Chapter 5-Old. Kelvin-Helmholtz Instability

Exercise 5.1

Read section 11.4.3 in the following textbook and derive symmetric Kelvin-Helmholtz instability occurred at a tangential discontinuity (TD) due to velocity shear on two sides of the TD.

Parks, G. K., *Physics of Space Plasmas: An Introduction,* Addison-Wesley Publ. Co., 1991.

Velocity shear at boundary of two mediums may be unstable to Kelvin-Helmholtz instability, which can result in large amplitude surface wave at the boundary. Kelvin-Helmholtz instability due to wind shear on water surface can lead to large amplitude surface wave. But the tension force of water surface can stabilize Kelvin-Helmholtz instability. Thus, large amplitude water wave can only be found when the wind speed is large enough to overcome the tension force. Kelvin-Helmholtz instability can also be found in the atmosphere. An island in ocean can disturb airflow above it. This disturbance can trigger Kelvin-Helmholtz instability and result in wavy cloud pattern downstream from the island. Twisting of auroral arcs is another example of Kelvin-Helmholtz instability.

In this lecture, we shall discuss MHD Kelvin-Helmholtz instability occurred at a tangential discontinuity, such as dawn and dusk flanks of magnetopause. The Kelvin-Helmholtz instability in this region can result in a mixing layer at low latitude boundary layer (LLBL).

Basic equations of an ideal MHD plasma:

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = (\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla)\rho + \rho \nabla \cdot \mathbf{V} = 0 \tag{5.1}$$

Momentum equation

$$\rho(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{5.2}$$

Energy equation

$$(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)(p\rho^{-\gamma}) = 0 \tag{5.3}$$

or

$$(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)p = \frac{\gamma p}{\rho}(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\rho = \frac{\gamma p}{\rho}[-\rho\nabla\cdot\mathbf{V}] = -\gamma p\nabla\cdot\mathbf{V} \tag{5.3'}$$

where Eq. (5.1) has been used to obtain Eq. (5.3').

Charge continuity equation

$$\nabla\cdot\mathbf{J} \approx 0$$

MHD Ohm's Law

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} \approx 0 \tag{5.4}$$

Maxwell's equations

$$\nabla\cdot\mathbf{E} \approx 0$$

$$\nabla\cdot\mathbf{B} = 0 \tag{5.5}$$

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \tag{5.6}$$

$$\nabla\times\mathbf{B} \approx \mu_0\mathbf{J} \tag{5.7}$$

Substituting Eq. (5.4) into Eq. (5.6) yields

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} = \nabla\times(\mathbf{V}\times\mathbf{B}) = -\mathbf{V}\cdot\nabla\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{V} + \mathbf{B}\cdot\nabla\mathbf{V} + \mathbf{V}\nabla\cdot\mathbf{B}$$

or

$$(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\mathbf{B} = -\mathbf{B}\nabla\cdot\mathbf{V} + \mathbf{B}\cdot\nabla\mathbf{V} \tag{5.6'}$$

where Eq. (5.5) has been used to obtain Eq. (5.6').

Substituting (5.7) into (5.2) yields

$$\rho(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = -\nabla p - \nabla\frac{B^2}{2\mu_0} + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0} \tag{5.2'}$$

Define total pressure

$$p_{tot} = p + \frac{B^2}{2\mu_0} \tag{5.8}$$

Eq. (5.2') can be rewritten as

$$\rho(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p_{tot} + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0} \tag{5.2''}$$

The time derivative of Eq. (5.8) in fluid element moving frame is

$$(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)p_{tot} = (\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)p + \frac{\mathbf{B}}{\mu_0}\cdot[(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\mathbf{B}] \tag{5.9}$$

Substituting Eqs. (5.3') and (5.6') into Eq. (5.9) yields

$$(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)p_{tot} = -(\gamma p + \frac{B^2}{\mu_0})\nabla\cdot\mathbf{V} + \frac{\mathbf{B}}{\mu_0}\cdot(\mathbf{B}\cdot\nabla\mathbf{V}) \tag{5.10}$$

We choose normal direction of the tangential discontinuity (TD) to be the x direction. We consider a TD located at $x = 0$. The background magnetic field and flow velocity are all in the tangent direction. To linearize above equations, we consider a small perturbation δA superimpose on a background equilibrium state $A_0(x)$ and assume the small perturbation δA is in the following form

