1}{2h^3}[-\frac{1}{24}(f_{i+4} - 2f_{i+2} + 2f_{i-2} - f_{i-4}) + \frac{4}{3}(f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2})] + O(h^4 f^{(7)})$$

$$f_i^{(4)} = \frac{1}{h^4}[-\frac{1}{48}(f_{i+4} - 4f_{i+2} + 6f_i - 4f_{i-2} + f_{i-4}) + \frac{4}{3}(f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2})] + O(h^4 f^{(8)})$$

where

$$\frac{4}{3} = \frac{2^2}{(2^2-1)}, \quad \frac{1}{6} = \frac{2^2}{(2^2-1)} \cdot \frac{1}{2^3}$$

The fifth order finite difference $O(h^6)$

$$f_i' = \frac{1}{2h}[+\frac{1}{180}(f_{i+4} - f_{i-4}) - \frac{2}{9}(f_{i+2} - f_{i-2}) + \frac{64}{45}(f_{i+1} - f_{i-1})] + O(h^6 f^{(7)})$$

$$f_i'' = \frac{1}{h^2}[\frac{1}{720}(f_{i+4} - 2f_i + f_{i-4}) - \frac{1}{9}(f_{i+2} - 2f_i + f_{i-2}) + \frac{64}{45}(f_{i+1} - 2f_i + f_{i-1})] + O(h^6 f^{(8)})$$

$$f_i^{(3)} = \frac{1}{2h^3}[+\frac{1}{2880}(f_{i+8} - 2f_{i+4} + 2f_{i-4} - f_{i-8}) - \frac{1}{18}(f_{i+4} - 2f_{i+2} + 2f_{i-2} - f_{i-4}) + \frac{64}{45}(f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2})] + O(h^6 f^{(9)})$$

$$f_i^{(4)} = \frac{1}{h^4}[+\frac{1}{11520}(f_{i+4} - 4f_{i+2} + 6f_i - 4f_{i-2} + f_{i-4}) - \frac{1}{36}(f_{i+4} - 4f_{i+2} + 6f_i - 4f_{i-2} + f_{i-4}) + \frac{64}{45}(f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2})] + O(h^6 f^{(10)})$$

where

$$\frac{64}{45} = \frac{2^4 \cdot 2^2}{(2^4-1)(2^2-1)}, \quad \frac{2}{9} = \frac{2^4 \cdot 2^2}{(2^4-1)(2^2-1)} \cdot \frac{2^2+1}{2^5}, \quad \frac{1}{180} = \frac{2^4 \cdot 2^2}{(2^4-1)(2^2-1)} \cdot \frac{1}{2^8}$$

Reference

Maron, M. J., and R. J. Lopez (1991), *Numerical Analysis: A Practical Approach,* Wadsworth Publ. Co., Belmont, California.