Interpolation Schemes

Let us consider a point with position x, or (x, y), or (x, y, z) in a 1-D, 2-D, or 3-D system, respectively. The interpolations of a field $f(\mathbf{x})$ at this point with respect to the near by grids are summarized below.

1. Linear interpolations

For $x_0 < x < x_1$, $y_0 < y < y_1$, $z_0 < z < z_1$, the linear interpolations of $f(\mathbf{x})$ can be written as

$$f(x) = f(x_0 + \delta x) = \sum_{i=0}^{1} a_i f(x_i) = \sum_{i=0}^{1} a_i f_i$$

$$f(x,y) = f(x_0 + \delta x, y_0 + \delta y) = \sum_{j=0}^{1} b_j (\sum_{i=0}^{1} a_i f(x_i, y_j)) = \sum_{j=0}^{1} b_j (\sum_{i=0}^{1} a_i f_{ij})$$

$$f(x,y,z) = f(x_0 + \delta x, y_0 + \delta y, z_0 + \delta z)$$

$$= \sum_{k=0}^{1} c_k [\sum_{j=0}^{1} b_j (\sum_{i=0}^{1} a_i f(x_i, y_j, z_k))] = \sum_{k=0}^{1} c_k [\sum_{j=0}^{1} b_j (\sum_{i=0}^{1} a_i f_{ijk})]$$

where

$a_0 = \dfrac{(x - x_1)}{(x_0 - x_1)}$	$b_0 = \dfrac{(y - y_1)}{(y_0 - y_1)}$	$c_0 = \dfrac{(z - z_1)}{(z_0 - z_1)}$
$a_1 = \dfrac{(x - x_0)}{(x_1 - x_0)}$	$b_1 = \dfrac{(y - y_0)}{(y_1 - y_0)}$	$c_1 = \dfrac{(z - z_0)}{(z_1 - z_0)}$

For simulations with equal-spacing grid size, we define the grid size in the x, y, and z directions to be h_x, h_y, and h_z, respectively. For convenience we also define

$D_{XL} = \frac{x - x_0}{h_x} = \frac{\delta x}{h_x}$ $D_{XR} = \frac{x - x_1}{h_x} = \frac{\delta x}{h_x} - 1$	$D_{YL} = \frac{y - y_0}{h_y} = \frac{\delta y}{h_y}$ $D_{YR} = \frac{y - y_1}{h_y} = \frac{\delta y}{h_y} - 1$	$D_{ZL} = \frac{z - z_0}{h_z} = \frac{\delta z}{h_z}$ $D_{ZR} = \frac{z - z_1}{h_z} = \frac{\delta z}{h_z} - 1$

It yields

$a_0 = -D_{XR}$ $a_1 = D_{XL}$	$b_0 = -D_{YR}$ $b_1 = D_{YL}$	$c_0 = -D_{ZR}$ $c_1 = D_{ZL}$

That is the interpolation used in the classical PIC code simulation, which is commonly written as

$$f(x_0+\delta x) = (1-\frac{\delta x}{h})f_0 + (\frac{\delta x}{h})f_1$$

$$f(x_0+\delta x, y_0+\delta y) = (1-\frac{\delta y}{h_y})[(1-\frac{\delta x}{h_x})f_{00} + (\frac{\delta x}{h_x})f_{10}] + (\frac{\delta y}{h_y})[(1-\frac{\delta x}{h_x})f_{01} + (\frac{\delta x}{h_x})f_{11}]$$

$$\begin{aligned} &f(x_0+\delta x, y_0+\delta y, z_0+\delta z) \\ &= (1-\frac{\delta z}{h_z})\{(1-\frac{\delta y}{h_y})[(1-\frac{\delta x}{h_x})f_{000} + (\frac{\delta x}{h_x})f_{100}] + (\frac{\delta y}{h_y})[(1-\frac{\delta x}{h_x})f_{010} + (\frac{\delta x}{h_x})f_{110}]\} \\ &\quad + (\frac{\delta z}{h_z})\{(1-\frac{\delta y}{h_y})[(1-\frac{\delta x}{h_x})f_{001} + (\frac{\delta x}{h_x})f_{101}] + (\frac{\delta y}{h_y})[(1-\frac{\delta x}{h_x})f_{011} + (\frac{\delta x}{h_x})f_{111}]\} \end{aligned}$$

2. Cubic interpolations

For $x_{-1} < x_0 < x < x_1 < x_2$, $y_{-1} < y_0 < y < y_1 < y_2$, $z_{-1} < z_0 < z < z_1 < z_2$, the cubic interpolations of $f(\mathbf{x})$ can be written as

$$f(x) = f(x_0 + \delta x) = \sum_{i=-1}^{2} a_i f(x_i) = \sum_{i=-1}^{2} a_i f_i$$

$$f(x,y) = f(x_0 + \delta x, y_0 + \delta y) = \sum_{j=-1}^{2} b_j (\sum_{i=-1}^{2} a_i f(x_i, y_j)) = \sum_{j=-1}^{2} b_j (\sum_{i=-1}^{2} a_i f_{ij})$$