$\delta A(x,y,t) = \delta\bar{A}(x)\exp[i(k_t y - \omega t)]$

so that we have

$$A(x,y,t) = A_0(x) + \delta\bar{A}(x)\exp[i(k_t y - \omega t)] \tag{5.11}$$

The equilibrium state of the TD satisfies

$$p_{0tot} = \text{constant} = p_0(x) + \frac{B_0^2(x)}{2\mu_0} = p_0(x) + \frac{B_{0y}^2(x) + B_{0z}^2(x)}{2\mu_0} \tag{5.12}$$

and

$$B_{0x} = 0 \tag{5.13}$$

$$\mathbf{V}_0 = \hat{y}V_{0y}(x) \tag{5.14}$$

Differentiating Eq. (5.12) once with respect to x yields

$$0 = \frac{dp_0(x)}{dx} + \frac{B_{0y}(x)}{\mu_0}\frac{dB_{0y}(x)}{dx} + \frac{B_{0z}(x)}{\mu_0}\frac{dB_{0z}(x)}{dx} \tag{5.15}$$

Applying Eq. (5.11) to Eq. (5.2'') yields

$$\rho_0(x)(-i)[\omega - k_t V_{0y}(x)]\delta\bar{V}_x(x) = -\frac{d\delta\bar{p}_{tot}(x)}{dx} + \frac{B_{0y}(ik_t)}{\mu_0}\delta\bar{B}_x(x) \tag{5.16}$$

$$\rho_0(x)(-i)[\omega - k_t V_{0y}(x)]\delta\bar{V}_y(x) = -ik_t\delta\bar{p}_{tot}(x) + \frac{B_{0y}(ik_t)}{\mu_0}\delta\bar{B}_y(x) + \frac{\delta\bar{B}_x(x)}{\mu_0}\frac{dB_{0y}(x)}{dx} - \rho_0(x)\delta\bar{V}_x(x)\frac{dV_{0y}(x)}{dx} \tag{5.17}$$

$$\rho_0(x)(-i)[\omega - k_t V_{0y}(x)]\delta\bar{V}_z(x) = 0 + \frac{B_{0y}(ik_t)}{\mu_0}\delta\bar{B}_z(x) + \frac{\delta\bar{B}_x(x)}{\mu_0}\frac{dB_{0z}(x)}{dx} \tag{5.18}$$

Applying Eq. (5.11) to Eq. (5.3') yields

$$(-i)[\omega - k_t V_{0y}(x)]\delta\overline{p}(x) = -\gamma p_0(x)[\frac{d\delta\overline{V}_x(x)}{dx} + ik_t\delta\overline{V}_y(x)] - \delta\overline{V}_x(x)\frac{dp_0(x)}{dx} \tag{5.19}$$

Applying Eq. (5.11) to Eq. (5.6') yields

$$(-i)[\omega - k_t V_{0y}(x)]\delta\overline{B}_x(x) = 0 + B_{0y}(x)(ik_t)\delta\overline{V}_x(x) \tag{5.20}$$

$$(-i)[\omega - k_t V_{0y}(x)]\delta\overline{B}_y(x) = -B_{0y}(x)[\frac{d\delta\overline{V}_x(x)}{dx} + ik_t\delta\overline{V}_y(x)] + B_{0y}(x)(ik_t)\delta\overline{V}_y(x) + \delta\overline{B}_x(x)\frac{dV_{0y}(x)}{dx} - \delta\overline{V}_x(x)\frac{dB_{0y}(x)}{dx} \tag{5.21}$$

$$(-i)[\omega - k_t V_{0y}(x)]\delta\overline{B}_z(x) = -B_{0z}(x)[\frac{d\delta\overline{V}_x(x)}{dx} + ik_t\delta\overline{V}_y(x)] + B_{0y}(x)(ik_t)\delta\overline{V}_z(x) - \delta\overline{V}_x(x)\frac{dB_{0z}(x)}{dx} \tag{5.22}$$

Applying Eq. (5.11) to Eq. (5.10) yields

$$(-i)[\omega - k_t V_{0y}(x)]\delta\overline{p}_{tot}(x) = -[\gamma p_0(x) + \frac{B_0^2(x)}{\mu_0}][\frac{d\delta\overline{V}_x(x)}{dx} + ik_t\delta\overline{V}_y(x)] + \frac{B_{0y}(x)}{\mu_0}B_{0y}(x)(ik_t)\delta\overline{V}_y(x) + \frac{B_{0z}(x)}{\mu_0}B_{0y}(x)(ik_t)\delta\overline{V}_z(x) + \frac{B_{0y}(x)}{\mu_0}\delta\overline{B}_x(x)\frac{dV_{0y}(x)}{dx} \tag{5.23}$$

Governing equations to be solved include

Eq. (5.16): $\{\delta\overline{V}_x, \delta\overline{B}_x, \frac{d\delta\overline{p}_{tot}}{dx}\}$

Eq. (5.17): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{B}_x, \delta\overline{B}_y, \delta\overline{p}_{tot}\}$

