Chapter 5. MHD Kelvin-Helmholtz Instability in a Compressible Plasma

5.1. Introduction

Velocity shear in a fluid can trigger velocity-shear instability, which is also called the Kelvin-Helmholtz (K-H) instability. Examples of the K-H instability in nature include wind-induced water waves, vortices caused by water flows with different velocities in a river, cloud vortices generated by air flow around an island, and the wavy and twisting auroral curtain due to the shear motion of plasmas on two sides of an auroral arc. In addition to these visible wavy or vortex structures, large-scale phenomena are also observed by spacecraft in the velocity-shear regions of space plasma both at the flank magnetopause (e.g., Ogilvie and Fitzenrieter, 1989; Chen and Kivelson, 1993; Chen et al., 1993; Fairfield, et al., 2000; 2003) and at the leading and trailing edges of high-speed solar wind streams (e.g., Belcher and Davis, 1971; Mavromichalaki et al., 1988; Neugebauer and Buti, 1990).

Nonlinear evolution of the K-H instability could result in vortices in saturation stage. However, due to the tension force of water surface, the wind-induced water waves usually show undulate structures without vortices. Similarly in the magnetohydrodynamic (MHD) plasma, when the background magnetic field is parallel or anti-parallel to the sheared flows, the magnetic tension force can stabilize the instability and reduce the growth rate.

The compressional effect in the K-H instability is negligible when the flow speeds on both sides of the TD are subsonic in the surface wave rest frame. The compressional effect becomes important when the flow speed on either side becomes supersonic, or supermagnetosonic in the surface wave rest frame. Supermagnetosonic velocity shears are commonly observed in the solar wind (e.g., Mavromichalaki et al., 1988; Neugebauer and Buti, 1990) and at the Earth's magnetopause (e.g., Chen and Kivelson, 1993). Numerical modeling of the nonlinear evolution of the K-H instability in compressible MHD plasma can help us to understand further the underline processes of the formation of the nonlinear disturbances in the solar wind and at the Earth's magnetopause.

Based on the eigen-value approximation (e.g., Press et al., 1988), linear wave analyses of the K-H instability with uniform growth rate in a sheared flow of finite thickness have been carried out by Blumen (1970), Blumen et al. (1975), Drazin and Davey (1977) for neutral fluid and by Miura (1982), Miura and Pritchett (1982) for the MHD plasma. It is shown that the unstable wave modes obtained from the eigen-value approximation depend on the boundary conditions of the eigen functions. For gas dynamic K-H instability, when the propagation speed of the surface wave is supersonic with respect to the background medium, unstable eigen mode cannot be found under fixed boundary condition (Blumen, 1970; Blumen et al., 1975), but can be found under uniform boundary condition (Drazin and Davey, 1977). For K-H instability in the MHD plasma, when the velocity shear is large enough, no unstable eigen mode can be found under fixed boundary condition (Miura and Pritchett, 1982). However, to our knowledge, eigen mode under uniform boundary condition has not been studied before for the K-H instability in the MHD plasma.

A brief review of the linear wave analyses of the K-H instability is given in Section 5.2. The non-local analysis based on the eigen-value approximation is reviewed in Section 5.3. Since it seems unrealistic to expect instabilities to grow uniformly in a non-uniform medium, we do not carry out the eigen-mode analysis to find out the unstable eigen mode under uniform boundary condition for the K-H instability in the MHD plasma. Instead, we propose a new non-local linear wave analysis scheme in Section 5.4. Based on the new analysis scheme, the instability in a non-uniform medium should have a non-uniform growth rate or damping rate, and the growth rate or damping rate should vary slowly with time. Thus, the numerical simulation is the best choice for studying both linear and nonlinear evolutions of K-H instabilities in the MHD plasma (e.g., Miura, 1982; 1984; 1987; 1990; 1992; 1997; 1999; Wu, 1986; Manuel and Samson, 1993; Thomas and Winske, 1993; Otto and Fairfield, 2000, Lai and Lyu, 2005)

Exercise 5.1

Read section 11.4.3 in the following textbook and derive incompressible Kelvin-Helmholtz instability occurred at a tangential discontinuity (TD) due to velocity shear on two sides of the TD.

Parks, G. K., *Physics of Space Plasmas: An Introduction,* Addison-Wesley Publ. Co., 1991.