$$f(x,y,z) = f(x_0 + \delta x, y_0 + \delta y, z_0 + \delta z)$$

$$= \sum_{k=-1}^{2} c_k [\sum_{j=-1}^{2} b_j (\sum_{i=-1}^{2} a_i f(x_i, y_j, z_k))] = \sum_{k=-1}^{2} c_k [\sum_{j=-1}^{2} b_j (\sum_{i=-1}^{2} a_i f_{ijk})]$$

where

$$a_{-1} = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_{-1}-x_0)(x_{-1}-x_1)(x_{-1}-x_2)}$$

$$a_0 = \frac{(x-x_{-1})(x-x_1)(x-x_2)}{(x_0-x_{-1})(x_0-x_1)(x_0-x_2)}$$

$$a_1 = \frac{(x-x_{-1})(x-x_0)(x-x_2)}{(x_1-x_{-1})(x_1-x_0)(x_1-x_2)}$$

$$a_2 = \frac{(x-x_{-1})(x-x_0)(x-x_1)}{(x_2-x_{-1})(x_2-x_0)(x_2-x_1)}$$

$$b_{-1} = \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_{-1}-y_0)(y_{-1}-y_1)(y_{-1}-y_2)}$$

$$b_0 = \frac{(y-y_{-1})(y-y_1)(y-y_2)}{(y_0-y_{-1})(y_0-y_1)(y_0-y_2)}$$

$$b_1 = \frac{(y-y_{-1})(y-y_0)(y-y_2)}{(y_1-y_{-1})(y_1-y_0)(y_1-y_2)}$$

$$b_2 = \frac{(y-y_{-1})(y-y_0)(y-y_1)}{(y_2-y_{-1})(y_2-y_0)(y_2-y_1)}$$

$$c_{-1} = \frac{(z-z_0)(z-z_1)(z-z_2)}{(z_{-1}-z_0)(z_{-1}-z_1)(z_{-1}-z_2)}$$

$$c_0 = \frac{(z-z_{-1})(z-z_1)(z-z_2)}{(z_0-z_{-1})(z_0-z_1)(z_0-z_2)}$$

$$c_1 = \frac{(z-z_{-1})(z-z_0)(z-z_2)}{(z_1-z_{-1})(z_1-z_0)(z_1-z_2)}$$

$$c_2 = \frac{(z-z_{-1})(z-z_0)(z-z_1)}{(z_2-z_{-1})(z_2-z_0)(z_2-z_1)}$$

For simulations with equal-spacing grid size, we define the grid size in the x, y, and z directions to be h_x, h_y, and h_z, respectively. For convenience we also define

$D_{XLL} = \dfrac{x - x_{-1}}{h_x} = 1 + \dfrac{\delta x}{h_x}$	$D_{YLL} = \dfrac{y - y_{-1}}{h_y} = 1 + \dfrac{\delta y}{h_y}$	$D_{ZLL} = \dfrac{z - z_{-1}}{h_z} = 1 + \dfrac{\delta z}{h_z}$
$D_{XL} = \dfrac{x - x_0}{h_x} = \dfrac{\delta x}{h_x}$	$D_{YL} = \dfrac{y - y_0}{h_y} = \dfrac{\delta y}{h_y}$	$D_{ZL} = \dfrac{z - z_0}{h_z} = \dfrac{\delta z}{h_z}$
$D_{XR} = \dfrac{x - x_1}{h_x} = \dfrac{\delta x}{h_x} - 1$	$D_{YR} = \dfrac{y - y_1}{h_y} = \dfrac{\delta y}{h_y} - 1$	$D_{ZR} = \dfrac{z - z_1}{h_z} = \dfrac{\delta z}{h_z} - 1$
$D_{XRR} = \dfrac{x - x_2}{h_x} = \dfrac{\delta x}{h_x} - 2$	$D_{YRR} = \dfrac{y - y_2}{h_y} = \dfrac{\delta y}{h_y} - 2$	$D_{ZRR} = \dfrac{z - z_2}{h_z} = \dfrac{\delta z}{h_z} - 2$

It yields

$a_{-1} = -\frac{1}{6}(D_{XL} D_{XR} D_{XRR})$	$b_{-1} = -\frac{1}{6}(D_{YL} D_{YR} D_{YRR})$	$c_{-1} = -\frac{1}{6}(D_{ZL} D_{ZR} D_{ZRR})$
$a_{0} = +\frac{1}{2}(D_{XLL} D_{XR} D_{XRR})$	$b_{0} = +\frac{1}{2}(D_{YLL} D_{YR} D_{YRR})$	$c_{0} = +\frac{1}{2}(D_{ZLL} D_{ZR} D_{ZRR})$
$a_{1} = -\frac{1}{2}(D_{XLL} D_{XL} D_{XRR})$	$b_{1} = -\frac{1}{2}(D_{YLL} D_{YL} D_{YRR})$	$c_{1} = -\frac{1}{2}(D_{ZLL} D_{ZL} D_{ZRR})$
$a_{2} = +\frac{1}{6}(D_{XLL} D_{XL} D_{XR})$	$b_{2} = +\frac{1}{6}(D_{YLL} D_{YL} D_{YR})$	$c_{2} = +\frac{1}{6}(D_{ZLL} D_{ZL} D_{ZR})$