Eq. (5.18): $\{\delta\overline{V}_z, \delta\overline{B}_x, \delta\overline{B}_z\}$

Eq. (5.20): $\{\delta\overline{V}_x, \delta\overline{B}_x\}$

Eq. (5.21): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{B}_x, \delta\overline{B}_y, \frac{d\delta\overline{V}_x}{dx}\}$

Eq. (5.22): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{V}_z, \delta\overline{B}_z, \frac{d\delta\overline{V}_x}{dx}\}$

Eq. (5.23): $\{\delta\overline{V}_y, \delta\overline{V}_z, \delta\overline{B}_x, \delta\overline{p}_{tot}, \frac{d\delta\overline{V}_x}{dx}\}$

Substituting Eq. (5.20) into Eqs. (5.16)~(5.18), (5.21) and (5.23) to eliminate $\delta\overline{B}_x$, it yields

Eq. (5.20) + Eq. (5.16) to eliminate $\delta\overline{B}_x$ ➔ Eq. (5.24): $\{\delta\overline{V}_x, \frac{d\delta\overline{p}_{tot}}{dx}\}$

Eq. (5.20) + Eq. (5.17) to eliminate $\delta\overline{B}_x$ ➔ Eq. (5.25): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{B}_y, \delta\overline{p}_{tot}\}$

Eq. (5.20) + Eq. (5.18) to eliminate $\delta\overline{B}_x$ ➔ Eq. (5.26): $\{\delta\overline{V}_z, \delta\overline{B}_z\}$

Eq. (5.20) + Eq. (5.21) to eliminate $\delta\overline{B}_x$ ➔ Eq. (5.27): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{B}_y, \frac{d\delta\overline{V}_x}{dx}\}$

Eq. (5.20) + Eq. (5.23) to eliminate $\delta\overline{B}_x$ ➔ Eq. (5.28): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{V}_z, \delta\overline{p}_{tot}, \frac{d\delta\overline{V}_x}{dx}\}$

We have eliminated $\delta\overline{B}_x$. Now we wish to eliminate $\delta\overline{B}_y$ and $\delta\overline{B}_z$.

Eq. (5.25) + Eq. (5.27) to eliminate $\delta\overline{B}_y$ ➔ Eq. (5.29): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{p}_{tot}, \frac{d\delta\overline{V}_x}{dx}\}$

Eq. (5.26) + Eq. (5.22) to eliminate $\delta\overline{B}_z$ ➔ Eq. (5.30): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{V}_z, \frac{d\delta\overline{V}_x}{dx}\}$

Indeed, we found that Eq. (5.30) is an expression without $\delta\overline{V}_x$. That is we have Eq. (5.30): $\{\delta\overline{V}_y, \delta\overline{V}_z, \frac{d\delta\overline{V}_x}{dx}\}$

In summary, after eliminating $\delta\overline{B}_x$, $\delta\overline{B}_y$, and $\delta\overline{B}_z$, we are left with the following four governing equations

Eq. (5.24): $\{\delta\overline{V}_x, \frac{d\delta\overline{p}_{tot}}{dx}\}$

Eq. (5.28): $\{\delta\overline{V}_y, \delta\overline{V}_z, \delta\overline{p}_{tot}, \frac{d\delta\overline{V}_x}{dx}\}$

Eq. (5.29): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{p}_{tot}, \frac{d\delta\overline{V}_x}{dx}\}$

Eq. (5.30): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{V}_z, \frac{d\delta\overline{V}_x}{dx}\}$

The following procedure is for constructing governing equation of $\delta\overline{p}_{tot}$

Eq. (5.28) + Eq. (5.30) to eliminate $\delta\overline{V}_z$ ➔ Eq. (5.31): $\{\delta\overline{V}_x, \delta\overline{V}_y, \delta\overline{p}_{tot}, \frac{d\delta\overline{V}_x}{dx}\}$

Eq. (5.29) + Eq. (5.31) to eliminate $\delta\overline{V}_y$ ➔ Eq. (5.32): $\{\delta\overline{V}_x, \delta\overline{p}_{tot}, \frac{d\delta\overline{V}_x}{dx}\}$

$\frac{d}{dx}$ Eq. (5.24) ➔ Eq. (5.33): $\{\frac{d\delta\overline{V}_x}{dx}, \frac{d\delta\overline{p}_{tot}}{dx}, \frac{d^2\delta\overline{p}_{tot}}{dx^2}\}$

Eq. (5.32) + Eq. (5.33) to eliminate $\frac{d\delta\overline{V}_x}{dx}$ ➔ Eq. (5.34): $\{\delta\overline{V}_x, \delta\overline{p}_{tot}, \frac{d\delta\overline{p}_{tot}}{dx}, \frac{d^2\delta\overline{p}_{tot}}{dx^2}\}$

Eq. (5.24) + Eq. (5.34) to eliminate $\delta\overline{V}_x$ ➔ Eq. (5.35): $\{\delta\overline{p}_{tot}, \frac{d\delta\overline{p}_{tot}}{dx}, \frac{d^2\delta\overline{p}_{tot}}{dx^2}\}$

The Eq. (5.35) is the governing equation of $\delta\overline{p}_{tot}$.