5.1. Linear analysis of K-H instability

Linear wave analyses of the Kelvin-Helmholtz (K-H) instability at a tangential discontinuity (TD) of an ideal MHD plasma are reviewed and discussed in this section. Basic equations of the ideal MHD plasma are as follows:

$$(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\rho=-\rho\nabla\cdot\mathbf{V} \tag{5.1}$$

$$\rho(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\mathbf{V}=-\nabla(p+\frac{B^2}{2\mu_0})+\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0} \tag{5.2}$$

$$(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)p=-\gamma p\nabla\cdot\mathbf{V} \tag{5.3}$$

$$(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\mathbf{B}=-\mathbf{B}\nabla\cdot\mathbf{V}+\mathbf{B}\cdot\nabla\mathbf{V} \tag{5.4}$$

$$\nabla\cdot\mathbf{B}=0 \tag{5.5}$$

where $\gamma=5/3$. The combination of equations (5.1) and (5.3) yields the adiabatic equation of state $(\partial/\partial t+\mathbf{V}\cdot\nabla)(p\rho^{-\gamma})=0$.

We define the total pressure to be the sum of the thermal pressure and the magnetic pressure. That is

$$p_{tot}=p+\frac{B^2}{2\mu_0} \tag{5.6}$$

Equation (5.2) can be rewritten as

$$\rho(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\mathbf{V}=-\nabla p_{tot}+\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0} \tag{5.2'}$$

The time derivative of equation (5.6) in the fluid rest frame is

$$(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)p_{tot}=(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)p+\frac{\mathbf{B}}{\mu_0}\cdot[(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)\mathbf{B}] \tag{5.7}$$

Substituting equations (5.3) and (5.4) into equation (5.7) yields

$$(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla)p_{tot}=-(\gamma p+\frac{B^2}{\mu_0})\nabla\cdot\mathbf{V}+\frac{\mathbf{B}}{\mu_0}\cdot(\mathbf{B}\cdot\nabla\mathbf{V}) \tag{5.8}$$

The background equilibrium state considered in this linear wave analysis is an MHD tangential discontinuity (TD). We choose a coordinate system such that the normal direction of the TD is along the x-axis and the TD is located at $x=0$. The background equilibrium fields of the TD are functions of x only. There is no x-component magnetic field or velocity field in the equilibrium state of the TD, i.e., $B_{0x}=0$ and $V_{0x}=0$. As a result, the equilibrium state satisfies equations (5.1), (5.3), (5.4), (5.5) and (5.8). Whereas, equations (5.6) and (5.2') at the equilibrium state yield

$$p_{0tot} = \text{constant} = p_0(x)+\frac{B_0^2(x)}{2\mu_0} = p_0(x)+\frac{B_{0y}^2(x)+B_{0z}^2(x)}{2\mu_0} \tag{5.9}$$

Taking the first derivative of equation (5.9) with respect to x yields

$$0=\frac{dp_0(x)}{dx}+\frac{B_{0y}(x)}{\mu_0}\frac{dB_{0y}(x)}{dx}+\frac{B_{0z}(x)}{\mu_0}\frac{dB_{0z}(x)}{dx} \tag{5.10}$$

For convenience, we choose a moving frame such that the background equilibrium velocity field is along the y direction, i.e., $\mathbf{V}_0=\hat{\mathbf{y}}V_{0y}(x)$.

We have assumed that the background equilibrium fields are functions of x only. We further assumed that the linear wave is uniform in the z direction. Let $A(x,y,t)$ be a field variable, which consists of a non-uniform background equilibrium component $A_0(x)$ and a small perturbation component $\delta A(x,y,t)$. Namely,

$$A(x,y,t)=A_0(x)+\delta A(x,y,t) \tag{5.11}$$

where the order of magnitude $O(\delta A/A)\approx\varepsilon<<1$.

Substituting equation (5.11) into equations (5.2'), (5.3), (5.4), and (5.8), and ignoring the second order (ε^2) and the higher order terms, we can be obtain the following linearized equations,

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta V_x=-\frac{1}{\rho_0}\frac{\partial\delta p_{tot}}{\partial x}+\frac{B_{0y}}{\mu_0\rho_0}\frac{\partial\delta B_x}{\partial y} \tag{5.12}$$

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta V_y=-\frac{1}{\rho_0}\frac{\partial\delta p_{tot}}{\partial y}+\frac{B_{0y}}{\mu_0\rho_0}\frac{\partial\delta B_y}{\partial y}+\frac{\delta B_x}{\mu_0\rho_0}\frac{dB_{0y}}{dx}-\delta V_x\frac{dV_{0y}}{dx} \tag{5.13}$$

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta V_z=\frac{B_{0y}}{\mu_0\rho_0}\frac{\partial\delta B_z}{\partial y}+\frac{\delta B_x}{\mu_0\rho_0}\frac{dB_{0z}}{dx} \tag{5.14}$$