3. Fifth-order interpolations

For $x_{-2} < x_{-1} < x_0 < x < x_1 < x_2 < x_3$, $y_{-2} < y_{-1} < y_0 < y < y_1 < y_2 < y_3$, $z_{-2} < z_{-1} < z_0 < z < z_1 < z_2 < z_3$, the 5th-order interpolations of $f(\mathbf{x})$ can be written as

$$f(x) = f(x_0 + \delta x) = \sum_{i=-2}^{3} a_i f(x_i) = \sum_{i=-2}^{3} a_i f_i$$

$$f(x,y) = f(x_0 + \delta x,\, y_0 + \delta y) = \sum_{j=-2}^{3} b_j (\sum_{i=-2}^{3} a_i f(x_i, y_j)) = \sum_{j=-2}^{3} b_j (\sum_{i=-2}^{3} a_i f_{ij})$$

$$f(x,y,z) = f(x_0 + \delta x,\, y_0 + \delta y,\, z_0 + \delta z)$$

$$= \sum_{k=-2}^{3} c_k [\sum_{j=-2}^{3} b_j (\sum_{i=-2}^{3} a_i f(x_i, y_j, z_k))] = \sum_{k=-2}^{3} c_k [\sum_{j=-2}^{3} b_j (\sum_{i=-2}^{3} a_i f_{ijk})]$$

where

$$a_{-2} = \frac{(x-x_{-1})(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_{-2}-x_{-1})(x_{-2}-x_0)(x_{-2}-x_1)(x_{-2}-x_2)(x_{-2}-x_3)}$$

$$a_{-1} = \frac{(x-x_{-2})(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_{-1}-x_{-2})(x_{-1}-x_0)(x_{-1}-x_1)(x_{-1}-x_2)(x_{-1}-x_3)}$$

$$a_0 = \frac{(x-x_{-2})(x-x_{-1})(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_{-2})(x_0-x_{-1})(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$a_1 = \frac{(x-x_{-2})(x-x_{-1})(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_{-2})(x_1-x_{-1})(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$a_2 = \frac{(x-x_{-2})(x-x_{-1})(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_{-2})(x_2-x_{-1})(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$a_3 = \frac{(x-x_{-2})(x-x_{-1})(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_{-2})(x_3-x_{-1})(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$b_{-2} = \frac{(y-y_{-1})(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_{-2}-y_{-1})(y_{-2}-y_0)(y_{-2}-y_1)(y_{-2}-y_2)(y_{-2}-y_3)}$$

$$b_{-1} = \frac{(y-y_{-2})(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_{-1}-y_{-2})(y_{-1}-y_0)(y_{-1}-y_1)(y_{-1}-y_2)(y_{-1}-y_3)}$$

$$b_0 = \frac{(y-y_{-2})(y-y_{-1})(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_{-2})(y_0-y_{-1})(y_0-y_1)(y_0-y_2)(y_0-y_3)}$$

$$b_1 = \frac{(y-y_{-2})(y-y_{-1})(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_{-2})(y_1-y_{-1})(y_1-y_0)(y_1-y_2)(y_1-y_3)}$$

$$b_2 = \frac{(y-y_{-2})(y-y_{-1})(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_{-2})(y_2-y_{-1})(y_2-y_0)(y_2-y_1)(y_2-y_3)}$$

$$b_3 = \frac{(y-y_{-2})(y-y_{-1})(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_{-2})(y_3-y_{-1})(y_3-y_0)(y_3-y_1)(y_3-y_2)}$$

$$c_{-2} = \frac{(z - z_{-1})(z - z_0)(z - z_1)(z - z_2)(z - z_3)}{(z_{-2} - z_{-1})(z_{-2} - z_0)(z_{-2} - z_1)(z_{-2} - z_2)(z_{-2} - z_3)}$$

$$c_{-1} = \frac{(z - z_{-2})(z - z_0)(z - z_1)(z - z_2)(z - z_3)}{(z_{-1} - z_{-2})(z_{-1} - z_0)(z_{-1} - z_1)(z_{-1} - z_2)(z_{-1} - z_3)}$$

$$c_0 = \frac{(z - z_{-2})(z - z_{-1})(z - z_1)(z - z_2)(z - z_3)}{(z_0 - z_{-2})(z_0 - z_{-1})(z_0 - z_1)(z_0 - z_2)(z_0 - z_3)}$$

$$c_1 = \frac{(z - z_{-2})(z - z_{-1})(z - z_0)(z - z_2)(z - z_3)}{(z_1 - z_{-2})(z_1 - z_{-1})(z_1 - z_0)(z_1 - z_2)(z_1 - z_3)}$$

$$c_2 = \frac{(z - z_{-2})(z - z_{-1})(z - z_0)(z - z_1)(z - z_3)}{(z_2 - z_{-2})(z_2 - z_{-1})(z_2 - z_0)(z_2 - z_1)(z_2 - z_3)}$$

$$c_3 = \frac{(z - z_{-2})(z - z_{-1})(z - z_0)(z - z_1)(z - z_2)}{(z_3 - z_{-2})(z_3 - z_{-1})(z_3 - z_0)(z_3 - z_1)(z_3 - z_2)}$$