The following procedure is for constructing governing equation of $\delta\overline{V}_x$

$\frac{d}{dx}$ Eq. (5.32) ➔ Eq. (5.36): $\{\delta\overline{V}_x, \frac{d\delta\overline{V}_x}{dx}, \frac{d^2\delta\overline{V}_x}{dx^2}, \frac{d\delta\overline{p}_{tot}}{dx}\}$

Eq. (5.24) + Eq. (5.36) to eliminate $\frac{d\delta\overline{p}_{tot}}{dx}$ ➔ Eq. (5.37): $\{\delta\overline{V}_x, \frac{d\delta\overline{V}_x}{dx}, \frac{d^2\delta\overline{V}_x}{dx^2}\}$

The Eq. (5.37) is the governing equation of $\delta\overline{V}_x$.

Similary, we can obtain the governing equations of other functions $\delta\overline{V}_y$, $\delta\overline{B}_x$, $\delta\overline{B}_y$, and $\delta\overline{B}_z$.

Here we briefly list Eqs. (5.24)~(5.37)

$$\boxed{\delta\overline{V}_x(x) = \frac{\frac{\omega}{k_t} - V_{0y}(x)}{\Pi_0(x)}[\frac{-i}{k_t}\frac{d}{dx}\delta\overline{p}_{tot}(x)]} \tag{5.24}$$

where

$$\Pi_0(x) \equiv \rho_0(x)\{[\frac{\omega}{k_t} - V_{0y}(x)]^2 - \frac{B_{0y}^2(x)}{\mu_0\rho_0(x)}\} \tag{5.24a}$$

$$\boxed{\begin{aligned}\rho_0(x)[\frac{\omega}{k_t} - V_{0y}(x)]^2\delta\overline{V}_y(x) &= [\frac{\omega}{k_t} - V_{0y}(x)]\delta\overline{p}_{tot}(x) - [\frac{\omega}{k_t} - V_{0y}(x)]\frac{B_{0y}(x)}{\mu_0}\delta\overline{B}_y(x) \\ &-i\{\frac{B_{0y}(x)}{\mu_0}\frac{1}{k_t}\frac{dB_{0y}(x)}{dx} + [\frac{\omega}{k_t} - V_{0y}(x)]\frac{\rho_0(x)}{k_t}\frac{dV_{0y}(x)}{dx}\}\delta\overline{V}_x(x)\end{aligned}} \tag{5.25}$$

$$\rho_0(x)[\frac{\omega}{k_t}-V_{0y}(x)]^2\delta\overline{V}_z(x) = -[\frac{\omega}{k_t}-V_{0y}(x)]\frac{B_{0y}(x)}{\mu_0}\delta\overline{B}_z(x)-i[\frac{B_{0y}(x)}{\mu_0}\frac{1}{k_t}\frac{dB_{0z}(x)}{dx}]\delta\overline{V}_x(x) \tag{5.26}$$

$$[\frac{\omega}{k_t}-V_{0y}(x)]\frac{B_{0y}(x)}{\mu_0}\delta\overline{B}_y(x) = -i\frac{B_{0y}^2(x)}{\mu_0}\frac{1}{k_t}\frac{d\delta\overline{V}_x(x)}{dx}-i[\frac{B_{0y}^2(x)}{\mu_0}\frac{\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{\frac{\omega}{k_t}-V_{0y}(x)}+\frac{B_{0y}(x)}{\mu_0}\frac{1}{k_t}\frac{dB_{0y}(x)}{dx}]\delta\overline{V}_x(x) \tag{5.27}$$

$$[\frac{\omega}{k_t}-V_{0y}(x)]\delta\overline{p}_{tot}(x)+i[\gamma p_0(x)+\frac{B_{0y}^2(x)+B_{0z}^2(x)}{\mu_0}]\frac{1}{k_t}\frac{d\delta\overline{V}_x(x)}{dx} +i\frac{B_{0y}^2(x)}{\mu_0}\frac{\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{\frac{\omega}{k_t}-V_{0y}(x)}\delta\overline{V}_x(x)-[\gamma p_0(x)+\frac{B_{0z}^2(x)}{\mu_0}]\delta\overline{V}_y(x) = -\frac{B_{0y}(x)B_{0z}(x)}{\mu_0}\delta\overline{V}_z(x) \tag{5.28}$$