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta p=-\gamma p_0(\frac{\partial\delta V_x}{\partial x}+\frac{\partial\delta V_y}{\partial y})-\delta V_x\frac{dp_0}{dx} \tag{5.15}$$

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta B_x = B_{0y}\frac{\partial \delta V_x}{\partial y} \tag{5.16}$$

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta B_y = -B_{0y}\frac{\partial \delta V_x}{\partial x}+\delta B_x\frac{dV_{0y}}{dx}-\delta V_x\frac{dB_{0y}}{dx} \tag{5.17}$$

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta B_z = -B_{0z}(\frac{\partial \delta V_x}{\partial x}+\frac{\partial \delta V_y}{\partial y})+B_{0y}\frac{\partial \delta V_z}{\partial y}-\delta V_x\frac{dB_{0z}}{dx} \tag{5.18}$$

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\delta p_{tot}}{\rho_0} = -(\frac{\gamma p_0}{\rho_0}+\frac{B_0^2}{\mu_0\rho_0})(\frac{\partial \delta V_x}{\partial x}+\frac{\partial \delta V_y}{\partial y})+\frac{B_{0y}}{\mu_0\rho_0}(B_{0y}\frac{\partial \delta V_y}{\partial y}+B_{0z}\frac{\partial \delta V_z}{\partial y}+\delta B_x\frac{dV_{0y}}{dx}) \tag{5.19}$$

By eliminating δV_x, δV_y, δV_z, δB_x, δB_y, δB_z, the above equations (5.12)~(5.19) can be rewritten into the following differential equation of δp_{tot},

$$\begin{aligned}
&-[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 - C_{I0y}^2\frac{\partial^2}{\partial y^2}][-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 + C_{S0}^2C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial^2\delta p_{tot}}{\partial x^2}\\
&+\{2\frac{dV_{0y}}{dx}\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y}) - \frac{dC_{I0y}^2}{dx}\frac{\partial^2}{\partial y^2}+\frac{1}{\rho_0}\frac{d\rho_0}{dx}[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 - C_{I0y}^2\frac{\partial^2}{\partial y^2}]\}\\
&\qquad [-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 + C_{S0}^2C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial \delta p_{tot}}{\partial x}\\
&-\{(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^4 + \frac{\partial^2}{\partial y^2}[-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 + C_{I0y}^2C_{S0}^2\frac{\partial^2}{\partial y^2}]\}\\
&\qquad [(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 - C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta p_{tot} = 0
\end{aligned} \tag{5.20}$$

where

$$C_{I0y}^2(x) = B_{0y}^2(x)/\mu_0\rho_0(x), \tag{5.21}$$

$$C_{S0}^2(x) = \gamma p_0(x)/\rho_0(x), \tag{5.22}$$

$$C_{A0}^2(x) = [B_{y0}^2(x)+B_{z0}^2(x)]/\mu_0\rho_0(x), \tag{5.23}$$

and $\gamma = 5/3$. Here $C_{I0y}(x)$ is the phase velocity of MHD intermediate-mode, which propagates along the y direction. Derivation of equation (5.20) can be found in the Appendix of Chapter 5.

5.3 K-H instabilities with uniform growth

Although, it seems unrealistic to expect instabilities to grow uniformly in a non-uniform medium, the equations (5.20) are commonly solved by means of an eigen-value approximation, in which both the wave frequency and the wave growth rate (or damping rate) are assumed to be uniform for a given tangential wavelength $2\pi/k_t$ (e.g., Blumen, 1970; Blumen et al., 1975, Drazin and Davey, 1977; Miura, 1982; Miura and Pritchett, 1982). As a result, for each k_t, a linear wave with non-uniform wave amplitude, which satisfies a unique profile along the x direction, is obtained. The profile of the wave amplitude is called the corresponding eigen function of the resulting eigen frequency. A brief review of the non-local analysis based on the eigen-value approximation is given below.

For linear waves with a uniform growth rate, or damping rate, the small perturbation δA with wavelength $2\pi / k_t$ can be rewritten into the following form

$$\delta A(x,y,t) = \delta \bar{A}\, f(x)\exp[i(k_t y - \omega t)] \tag{5.24}$$

Substituting equation (5.24) into equations (5.20), it yields

$$\frac{1}{k_t^2}\frac{d^2 f(x)}{dx^2} - \frac{1}{k_t}\frac{d f(x)}{dx} R_0(x,\omega,k_t) + f(x)F_0(x,\omega,k_t) = 0 \tag{5.25}$$