For simulations with equal-spacing grid size h_x, h_y, and h_z, we define

$D_{XLLL} = \frac{x - x_{-2}}{h_x} = 2 + \frac{\delta x}{h_x}$	$D_{YLLL} = \frac{y - y_{-2}}{h_y} = 2 + \frac{\delta y}{h_y}$	$D_{ZLLL} = \frac{z - z_{-2}}{h_z} = 2 + \frac{\delta z}{h_z}$
$D_{XLL} = \frac{x - x_{-1}}{h_x} = 1 + \frac{\delta x}{h_x}$	$D_{YLL} = \frac{y - y_{-1}}{h_y} = 1 + \frac{\delta y}{h_y}$	$D_{ZLL} = \frac{z - z_{-1}}{h_z} = 1 + \frac{\delta z}{h_z}$
$D_{XL} = \frac{x - x_0}{h_x} = \frac{\delta x}{h_x}$	$D_{YL} = \frac{y - y_0}{h_y} = \frac{\delta y}{h_y}$	$D_{ZL} = \frac{z - z_0}{h_z} = \frac{\delta z}{h_z}$
$D_{XR} = \frac{x - x_1}{h_x} = \frac{\delta x}{h_x} - 1$	$D_{YR} = \frac{y - y_1}{h_y} = \frac{\delta y}{h_y} - 1$	$D_{ZR} = \frac{z - z_1}{h_z} = \frac{\delta z}{h_z} - 1$
$D_{XRR} = \frac{x - x_2}{h_x} = \frac{\delta x}{h_x} - 2$	$D_{YRR} = \frac{y - y_2}{h_y} = \frac{\delta y}{h_y} - 2$	$D_{ZRR} = \frac{z - z_2}{h_z} = \frac{\delta z}{h_z} - 2$
$D_{XRRR} = \frac{x - x_3}{h_x} = \frac{\delta x}{h_x} - 3$	$D_{YRRR} = \frac{y - y_3}{h_y} = \frac{\delta y}{h_y} - 3$	$D_{ZRRR} = \frac{z - z_3}{h_z} = \frac{\delta z}{h_z} - 3$