$$\rho_0(x)[\frac{\omega}{k_t}-V_{0y}(x)]^2\delta\overline{V}_y(x) = [\frac{\omega}{k_t}-V_{0y}(x)]\delta\overline{p}_{tot}(x)+i\frac{B_{0y}^2(x)}{\mu_0}\frac{1}{k_t}\frac{d\delta\overline{V}_x(x)}{dx}-i\delta\overline{V}_x(x)\Pi_0(x)\frac{\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{\frac{\omega}{k_t}-V_{0y}(x)} \tag{5.29}$$

$$\Pi_0(x)\delta\overline{V}_z(x) = i\frac{B_{0y}(x)B_{0z}(x)}{\mu_0}\frac{1}{k_t}\frac{d\delta\overline{V}_x(x)}{dx}-\frac{B_{0y}(x)B_{0z}(x)}{\mu_0}\delta\overline{V}_y(x) \tag{5.30}$$

$$[\frac{\omega}{k_t}-V_{0y}(x)]\delta\overline{p}_{tot}(x)+i\{\gamma p_0(x)+\frac{B_{0y}^2(x)}{\mu_0}+\frac{\frac{B_{0z}^2(x)}{\mu_0}[\frac{\omega}{k_t}-V_{0y}(x)]^2}{[\frac{\omega}{k_t}-V_{0y}(x)]^2-\frac{B_{0y}^2(x)}{\mu_0\rho_0(x)}}\}\frac{1}{k_t}\frac{d\delta\overline{V}_x(x)}{dx} +i\frac{B_{0y}^2(x)}{\mu_0}\frac{\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{\frac{\omega}{k_t}-V_{0y}(x)}\delta\overline{V}_x(x)-\{\gamma p_0(x)+\frac{\frac{B_{0z}^2(x)}{\mu_0}[\frac{\omega}{k_t}-V_{0y}(x)]^2}{[\frac{\omega}{k_t}-V_{0y}(x)]^2-\frac{B_{0y}^2(x)}{\mu_0\rho_0(x)}}\}\delta\overline{V}_y(x)=0 \tag{5.31}$$

$$\delta\bar{p}_{tot}(x)\left[\frac{\frac{\omega}{k_t}-V_{0y}(x)}{\Pi_0(x)}\right]F_0(x)+\frac{i}{k_t}\frac{d\delta\bar{V}_x(x)}{dx}+i\delta\bar{V}_x(x)\frac{\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{\frac{\omega}{k_t}-V_{0y}(x)}=0 \tag{5.32}$$

where

$$F_0(x)=\frac{[\frac{\omega}{k_t}-V_{0y}(x)]^2}{\frac{\gamma p_0(x)}{\rho_0(x)}+\frac{B_{0y}^2(x)+B_{0z}^2(x)}{\mu_0\rho_0(x)}-\frac{B_{0y}^2(x)}{\mu_0\rho_0(x)}\frac{\frac{\gamma p_0(x)}{\rho_0(x)}}{[\frac{\omega}{k_t}-V_{0y}(x)]^2}}-1 \tag{5.32a}$$

$$\frac{i}{k_t}\frac{d}{dx}\delta\bar{V}_x(x)=\frac{1}{k_t}\frac{d}{dx}\left[\frac{\frac{\omega}{k_t}-V_{0y}(x)}{\Pi_0(x)}\right]\left[\frac{1}{k_t}\frac{d\delta\bar{p}_{tot}(x)}{dx}\right]+\frac{\frac{\omega}{k_t}-V_{0y}(x)}{\Pi_0(x)}\left[\frac{1}{k_t^2}\frac{d^2\delta\bar{p}_{tot}(x)}{dx^2}\right] \tag{5.33}$$

$$\begin{aligned}&\delta\bar{p}_{tot}(x)\left[\frac{\frac{\omega}{k_t}-V_{0y}(x)}{\Pi_0(x)}\right]F_0(x)+\frac{1}{k_t}\frac{d\delta\bar{p}_{tot}(x)}{dx}\frac{1}{k_t}\frac{d}{dx}\left[\frac{\frac{\omega}{k_t}-V_{0y}(x)}{\Pi_0(x)}\right]\\&+\frac{1}{k_t^2}\frac{d^2\delta\bar{p}_{tot}(x)}{dx^2}\frac{\frac{\omega}{k_t}-V_{0y}(x)}{\Pi_0(x)}+i\delta\bar{V}_x(x)\frac{\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{\frac{\omega}{k_t}-V_{0y}(x)}=0\end{aligned} \tag{5.34}$$