where

$$R_0(x,\omega,k_t) = \frac{\dfrac{1}{k_t}\dfrac{d}{dx}\left\{\rho_0(x)\{[\dfrac{\omega}{k_t} - V_{0y}(x)]^2 - C_{I0y}^2(x)\}\right\}}{\rho_0(x)\{[\dfrac{\omega}{k_t} - V_{0y}(x)]^2 - C_{I0y}^2(x)\}} \tag{5.26}$$

$$F_0(x,\omega,k_t) = \frac{\{[\dfrac{\omega}{k_t} - V_{0y}(x)]^2 - C_{F0y}^2(x)\}\{[\dfrac{\omega}{k_t} - V_{0y}(x)]^2 - C_{SL0y}^2(x)\}}{[\dfrac{\omega}{k_t} - V_{0y}(x)]^2[C_{F0y}^2(x) + C_{SL0y}^2(x)] - C_{F0y}^2(x)C_{SL0y}^2(x)} \tag{5.27}$$

$$C_{F0y}^2(x) = \frac{1}{2}\left\{C_{A0}^2(x) + C_{S0}^2(x) + \sqrt{[C_{A0}^2(x) + C_{S0}^2(x)]^2 - 4C_{I0y}^2(x)C_{S0}^2(x)}\right\}, \tag{5.28}$$

$$C_{SL0y}^2(x) = \frac{1}{2}\left\{C_{A0}^2(x) + C_{S0}^2(x) - \sqrt{[C_{A0}^2(x) + C_{S0}^2(x)]^2 - 4C_{I0y}^2(x)C_{S0}^2(x)}\right\}, \tag{5.29}$$

and $C_{I0y}(x)$, $C_{S0}(x)$, $C_{A0}(x)$ are given in equations (5.21), (5.22), (5.23). Here $C_{F0y}(x)$, $C_{I0y}(x)$, and $C_{SL0y}(x)$ are the phase velocities of MHD fast-mode, intermediate-mode, and slow-mode waves, respectively, which propagate along the y direction.

For gas dynamics, $B_{0y}(x) = B_{0z}(x) = 0$, and $\delta p_{tot}(x) = \delta p(x)$. Thus, equations (5.25) is reduced to

$$\frac{1}{k_t^2}\frac{d^2 f(x)}{dx^2} - \frac{1}{k_t}\frac{d f(x)}{dx}R_{n0}(x,\omega,k_t) + f(x)F_{n0}(x,\omega,k_t) = 0 \qquad (5.25n)$$

where

$$R_{n0}(x,\omega,k_t) = \frac{\frac{1}{k_t}\frac{d}{dx}\{\rho_0(x)[\frac{\omega}{k_t} - V_{0y}(x)]^2\}}{\rho_0(x)[\frac{\omega}{k_t} - V_{0y}(x)]^2} \qquad (5.26n)$$

$$F_{n0}(x,\omega,k_t) = \frac{[\frac{\omega}{k_t} - V_{0y}(x)]^2 - C_{S0}^2}{C_{S0}^2} \qquad (5.27n)$$

Note that equation (5.25) is similar to the equation (23) in the paper by Miura and Pritchett (1982). Equation (5.25n) is similar to the equation (7) in the paper by Blumen (1970). But the equation (7) in the paper by Blumen (1970) is derived under the assumption of uniform background density.

Equations (5.25n) and (5.25) can be solved based on an eigen-value approximation (e.g., Press et al., 1988), in which the eigen value $\omega = \omega_r + i\omega_i$ is assumed to be uniform but the corresponding eigen function is non-uniform along the x direction (e.g., Blumen, 1970; Blumen et al., 1975, Drazin and Davey, 1977; Miura, 1982; Miura and Pritchett, 1982). Namely, the second order ordinary differential equations (5.25) and (5.25n) with a uniform eigen value $\omega = \omega_r + i\omega_i$, can be rewritten into a system of first order ordinary differential equations (ODEs), i.e.,

$$\frac{dy_1(x)}{dx} = G[x, y_1(x), y_2(x), y_3(x); k_t] \qquad (5.30)$$

$$\frac{dy_2(x)}{dx} = y_1(x) \qquad (5.31)$$

$$\frac{dy_3(x)}{dx} = 0 \qquad (5.32)$$

where $y_1(x) = d f(x)/dx$, $y_2(x) = f(x)$, $y_3(x) = \omega$, and for K-H instability in MHD plasma

$$G(x, y_1, y_2, y_3; k_t) = y_1 k_t R_0(x, y_3, k_t) - y_2 k_t^2 F_0(x, y_3, k_t) \qquad (5.33)$$

for K-H instability in gas dynamics,

$$G(x, y_1, y_2, y_3; k_t) = y_1 k_t R_{n0}(x, y_3, k_t) - y_2 k_t^2 F_{n0}(x, y_3, k_t) \qquad (5.33n)$$