$$a_{-2} = -\frac{1}{120}(D_{XLL} D_{XL} D_{XR} D_{XRR} D_{XRRR})$$

$$a_{-1} = +\frac{1}{24}(D_{XLLL} D_{XL} D_{XR} D_{XRR} D_{XRRR})$$

$$a_{0} = -\frac{1}{12}(D_{XLLL} D_{XLL} D_{XR} D_{XRR} D_{XRRR})$$

$$a_{1} = +\frac{1}{12}(D_{XLLL} D_{XLL} D_{XL} D_{XRR} D_{XRRR})$$

$$a_{2} = -\frac{1}{24}(D_{XLLL} D_{XLL} D_{XL} D_{XR} D_{XRRR})$$

$$a_{3} = +\frac{1}{120}(D_{XLLL} D_{XLL} D_{XL} D_{XR} D_{XRR})$$

$$b_{-2} = -\frac{1}{120}(D_{YLL} D_{YL} D_{YR} D_{YRR} D_{YRRR})$$

$$b_{-1} = +\frac{1}{24}(D_{YLLL} D_{YL} D_{YR} D_{YRR} D_{YRRR})$$

$$b_{0} = -\frac{1}{12}(D_{YLLL} D_{YLL} D_{YR} D_{YRR} D_{YRRR})$$

$$b_{1} = +\frac{1}{12}(D_{YLLL} D_{YLL} D_{YL} D_{YRR} D_{YRRR})$$

$$b_{2} = -\frac{1}{24}(D_{YLLL} D_{YLL} D_{YL} D_{YR} D_{YRRR})$$

$$b_{3} = +\frac{1}{120}(D_{YLLL} D_{YLL} D_{YL} D_{YR} D_{YRR})$$

$$c_{-2} = -\frac{1}{120}(D_{ZLL} D_{ZL} D_{ZR} D_{ZRR} D_{ZRRR})$$

$$c_{-1} = +\frac{1}{24}(D_{ZLLL} D_{ZL} D_{ZR} D_{ZRR} D_{ZRRR})$$

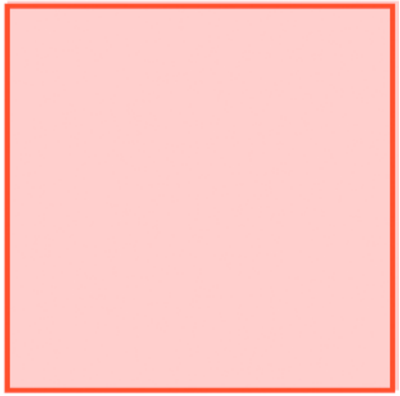
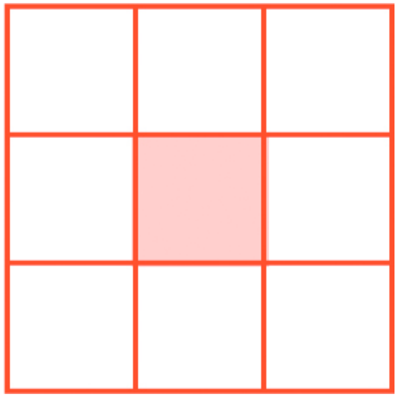
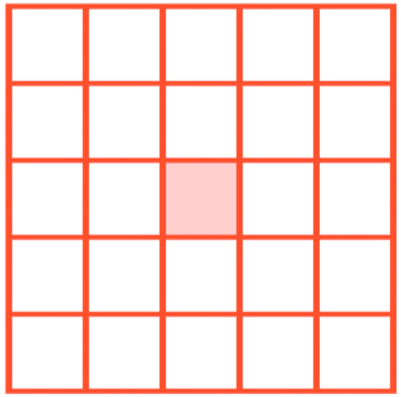
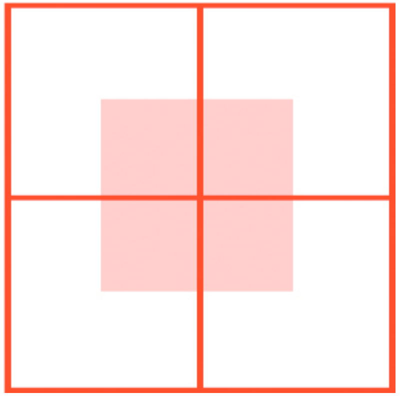
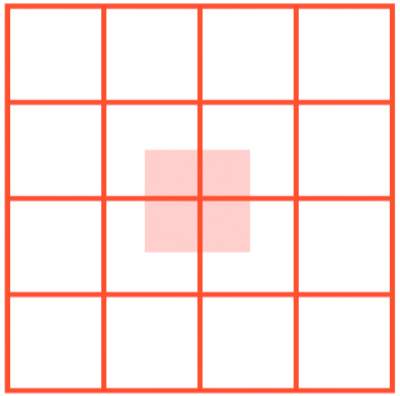
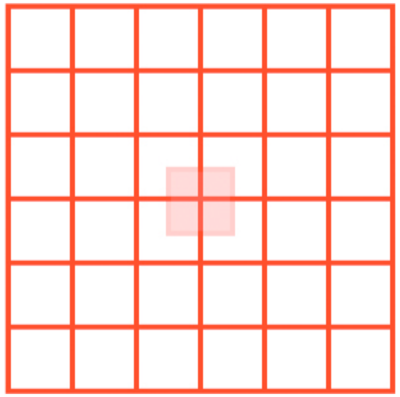
$$c_{0} = -\frac{1}{12}(D_{ZLLL} D_{ZLL} D_{ZR} D_{ZRR} D_{ZRRR})$$

$$c_{1} = +\frac{1}{12}(D_{ZLLL} D_{ZLL} D_{ZL} D_{ZRR} D_{ZRRR})$$

$$c_{2} = -\frac{1}{24}(D_{ZLLL} D_{ZLL} D_{ZL} D_{ZR} D_{ZRRR})$$

$$c_{3} = +\frac{1}{120}(D_{ZLLL} D_{ZLL} D_{ZL} D_{ZR} D_{ZRR})$$

Grids in the expansion of $\delta[\mathbf{x}-\mathbf{x}_{\alpha,k}(t)]$ with $\mathbf{x}_{\alpha,k}$ located in the shaded region

PIC Code & This Study	1st-order 	3rd-order 	5th-order 
UCLA Finite-Size Particle Code	1st- and 2nd-order 	3rd- and 4th-order 	5th- and 6th-order 

4. Benchmarks

Figure 1 shows the time variations of the total energy of particle code simulations, which are built based on (a) the first-order UCLA-particle-code-like, (b) the first-order PIC-code-like, (c) the third-order, and (d) the fifth-order deposition-interpolation schemes.

Figure 2 shows the time variations of the total energy of particle code simulations, which are built based on (a) the third-order and (b) the fifth-order deposition-interpolation schemes.

Obviously the numerical errors have been greatly reduced in the simulations with higher order deposition-interpolation schemes.

