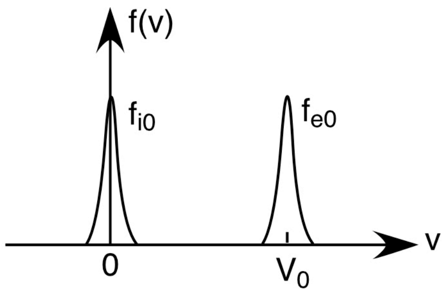
Chapter 4. Two-Stream Instabilities

Two-stream instability occurs when there are counter-streaming plasma flow in velocity space. Let us consider a field-free two-fluid plasma system, which consists of a cold ion fluid and a cold electron fluid. The cold ion fluid is at rest ($\mathbf{V}_{i0} = 0$) with uniform number density n_0. The cold electron fluid moves at velocity $\mathbf{V}_{e0} = V_0\hat{x}$, with number density n_0. It will be shown in this lecture that such a plasma system is unstable to some electrostatic waves that propagated in x-direction. It should be noted that such a two-stream plasma can lead to strong electric current in –x direction. As a result, the background field should not be field-free. To overcome this difficulty, we can consider a system with two counter-streaming electrons and one ion fluid at rest, or a system with two counter-streaming ions and one electron fluid at rest, or a system with two counter-streaming electrons and two counter-streaming ions. Procedures to obtain electrostatic wave dispersion relation and instability growth rate in such systems are similar to the one discussed in this lecture. Wave dispersion relation and instability analysis in these systems will leave as exercises for the students to explore.



For one ion fluid at rest and one electron fluid with velocity $\mathbf{V}_{e0} = V_0\hat{x}$, the field structure must be of two- or three-dimension. However, for simplicity, we shall consider "local approximation" and assume $\nabla = (\partial/\partial x)\hat{x}$ in a finite extended column along x-axis. For electrostatic waves, we have $\mathbf{E}_1 = -\nabla\Phi_1 = E_{x1}\hat{x}$. The linearized electrostatic two-fluid equations are

Linearized continuity equations

$$\frac{\partial n_{i1}}{\partial t} + n_0\frac{\partial V_{i1x}}{\partial x} = 0 \tag{4.1}$$

$$\frac{\partial n_{e1}}{\partial t} + V_0\frac{\partial n_{e1}}{\partial x} + n_0\frac{\partial V_{e1x}}{\partial x} = 0 \tag{4.2}$$

Linearized momentum equations

$$n_0 m_i \frac{\partial V_{i1x}}{\partial t} = e n_0 E_{1x} \tag{4.3}$$

$$n_0 m_e \left(\frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial x}\right) V_{e1x} = -e n_0 E_1 \tag{4.4}$$

Poisson equation

$$\frac{\partial}{\partial x} E_1 = \frac{e}{\varepsilon_0}(n_{i1} - n_{e1}) \tag{4.5}$$

We assume a plane-wave type of linear perturbation: $A_1(x,t) = \mathrm{Re}\{\tilde{A}_1(k,\omega)\exp[i(kx-\omega t)]\}$.

Fourier and Laplace transform of Eq. (4.1)-(4.5), yields

$$-i\omega \tilde{n}_{i1} + n_0 ik\tilde{V}_{i1x} = 0 \tag{4.1a}$$

$$-i(\omega - V_0 k)\tilde{n}_{e1} + n_0 ik\tilde{V}_{e1x} = 0 \tag{4.2a}$$

$$n_0 m_i(-i\omega)\tilde{V}_{i1x} = e n_0 \tilde{E}_{1x} \tag{4.3a}$$

$$n_0 m_e(-i)(\omega - V_0 k)\tilde{V}_{e1x} = -e n_0 \tilde{E}_1 \tag{4.4a}$$

$$ik\tilde{E}_1 = \frac{e}{\varepsilon_0}(\tilde{n}_{i1} - \tilde{n}_{e1}) \tag{4.5a}$$

There are two ways to determine dispersion relation of this system.

<u>Method 1</u>

Substituting (4.5a) into (4.3a) and (4.4a), then substituting the resulting equation into (4.1a) and (4.2a) yields

$$\begin{pmatrix} 1-\dfrac{\omega^2}{\omega_{pi}^2} & -1 \\ -1 & 1-\dfrac{(\omega-kV_0)^2}{\omega_{pe}^2} \end{pmatrix} \begin{pmatrix} \tilde{n}_{i1} \\ \tilde{n}_{e1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $\tilde{n}_{i1}$ and $\tilde{n}_{e1}$ have non-trivial solutions then

$$\det\begin{pmatrix} 1-\dfrac{\omega^2}{\omega_{pi}^2} & -1 \\ -1 & 1-\dfrac{(\omega-kV_0)^2}{\omega_{pe}^2} \end{pmatrix} = \left(\frac{\omega}{\omega_{pi}}\right)^2\left(\frac{\omega-kV_0}{\omega_{pe}}\right)^2 - \left(\frac{\omega}{\omega_{pi}}\right)^2 - \left(\frac{\omega-kV_0}{\omega_{pe}}\right)^2 = 0 \tag{4.6}$$

Method 2

Substituting (4.3a) and (4.4a) into (4.1a) and (4.2a), respectively, then substituting resulting equations into (4.5) yields

$$\varepsilon(k,\omega) ik\tilde{E}_{1x} = 0$$

where

$$\varepsilon(k,\omega) = 1 - (\frac{\omega_{pi}}{\omega})^2 - (\frac{\omega_{pe}}{\omega - kV_0})^2 = 0 \tag{4.7}$$

It will be shown that Eq. (4.6) obtained in Method 1 is useful for finding numerical solutions of different wave modes and growth rate of different wave mode. Whereas, Eq. (4.7) obtained in Method 2 is useful for determining solution space of unstable wave modes analytically.