For a given k_t, we can use the standard shooting method (e.g., Press et al., 1988) to find a set of eigen-mode solutions, which satisfy the fixed boundary conditions that $y_2(x_L) = y_2(x_R) = 0$ (e.g., Blumen, 1970; Blumen et al., 1975; Miura and Pritchett, 1982), where x_L and x_R are the left and the right boundaries of the system, respectively. Likewise, for a given k_t, one can use the standard shooting method to find a set of eigen-mode solutions, which satisfy the uniform boundary conditions that $y_1(x_L) = y_1(x_R) = 0$ (e.g., Drazin and Davey, 1977). To our knowledge, the eigen-mode solutions of K-H instability in MHD plasma, which satisfy the uniform boundary conditions with $y_1(x_L) = y_1(x_R) = 0$, have not been studied before.

However, in our opinion, the uniform-growth-rate assumption used in equation (5.32), is unlikely to be fulfilled in a non-uniform medium, unless the initial perturbation is chosen exactly equal to one of the eigen functions, or equal to a linear combinations of the eigen functions. Since there is only one eigen mode for each given k_t, if the initial perturbations, with tangential wavelength equal to $2\pi / k_t$, but with a perturbation profile, $f(x)$, different from the corresponding eigen function, it would be impossible to decompose this initial profile into a linear combination of the eigen functions. Indeed, only perturbations with uniform growth rate can be decomposed into a linear combination of these eigen functions. Note that, even if the initial perturbations are chosen exactly equal to a linear combinations of these eigen modes, the fastest growing eigen mode might not be the dominate wave mode in the system, unless the system could keep in a linear state before the most unstable eigen mode is fully developed, because the concept of superimposition of wave modes is valid only for linear wave analysis.

5.4 K-H instabilities with non-uniform growth rate

We propose a new non-local or global analysis in this section to study K-H instability with non-uniform growth rate. The proposed non-local analysis procedure is applicable to other instabilities in a non-uniform medium (e.g., Lui et al., 1995; Yoon et al., 1996).

For linear waves with a non-uniform growth rate, or damping rate, the small perturbation δA with wavelength $2\pi / k_t$ can be rewritten into the following form

$$\delta A(x,y,t) = \delta \bar{A} f(x) \exp\{i[k_t y - \omega(x) t]\} \tag{5.34}$$

where $\omega(x) = \omega_r(x) + i\omega_i(x)$.

Substituting (5.34) into equation (5.20), it yields

$$
\begin{aligned}
&\frac{1}{k_t^2}[\frac{d^2 f}{dx^2} - if\frac{d^2\omega}{dx^2}t - 2i\frac{df}{dx}\frac{d\omega}{dx}t - f(\frac{d\omega}{dx})^2 t^2] \\
&-\frac{1}{k_t^2}[\frac{2\frac{dV_{0y}}{dx}(V_{0y}-\frac{\omega}{k_t}) - \frac{dC_{I0y}^2}{dx}}{(V_{0y}-\frac{\omega}{k_t})^2 - C_{I0y}^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dx}][\frac{df}{dx} - if\frac{d\omega}{dx}t] \\
&+\frac{[(V_{0y}-\frac{\omega}{k_t})^4 - (C_{S0}^2 + C_{A0}^2)(V_{0y}-\frac{\omega}{k_t})^2 + C_{S0}^2 C_{I0y}^2]}{[(C_{S0}^2 + C_{A0}^2)(V_{0y}-\frac{\omega}{k_t})^2 - C_{S0}^2 C_{I0y}^2]} f = 0
\end{aligned}
\tag{5.35}
$$

For $|t\omega_i| << 1$, equation (5.35) can be approximately rewritten into the following form

$$
\begin{aligned}
&\frac{1}{k_t^2}\frac{d^2 f}{dx^2} - \frac{1}{k_t^2}[\frac{2\frac{dV_{0y}}{dx}(V_{0y}-\frac{\omega}{k_t}) - \frac{dC_{I0y}^2}{dx}}{(V_{0y}-\frac{\omega}{k_t})^2 - C_{I0y}^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dx}]\frac{df}{dx} \\
&+\frac{[(V_{0y}-\frac{\omega}{k_t})^2 - C_{F0y}^2][(V_{0y}-\frac{\omega}{k_t})^2 - C_{SL0y}^2]}{(C_{F0y}^2 + C_{SL0y}^2)(V_{0y}-\frac{\omega}{k_t})^2 - C_{F0y}^2 C_{SL0y}^2} f \approx 0
\end{aligned}
\tag{5.36}
$$

where $C_{F0y}(x)$, $C_{I0y}(x)$, and $C_{SL0y}(x)$ are the phase velocities of MHD fast-mode, intermediate-mode, and slow-mode waves, respectively, which propagate along the y direction. They have been defined in equations (5.28), (5.21), and (5.29), respectively.