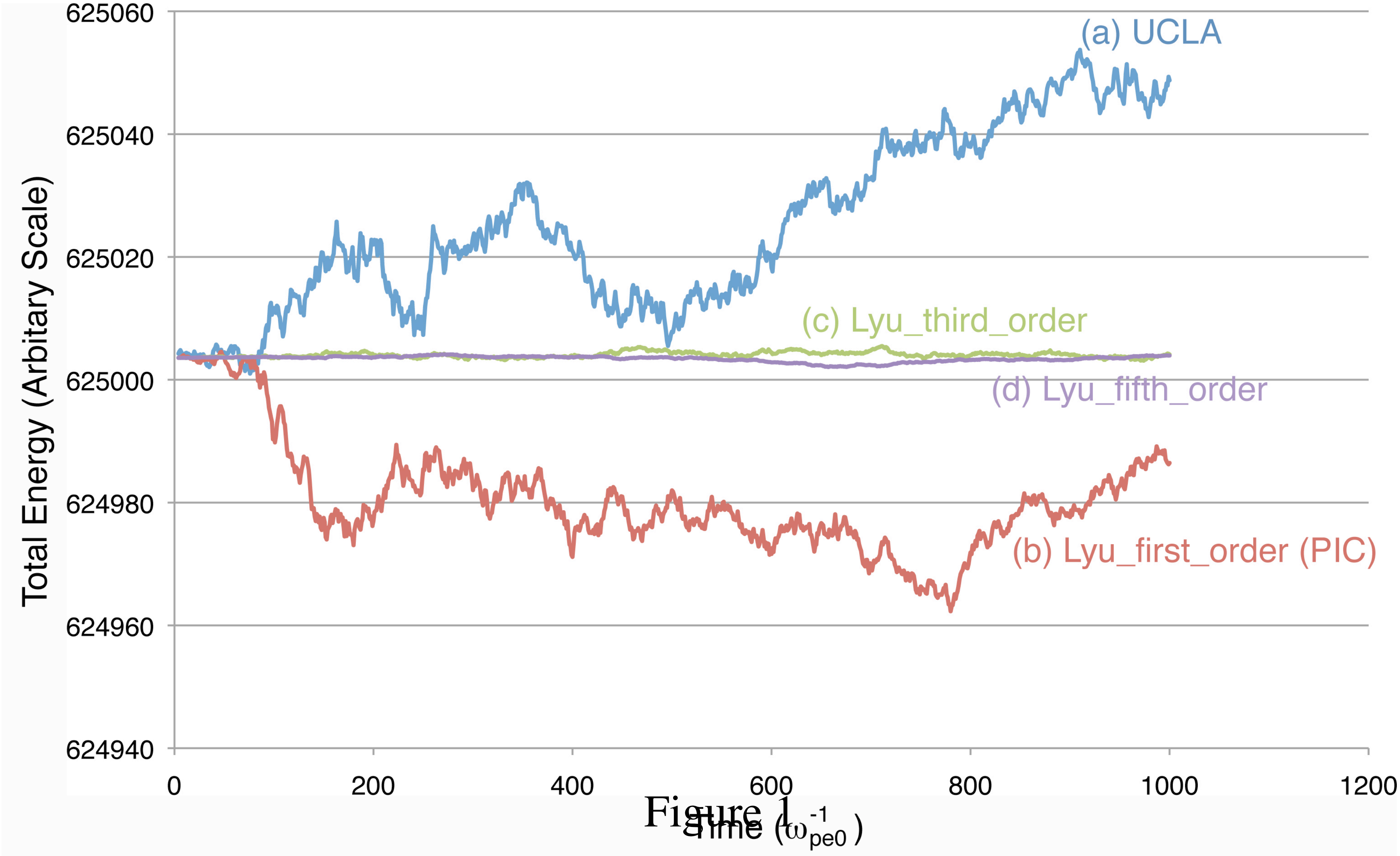

625060
625040
625020
625000
624980
624960
624940
Total Energy (Arbitary Scale)
0
200
400
600
800
1000
1200
Time (ω_{pe0}^{-1})
(a) UCLA
(c) Lyu_third_order
(d) Lyu_fifth_order
(b) Lyu_first_order (PIC)