$$\frac{1}{k_t^2}\frac{d^2\delta\bar{p}_{tot}(x)}{dx^2}+\frac{1}{k_t}\frac{d\delta\bar{p}_{tot}(x)}{dx}\Pi_0(x)\frac{1}{k_t}\frac{d}{dx}\left[\frac{1}{\Pi_0(x)}\right]+\delta\bar{p}_{tot}(x)F_0(x)=0 \tag{5.35}$$

$$\frac{i}{k_t}\frac{d\delta\bar{p}_{tot}(x)}{dx}=\frac{1}{k_t}\frac{d}{dx}\left\{\frac{[\frac{\omega}{k_t}-V_{0y}(x)]\frac{1}{k_t}\frac{d\delta\bar{V}_x(x)}{dx}+\delta\bar{V}_x(x)\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{[\frac{\omega}{k_t}-V_{0y}(x)]^2\frac{F_0(x)}{\Pi_0(x)}}\right\} \tag{5.36}$$

$$\frac{\Pi_0(x)}{\frac{\omega}{k_t}-V_{0y}(x)}\delta\bar{V}_x(x)+\frac{1}{k_t}\frac{d}{dx}\left\{\frac{[\frac{\omega}{k_t}-V_{0y}(x)]\frac{1}{k_t}\frac{d\delta\bar{V}_x(x)}{dx}+\delta\bar{V}_x(x)\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{[\frac{\omega}{k_t}-V_{0y}(x)]^2\frac{F_0(x)}{\Pi_0(x)}}\right\}=0 \tag{5.37}$$

In Summary:

The governing equation of $\delta\overline{p}_{tot}$ is

$$\frac{1}{k_t^2}\frac{d^2\delta\overline{p}_{tot}(x)}{dx^2}+\frac{1}{k_t}\frac{d\delta\overline{p}_{tot}(x)}{dx}\Pi_0(x)\frac{1}{k_t}\frac{d}{dx}[\frac{1}{\Pi_0(x)}]+\delta\overline{p}_{tot}(x)F_0(x)=0 \quad (5.35)$$

The governing equation of $\delta\overline{V}_x$ is

$$\frac{\Pi_0(x)}{\frac{\omega}{k_t}-V_{0y}(x)}\delta\overline{V}_x(x)+\frac{1}{k_t}\frac{d}{dx}\{\frac{[\frac{\omega}{k_t}-V_{0y}(x)]\frac{1}{k_t}\frac{d\delta\overline{V}_x(x)}{dx}+\delta\overline{V}_x(x)\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{[\frac{\omega}{k_t}-V_{0y}(x)]^2\frac{F_0(x)}{\Pi_0(x)}}\}=0 \quad (5.37)$$

where

$$\Pi_0(x)\equiv\rho_0(x)\{[\frac{\omega}{k_t}-V_{0y}(x)]^2-\frac{B_{0y}^2(x)}{\mu_0\rho_0(x)}\} \quad (5.24a)$$

$$F_0(x)=\frac{[\frac{\omega}{k_t}-V_{0y}(x)]^2}{\frac{\gamma p_0(x)}{\rho_0(x)}+\frac{B_{0y}^2(x)+B_{0z}^2(x)}{\mu_0\rho_0(x)}-\frac{B_{0y}^2(x)}{\mu_0\rho_0(x)}\frac{\frac{\gamma p_0(x)}{\rho_0(x)}}{[\frac{\omega}{k_t}-V_{0y}(x)]^2}}-1 \quad (5.32a)$$

Remark 1:

$F_0(x)$ in Eq. (5.32a) can be written as

$$F_0(x)=\frac{[\frac{\omega}{k_t}-V_{0y}(x)]^4}{R_0(x)}-1$$

where

$$R_0(x)=\{\frac{\gamma p_0(x)}{\rho_0(x)}+\frac{B_{0y}^2(x)+B_{0z}^2(x)}{\mu_0\rho_0(x)}\}[\frac{\omega}{k_t}-V_{0y}(x)]^2-\frac{B_{0y}^2(x)}{\mu_0\rho_0(x)}\frac{\gamma p_0(x)}{\rho_0(x)}$$

It can be shown that

$$[\frac{\omega}{k_t}-V_{0y}(x)]^4-R_0(x)$$

$$=[\frac{\omega}{k_t}-V_{0y}(x)]^4-\{\frac{\gamma p_0(x)}{\rho_0(x)}+\frac{B_{0y}^2(x)+B_{0z}^2(x)}{\mu_0\rho_0(x)}\}[\frac{\omega}{k_t}-V_{0y}(x)]^2-\frac{B_{0y}^2(x)}{\mu_0\rho_0(x)}\frac{\gamma p_0(x)}{\rho_0(x)}$$

$$=\{[\frac{\omega}{k_t}-V_{0y}(x)]^2-V_{F0y}^2(x)\}\{[\frac{\omega}{k_t}-V_{0y}(x)]^2-V_{SL0y}^2(x)\}$$

where $V_{F0y}(x)$ and $V_{SL0y}(x)$ are, respectively, the phase velocity of fast-mode and slow-mode waves which propagate parallel or anti-parallel to the surface wave direction ($\mathbf{k}_t = k_t\hat{y}$).