By solving equation (5.36), we can obtain a non-uniform solution of $\omega(x)$ for a given set of k_t, $f(x)$, $V_{0y}(x)$, $C_{F0y}(x)$, $C_{I0y}(x)$, and $C_{SL0y}(x)$. Solution of equation (5.36) is only good for $|t\omega_i| << 1$. After a time period t_0, where $\max(|t_0\omega_i|) \approx 0.1$, we should choose the latest profile, $f_{new}(x) = f(x)\exp[\omega_i(x)t_0]$, as a new starting point to solve $\omega(x)$ for the proceeding time period. One can repeat this procedure as long as the wave amplitude is of small amplitude, so that the concept of superimposition of wave modes is valid in this linear wave analysis. The proposed non-local linear wave analysis procedure is good for studying instabilities in a non-uniform medium with non-uniform growth rate or damping rate, which varies slowly with time.

Exercise 5.2

Show that for uniform background medium at region 1 ($x < 0$) and region 2 ($x > 0$) and with zero-thickness transition layer at $x = 0$, equation (5.36) is reduced to

$$(\rho_{01} + \rho_{02})\omega^2 - 2\omega(\rho_{01}V_{0y1} + \rho_{02}V_{0y2}) + \rho_{01}(V_{0y1}^2 - C_{I0y1}^2) + \rho_{02}(V_{0y2}^2 - C_{I0y2}^2) = 0 \tag{5.37}$$

K-H instability can take place if

$$(V_{0y2} - V_{0y1})^2 > C_{I0y1}^2(\frac{\rho_{01}}{\rho_{02}} + 1) + C_{I02}^2(\frac{\rho_{02}}{\rho_{01}} + 1) \tag{5.38}$$

which is consistent with Chandrasekhar's results (Chandrasekhar, 1961).

Exercise 5.3

Collect observational data of magnetopause crossing during northward IMF (Interplanetary Magnetic Field). Find evidence of Kelvin-Helmholtz (K-H) instability at Earth's magnetopause. Find evidence of dawn-dusk asymmetric development of the K-H instability. Qualitatively determine ionospheric feedback effect on the K-H instability occurred at Earth's magnetopause.

Exercise 5.4

Study generation of Kelvin-Helmholtz (K-H) instability in discrete auroral arcs. Determine differences in evolution and polarization direction of vortexes in active auroral arcs observed in northern hemisphere and those observed in southern hemisphere.

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Appendix of Chapter 5: Derivation of equation (5.20)

Applying the differential operator $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})$ to equations (5.12), (5.13), (5.14), (5.17) and (5.19), then substituting equation (5.16) into the resulting equations to eliminate $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta B_x$, it yields

$$[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta V_x=-\frac{1}{\rho_0}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial \delta p_{tot}}{\partial x} \tag{5.12a}$$

$$\begin{aligned}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\delta V_y=&-\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\delta p_{tot}}{\rho_0}+\frac{B_{0y}}{\mu_0\rho_0}\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta B_y\\&+[\frac{B_{0y}}{\mu_0\rho_0}\frac{dB_{0y}}{dx}\frac{\partial}{\partial y}-\frac{dV_{0y}}{dx}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})]\delta V_x\end{aligned} \tag{5.13a}$$

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\delta V_z=\frac{B_{0y}}{\mu_0\rho_0}\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta B_z+\frac{B_{0y}}{\mu_0\rho_0}\frac{dB_{0z}}{dx}\frac{\partial \delta V_x}{\partial y} \tag{5.14a}$$

$$(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\delta B_y=-B_{0y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial \delta V_x}{\partial x}+[\frac{dV_{0y}}{dx}B_{0y}\frac{\partial}{\partial y}-\frac{dB_{0y}}{dx}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})]\delta V_x \tag{5.17a}$$

$$\begin{aligned}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\frac{\delta p_{tot}}{\rho_0}=&-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial \delta V_x}{\partial x}\\&-(C_{S0}^2+C_{A0}^2-C_{I0y}^2)\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta V_y+\frac{B_{0y}}{\mu_0\rho_0}B_{0z}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial \delta V_z}{\partial y}+\frac{dV_{0y}}{dx}C_{I0y}^2\frac{\partial \delta V_x}{\partial y}\end{aligned} \tag{5.19a}$$