Figure 1

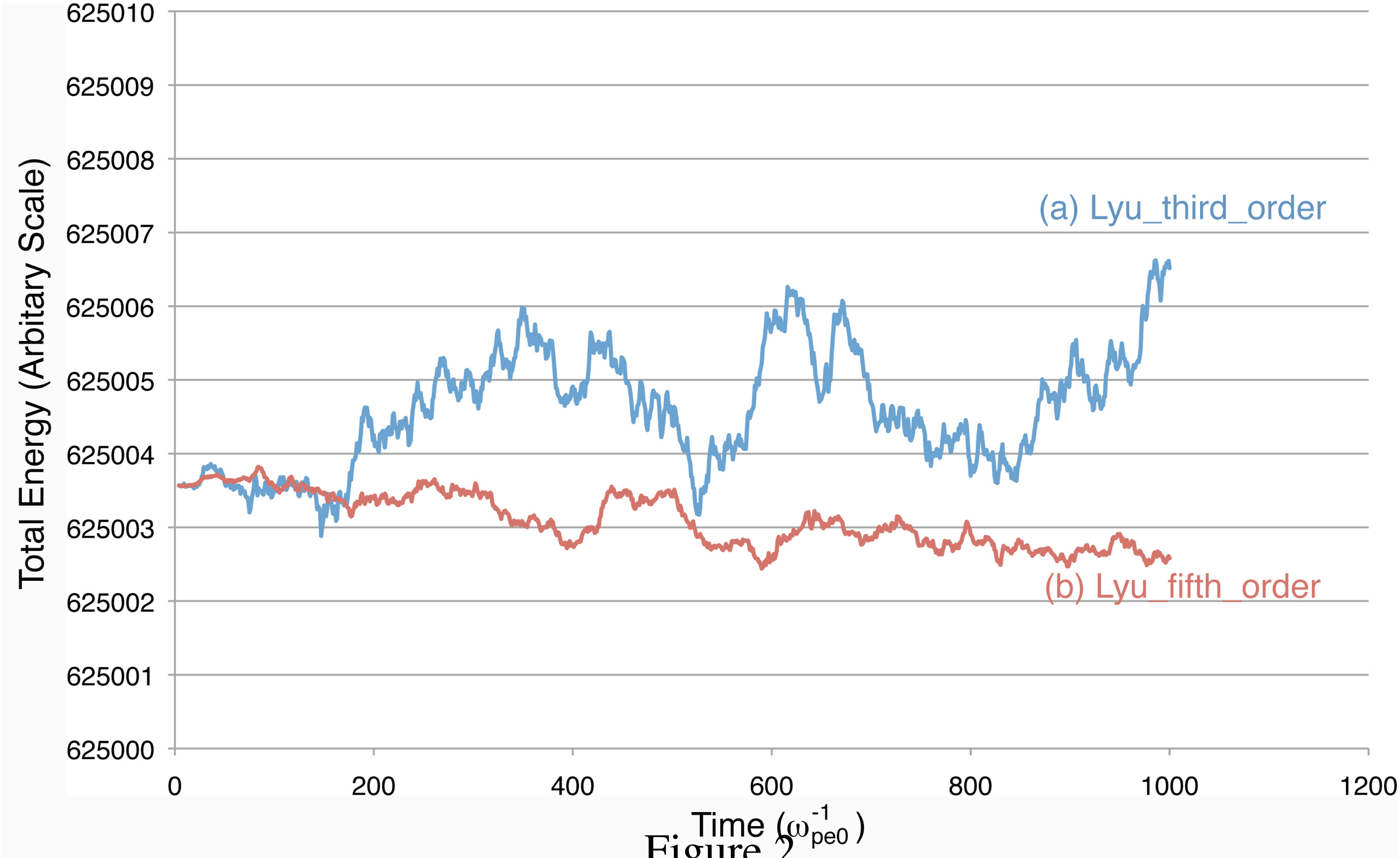

625010
625009
625008
625007
625006
625005
625004
625003
625002
625001
625000
Total Energy (Arbitary Scale)
(a) Lyu_third_order
(b) Lyu_fifth_order
0
200
400
600
800
1000
1200
Time (ω_{pe0}^{-1})

Figure 2

Different Point of Views:

Deposition

vs.

Interpolation / Distribution

太空科學研究所

一度空間之
一階與三階內差／分散法

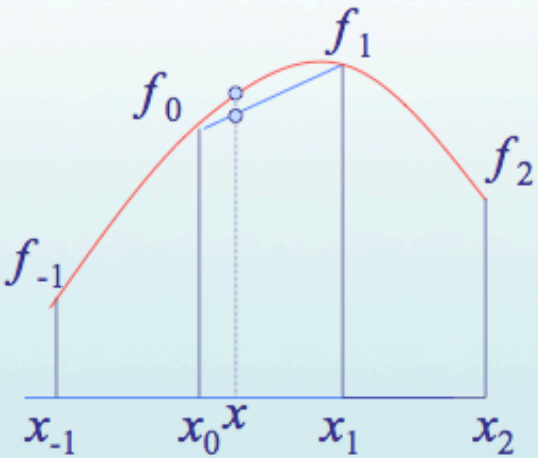
- 一階： $f(x) = (\frac{x_1 - x}{x_1 - x_0}) f_0 + (\frac{x - x_0}{x_1 - x_0}) f_1 + O(h^2 f^{(2)}) = b\, f_0 + a\, f_1 + O(h^2 f^{(2)})$
- 三階： $f(x) = a_{-1} f_{-1} + a_0 f_0 + a_1 f_1 + a_2 f_2 + O(h^4 f^{(4)})$

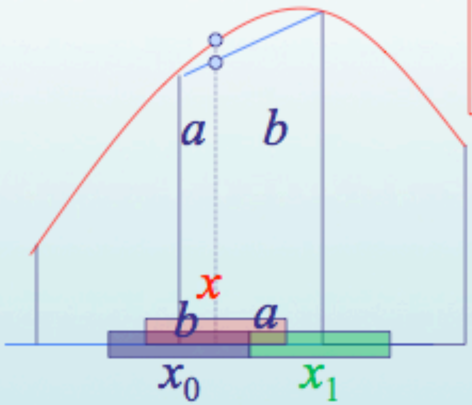
$$a_{-1} = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_{-1} - x_0)(x_{-1} - x_1)(x_{-1} - x_2)}$$

$$a_0 = \frac{(x - x_{-1})(x - x_1)(x - x_2)}{(x_0 - x_{-1})(x_0 - x_1)(x_0 - x_2)}$$

$$a_1 = \frac{(x - x_{-1})(x - x_0)(x - x_2)}{(x_1 - x_{-1})(x_1 - x_0)(x_1 - x_2)}$$

$$a_2 = \frac{(x - x_{-1})(x - x_0)(x - x_1)}{(x_2 - x_{-1})(x_2 - x_0)(x_2 - x_1)}$$