Thus $F_0(x) > 0$ indicates

$$[\frac{\omega}{k_t} - V_{0y}(x)]^2 > V_{F0y}^2(x)$$

or

$$[\frac{\omega}{k_t} - V_{0y}(x)]^2 < V_{SL0y}^2(x)$$

Remark 2:

$\Pi_0(x)$ in Eq. (5.24a) can be written as

$$\Pi_0(x) = \rho_0(x)\{[\frac{\omega}{k_t} - V_{0y}(x)]^2 - V_{A0y}^2(x)\}$$

where $V_{A0y}(x)$ is the phase velocity of shear Alfven wave which propagates parallel or anti-parallel to the surface wave direction ($\mathbf{k}_t = k_t\hat{y}$).

Thus $\Pi_0(x) > 0$ indicates

$$[\frac{\omega}{k_t} - V_{0y}(x)]^2 > V_{A0y}^2(x)$$

Remark 3:

For neutral fluid dynamics $B_{0y}(x) = B_{0z}(x) = 0$, it yields

$$\Pi_0(x) = \rho_0(x)[\frac{\omega}{k_t} - V_{0y}(x)]^2,$$

$$F_0(x) = \frac{[\frac{\omega}{k_t} - V_{0y}(x)]^2}{\frac{\gamma p_0(x)}{\rho_0(x)}} - 1,$$

and $\delta\overline{p}_{tot}(x) = \delta\overline{p}(x)$. Thus, Eqs. (5.35) and (5.37) are reduced to

$$\frac{1}{k_t^2}\frac{d^2\delta\overline{p}(x)}{dx^2} + \frac{1}{k_t}\frac{d\delta\overline{p}(x)}{dx}\rho_0(x)[\frac{\omega}{k_t} - V_{0y}(x)]^2\frac{1}{k_t}\frac{d}{dx}\{\frac{1}{\rho_0(x)[\frac{\omega}{k_t} - V_{0y}(x)]^2}\} + \delta\overline{p}(x)\{\frac{[\frac{\omega}{k_t} - V_{0y}(x)]^2}{\frac{\gamma p_0(x)}{\rho_0(x)}} - 1\} = 0 \qquad (5.35n)$$

$$\rho_0(x)[\frac{\omega}{k_t} - V_{0y}(x)]\delta\overline{V}_x(x) + \frac{1}{k_t}\frac{d}{dx}\{\rho_0(x)\frac{[\frac{\omega}{k_t} - V_{0y}(x)]\frac{1}{k_t}\frac{d\delta\overline{V}_x(x)}{dx} + \delta\overline{V}_x(x)\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{\frac{[\frac{\omega}{k_t} - V_{0y}(x)]^2}{\frac{\gamma p_0(x)}{\rho_0(x)}} - 1}\} = 0 \qquad (5.37n)$$

Eqs. (5.35n) and (5.37n) are similar to Eqs. (7) and (8), respectively, in the classical paper by Blumen (1970), in which density distribution are assumed to be uniform in space.

Exercise 5.2

Read the following classical papers, which study K-H instability in neutral fluid:

Blumen, W., Shear layer instability of an inviscid compressible fluid, *J. Fluid Mech., 40*, 769, 1970

Blumen, W., P. G. Drazin, and D. F. Billings, Shear layer instability of an inviscid

compressible fluid. Part 2, *J. Fluid Mech., 71*, 305, 1975.

Drazin, P. G., and A. Davey, Shear layer instability of an inviscid compressible fluid. Part 3, *J. Fluid Mech., 82*, 255, 1977.

Exercise 5.3

Read the following classical paper by Miura and Pritchett (1982):

Miura, A. and P. L. Pritchett, Nonlocal stability analysis of the MHD Kelvin-Helmholtz instability in a compressible plasma, *J. Geophys. Res., 87*, 7431, 1982.