Applying the differential operator $(\frac{\partial}{\partial x})$ to equation (5.12a), it yields

$$\begin{aligned}[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial \delta V_x}{\partial x}=&-[2(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{dV_{0y}}{dx}\frac{\partial}{\partial y}-\frac{dC_{I0y}^2}{dx}\frac{\partial^2}{\partial y^2}]\delta V_x\\&-\frac{1}{\rho_0}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial^2 \delta p_{tot}}{\partial x^2}-\frac{1}{\rho_0}\frac{dV_{0y}}{dx}\frac{\partial}{\partial y}\frac{\partial \delta p_{tot}}{\partial x}+\frac{1}{\rho_0}(\frac{1}{\rho_0}\frac{d\rho_0}{dx})(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial \delta p_{tot}}{\partial x}\end{aligned} \tag{5.12b}$$

Applying the differential operator $[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]$ to equation (5.12b) then substituting equation (5.12a) into the resulting equation to eliminate $[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta V_x$, it yields

$$
\begin{aligned}
&[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]^2\frac{\partial\delta V_x}{\partial x}=-\frac{1}{\rho_0}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial^2\delta p_{tot}}{\partial x^2}\\
&+\frac{1}{\rho_0}\{\frac{dV_{0y}}{dx}\frac{\partial}{\partial y}[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2+C_{I0y}^2\frac{\partial^2}{\partial y^2}]-\frac{dC_{I0y}^2}{dx}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial^2}{\partial y^2}\}\frac{\partial\delta p_{tot}}{\partial x}\\
&+\frac{1}{\rho_0}(\frac{1}{\rho_0}\frac{d\rho_0}{dx})(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial\delta p_{tot}}{\partial x}
\end{aligned}
\tag{5.12c}
$$

Applying the differential operator $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})$ to equation (5.13a), then substituting equation (5.17a) into the resulting equation to eliminate $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\delta B_y$, and then substituting equation (5.12a) into the resulting equation to eliminate $[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta V_x$, it yields

$$
\begin{aligned}
&(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^3\delta V_y=-\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\frac{\delta p_{tot}}{\rho_0}-C_{I0y}^2\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial\delta V_x}{\partial x}\\
&+\frac{1}{\rho_0}\frac{dV_{0y}}{dx}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial\delta p_{tot}}{\partial x}
\end{aligned}
\tag{5.13b}
$$

Applying the differential operator $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2$ to equation (5.18), then substituting equation (5.14a) into the resulting equation to eliminate $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\delta V_z$, and then substituting equation (5.12a) into the resulting equation to eliminate $[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta V_x$, it yields

$$
\begin{aligned}
&(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta B_z=-B_{0z}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\frac{\partial\delta V_x}{\partial x}\\
&-B_{0z}\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\delta V_y+\frac{1}{\rho_0}\frac{dB_{0z}}{dx}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial\delta p_{tot}}{\partial x}
\end{aligned}
\tag{5.18a}
$$

Applying the differential operator $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})$ to equation (5.18a) then substituting equation (5.13b) into the resulting equation to eliminate $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^3\delta V_y$, it yields

$$\begin{aligned}
&(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta B_z \\
&=-B_{0z}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial\delta V_x}{\partial x} \\
&+B_{0z}\frac{\partial^2}{\partial y^2}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\frac{\delta p_{tot}}{\rho_0} \\
&+\frac{1}{\rho_0}[\frac{dB_{0z}}{dx}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-B_{0z}\frac{dV_{0y}}{dx}\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})]\frac{\partial\delta p_{tot}}{\partial x}
\end{aligned} \tag{5.18b}$$

Applying the differential operator $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})$ to equation (5.19a) then substituting equation (5.14a) into the resulting equation to eliminate $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\delta V_z$, it yields

$$\begin{aligned}
&(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^3\frac{\delta p_{tot}}{\rho_0}=-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\frac{\partial\delta V_x}{\partial x} \\
&+[\frac{B_{0z}}{\mu_0\rho_0}\frac{dB_{0z}}{dx}C_{I0y}^2\frac{\partial^2}{\partial y^2}+\frac{dV_{0y}}{dx}C_{I0y}^2\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})]\delta V_x \\
&-(C_{S0}^2+C_{A0}^2-C_{I0y}^2)\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\delta V_y+\frac{B_{0z}}{\mu_0\rho_0}C_{I0y}^2\frac{\partial^2}{\partial y^2}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\delta B_z
\end{aligned} \tag{5.19b}$$