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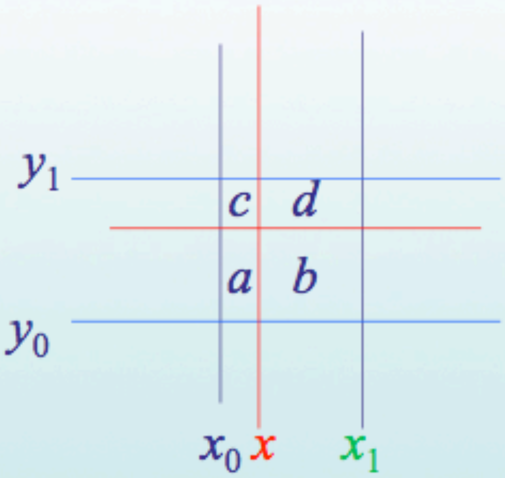
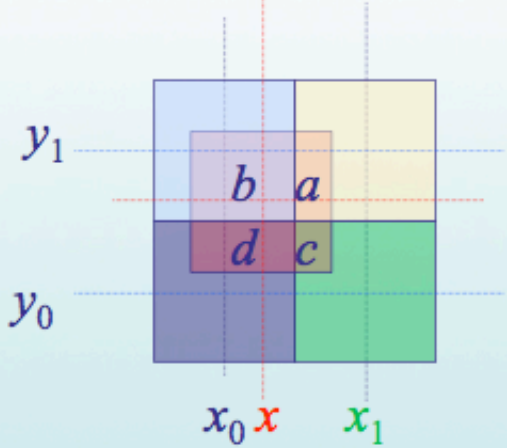
二度空間之 一階內差／分散法

- 一階：

以下Particle-in-Cell的看法，看似好懂，其實不易推廣到高階

$$f(x) = d\,f_{00} + c\,f_{10} + b\,f_{01} + a\,f_{11}$$
$$= \left(\frac{x_1 - x}{x_1 - x_0}\right)\left(\frac{y_1 - y}{y_1 - y_0}\right) f_{00} + \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{y_1 - y}{y_1 - y_0}\right) f_{10}$$
$$+ \left(\frac{x_1 - x}{x_1 - x_0}\right)\left(\frac{y - y_0}{y_1 - y_0}\right) f_{01} + \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{y - y_0}{y_1 - y_0}\right) f_{11}$$





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Summary of the Higher-Order Interpolations or Distributions

Cubic interpolations For $x_{-1} < x_0 < x < x_1 < x_2$	*Fifth-order interpolations* For $x_{-2} < x_{-1} < x_0 < x < x_1 < x_2 < x_3$
$f(x) = \sum_{i=-1}^{2} a_i f(x_i)$ $f(x,y) = \sum_{j=-1}^{2} b_j (\sum_{i=-1}^{2} a_i f(x_i, y_j))$ $f(x,y,z) = \sum_{k=-1}^{2} c_k [\sum_{j=-1}^{2} b_j (\sum_{i=-1}^{2} a_i f(x_i, y_j, z_k))]$	$f(x) = \sum_{i=-2}^{3} a_i f(x_i)$ $f(x,y) = \sum_{j=-2}^{3} b_j (\sum_{i=-2}^{3} a_i f(x_i, y_j))$ $f(x,y,z) = \sum_{k=-2}^{3} c_k [\sum_{j=-2}^{3} b_j (\sum_{i=-2}^{3} a_i f(x_i, y_j, z_k))]$
where	where
$a_{-1} = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_{-1}-x_0)(x_{-1}-x_1)(x_{-1}-x_2)}$ $a_0 = \frac{(x-x_{-1})(x-x_1)(x-x_2)}{(x_0-x_{-1})(x_0-x_1)(x_0-x_2)}$ $a_1 = \frac{(x-x_{-1})(x-x_0)(x-x_2)}{(x_1-x_{-1})(x_1-x_0)(x_1-x_2)}$ $a_2 = \frac{(x-x_{-1})(x-x_0)(x-x_1)}{(x_2-x_{-1})(x_2-x_0)(x_2-x_1)}$	$a_{-2} = \frac{(x-x_{-1})(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_{-2}-x_{-1})(x_{-2}-x_0)(x_{-2}-x_1)(x_{-2}-x_2)(x_{-2}-x_3)}$ $a_{-1} = \frac{(x-x_{-2})(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_{-1}-x_{-2})(x_{-1}-x_0)(x_{-1}-x_1)(x_{-1}-x_2)(x_{-1}-x_3)}$ $a_0 = \frac{(x-x_{-2})(x-x_{-1})(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_{-2})(x_0-x_{-1})(x_0-x_1)(x_0-x_2)(x_0-x_3)}$ $a_1 = \frac{(x-x_{-2})(x-x_{-1})(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_{-2})(x_1-x_{-1})(x_1-x_0)(x_1-x_2)(x_1-x_3)}$ $a_2 = \frac{(x-x_{-2})(x-x_{-1})(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_{-2})(x_2-x_{-1})(x_2-x_0)(x_2-x_1)(x_2-x_3)}$ $a_3 = \frac{(x-x_{-2})(x-x_{-1})(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_{-2})(x_3-x_{-1})(x_3-x_0)(x_3-x_1)(x_3-x_2)}$
so are the b_j and c_k.	so are the b_j and c_k.