Remark 4:

For surface wave propagating perpendicular to the ambient magnetic field, i.e., $B_{0y}(x)=0$. It yields

$\Pi_0(x)=\rho_0(x)[\frac{\omega}{k_t}-V_{0y}(x)]^2$, and

$$F_0(x)=\frac{[\frac{\omega}{k_t}-V_{0y}(x)]^2}{\frac{\gamma p_0(x)}{\rho_0(x)}+\frac{B_{0z}^2(x)}{\mu_0\rho_0(x)}}-1.$$

Thus, Eqs. (5.35) and (5.37) are reduced to

$$\frac{1}{k_t^2}\frac{d^2\delta\bar{p}_{tot}(x)}{dx^2}+\frac{1}{k_t}\frac{d\delta\bar{p}_{tot}(x)}{dx}\rho_0(x)[\frac{\omega}{k_t}-V_{0y}(x)]^2\frac{1}{k_t}\frac{d}{dx}\{\frac{1}{\rho_0(x)[\frac{\omega}{k_t}-V_{0y}(x)]^2}\} +\delta\bar{p}_{tot}(x)\{\frac{[\frac{\omega}{k_t}-V_{0y}(x)]^2}{\frac{\gamma p_0(x)}{\rho_0(x)}+\frac{B_{0z}^2(x)}{\mu_0\rho_0(x)}}-1\}=0 \tag{5.35⊥}$$

$$\rho_0(x)[\frac{\omega}{k_t}-V_{0y}(x)]\delta\bar{V}_x(x) +\frac{1}{k_t}\frac{d}{dx}\{\rho_0(x)\frac{[\frac{\omega}{k_t}-V_{0y}(x)]\frac{1}{k_t}\frac{d\delta\bar{V}_x(x)}{dx}+\delta\bar{V}_x(x)\frac{1}{k_t}\frac{dV_{0y}(x)}{dx}}{\frac{[\frac{\omega}{k_t}-V_{0y}(x)]^2}{\frac{\gamma p_0(x)}{\rho_0(x)}+\frac{B_{0z}^2(x)}{\mu_0\rho_0(x)}}-1}\}=0 \tag{5.37⊥}$$

Eqs. (5.35⊥) and (5.37⊥) are similar to Eqs. (5.35n) and (5.37n). Thus, solutions obtained by Drazin and Davey (1997) are applicable to Eqs. (5.35⊥) and (5.37⊥).

Remark 5:

For uniform background,

$$\Pi_0 = \rho_0\{[\frac{\omega}{k_t} - V_{0y}]^2 - \frac{B_{0y}^2}{\mu_0\rho_0}\}$$

$$F_0 = \frac{[\frac{\omega}{k_t} - V_{0y}]^2}{\frac{\gamma p_0}{\rho_0} + \frac{B_{0y}^2 + B_{0z}^2}{\mu_0\rho_0} - \frac{B_{0y}^2}{\mu_0\rho_0}\frac{\frac{\gamma p_0}{\rho_0}}{[\frac{\omega}{k_t} - V_{0y}]^2}} - 1$$

$$\frac{d}{dx}[\frac{1}{\Pi_0}] = 0, \quad \frac{dV_{0y}}{dx} = 0$$

Eqs. (5.35) and (5.37) are reduced to

$$\frac{1}{k_t^2}\frac{d^2\delta\bar{p}_{tot}(x)}{dx^2} + \delta\bar{p}_{tot}(x)F_0 = 0 \qquad (5.35u)$$

$$\frac{\Pi_0}{\frac{\omega}{k_t} - V_{0y}}\{\delta\bar{V}_x(x) + \frac{1}{F_0}\frac{1}{k_t^2}\frac{d^2\delta\bar{V}_x(x)}{dx^2}\} = 0 \qquad (5.37u)$$

Thus, for $\delta\bar{p}_{tot}(x) \neq 0$, it yields

$$F_0 + \frac{k_{xi}^2 - k_{xr}^2}{k_t^2} = 0$$

That is fast-mode or slow-mode. For the shear Alfven mode, we have

$\delta\bar{p}_{tot}(x) = 0$

For $\delta\bar{V}_x(x) \neq 0$, it yields

$$\Pi_0 = 0 \;\Rightarrow [\frac{\omega}{k_t} - V_{0y}]^2 = \frac{B_{0y}^2}{\mu_0\rho_0} \quad \text{(Shear-Alfven mode)}$$

or

$$F_0 + \frac{k_{xi}^2 - k_{xr}^2}{k_t^2} = 0 \Rightarrow \{(\frac{\omega}{k_t} - V_{0y})^2 - V_{F0y}^2\}\{(\frac{\omega}{k_t} - V_{0y})^2 - V_{SL0y}^2\} = \frac{k_{xr}^2 - k_{xi}^2}{k_t^2}R_0$$

where

$$R_0 = (\frac{\gamma p_0}{\rho_0} + \frac{B_{0y}^2 + B_{0z}^2}{\mu_0\rho_0})(\frac{\omega}{k_t} - V_{0y})^2 - \frac{B_{0y}^2}{\mu_0\rho_0}\frac{\gamma p_0}{\rho_0}$$