Applying the differential operator $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})$ to equation (5.19b) then substituting equation (5.13b) into the resulting equation to eliminate $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^3\delta V_y$, it yields

$$\begin{aligned}
&(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^4\frac{\delta p_{tot}}{\rho_0} \\
&=[-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^3+(C_{S0}^2+C_{A0}^2-C_{I0y}^2)C_{I0y}^2\frac{\partial^2}{\partial y^2}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})]\frac{\partial\delta V_x}{\partial x} \\
&+[\frac{B_{0z}}{\mu_0\rho_0}\frac{dB_{0z}}{dx}C_{I0y}^2\frac{\partial^2}{\partial y^2}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})+\frac{dV_{0y}}{dx}C_{I0y}^2\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2]\delta V_x \\
&+(C_{S0}^2+C_{A0}^2-C_{I0y}^2)\frac{\partial^2}{\partial y^2}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\frac{\delta p_{tot}}{\rho_0} \\
&-\frac{1}{\rho_0}(C_{S0}^2+C_{A0}^2-C_{I0y}^2)\frac{dV_{0y}}{dx}\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})\frac{\partial\delta p_{tot}}{\partial x} \\
&+\frac{B_{0z}}{\mu_0\rho_0}C_{I0y}^2\frac{\partial^2}{\partial y^2}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\delta B_z
\end{aligned} \tag{5.19c}$$

Applying the differential operator $[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]$ to equation (5.19c), then substituting equation (5.18b) into the resulting equation to eliminate $(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta B_z$, and then substituting equation (5.12a) into the resulting equation to eliminate $[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta V_x$, it yields

$$
\begin{aligned}
&(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2\{(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^4+\frac{\partial^2}{\partial y^2}[-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2+C_{I0y}^2C_{S0}^2\frac{\partial^2}{\partial y^2}\}\frac{\delta p_{tot}}{\rho_0}\\
&=[-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^3+C_{S0}^2C_{I0y}^2\frac{\partial^2}{\partial y^2}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})]\\
&\qquad[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial\delta V_x}{\partial x}\\
&+\frac{1}{\rho_0}\frac{dV_{0y}}{dx}\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})[-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2+C_{S0}^2C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial\delta p_{tot}}{\partial x}
\end{aligned}
\tag{5.19d}
$$

Applying the differential operator $[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]$ to equation (5.19d) then substituting equation (5.12c) into the resulting equation to eliminate $[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]^2\frac{\partial\delta V_x}{\partial x}$, it yields

$$
\begin{aligned}
&-\frac{1}{\rho_0}[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\\
&\qquad(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2[-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2+C_{S0}^2C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial^2\delta p_{tot}}{\partial x^2}\\
&+\frac{1}{\rho_0}\{2\frac{dV_{0y}}{dx}\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})-\frac{dC_{I0y}^2}{dx}\frac{\partial^2}{\partial y^2}+\frac{1}{\rho_0}\frac{d\rho_0}{dx}[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\}\\
&\qquad(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2[-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2+C_{S0}^2C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial\delta p_{tot}}{\partial x}\\
&-\frac{1}{\rho_0}\{(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^4+\frac{\partial^2}{\partial y^2}[-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2+C_{I0y}^2C_{S0}^2\frac{\partial^2}{\partial y^2}]\}\\
&\qquad(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2-C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta p_{tot}=0
\end{aligned}
\tag{5.19e}
$$

Applying the integration operator $\rho_0(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^{-2}$ to equation (5.19e) yields

$$
\begin{aligned}
&-[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 - C_{I0y}^2\frac{\partial^2}{\partial y^2}] \\
&\qquad [-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 + C_{S0}^2 C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial^2 \delta p_{tot}}{\partial x^2} \\
&+\{2\frac{dV_{0y}}{dx}\frac{\partial}{\partial y}(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y}) - \frac{dC_{I0y}^2}{dx}\frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dx}[(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 - C_{I0y}^2\frac{\partial^2}{\partial y^2}]\} \\
&\qquad [-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 + C_{S0}^2 C_{I0y}^2\frac{\partial^2}{\partial y^2}]\frac{\partial \delta p_{tot}}{\partial x} \\
&-\{(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^4 + \frac{\partial^2}{\partial y^2}[-(C_{S0}^2+C_{A0}^2)(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 + C_{I0y}^2 C_{S0}^2\frac{\partial^2}{\partial y^2}]\} \\
&\qquad [(\frac{\partial}{\partial t}+V_{0y}\frac{\partial}{\partial y})^2 - C_{I0y}^2\frac{\partial^2}{\partial y^2}]\delta p_{tot} = 0
\end{aligned}
\tag{5.20}
$$