Let $x = \omega / \omega_{pe}$, $\alpha = kV_0 / \omega_{pe}$, then (4.7) becomes

$$1 - \frac{m_e}{m_i}\frac{1}{x^2} - \frac{1}{(x-\alpha)^2} = 0 \tag{4.8}$$

To estimate solution of Eq. (4.8), let us consider the following function

$$f(x) = \frac{m_e}{m_i}\frac{1}{x^2} + \frac{1}{(x-\alpha)^2} \tag{4.9}$$

Figure 4.1 sketches (a) $y = 1/x^2$, (b) $y = 1/(x-\alpha)^2$, and (c) $y = f(x) = \frac{m_e}{m_i}\frac{1}{x^2} + \frac{1}{(x-\alpha)^2}$.

Solutions of (4.8) are the intersections of $y = 1$ and $y = f(x)$. Instability occurs when Eq. (4.8) has complex roots. It occurs when the local minimum of $y = f(x)$ for $0 < x < \alpha$ is greater than 1. Let local minimum of $y = f(x)$ is located at $x = x_A$, then $f'(x_A) = -2\frac{m_e}{m_i}\frac{1}{x_A^3} - 2\frac{1}{(x_A - \alpha)^3} = 0$, or

$$x_A = \frac{\alpha}{1 + \sqrt[3]{m_i / m_e}} = \frac{\alpha}{1 + A} \approx 0.075\alpha \tag{4.10}$$

where $A = \sqrt[3]{m_i / m_e} \approx 12.25$

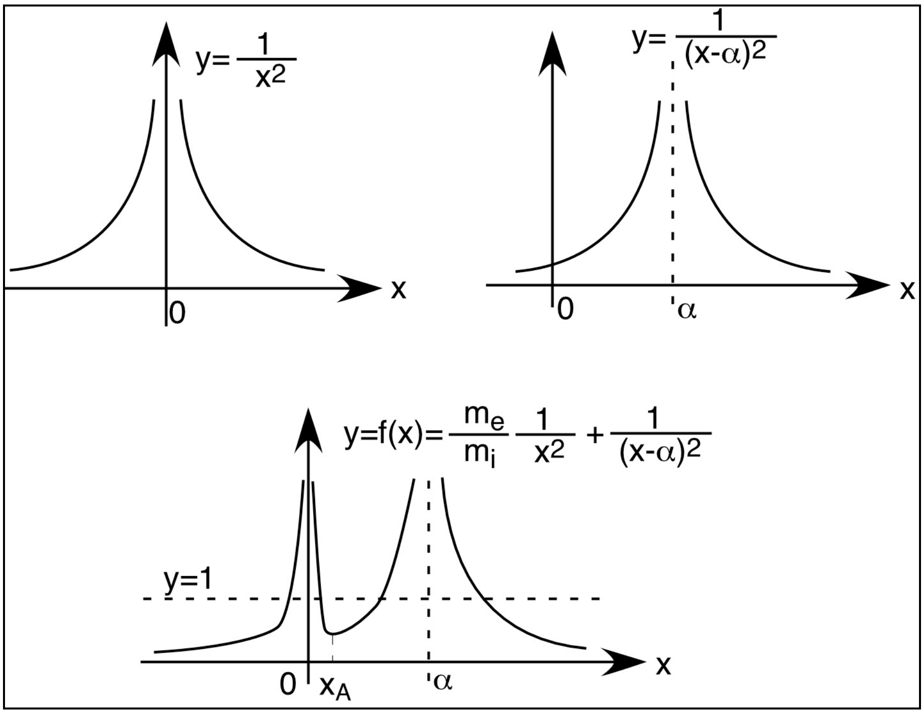


Figure 4.1 Sketches of (a) $y = 1/x^2$, (b) $y = 1/(x-\alpha)^2$, and (c) $y = f(x) = \dfrac{m_e}{m_i}\dfrac{1}{x^2} + \dfrac{1}{(x-\alpha)^2}$. Two-stream instability occurs when $f(x = x_A) > 1$

Thus, instability condition becomes

$$f(x_A) = \frac{m_e}{m_i}\frac{1}{x_A^2} + \frac{1}{(x_A-\alpha)^2} = \frac{1}{A^3}\frac{(1+A)^2}{\alpha^2} + \frac{(1+A)^2}{\alpha^2 A^2} = \frac{(1+A)^3}{\alpha^2 A^3} > 1 \text{or}$$

$$\alpha^2 < (\frac{1+A}{A})^3 \approx (\frac{13.25}{12.25})^3 \approx 1.265 \tag{4.11}$$

which yields

$$\alpha < 1.12486 \quad \text{or} \quad kV_0 < 1.12486\omega_{pe}. \tag{4.12}$$

Eq. (4.12) determines solution space of unstable wave modes, but does not tell us what is the most unstable wave mode. The most unstable wave mode can only be obtained by directly solving Eq. (4.6). We can rewrite Eq. (4.6) into the following form,

$$x^2(x-\alpha)^2 - x^2 - \frac{m_e}{m_i}(x-\alpha)^2 = 0$$

or

$$x^4 - 2\alpha x^3 + x^2(\alpha^2 - 1 - \frac{m_e}{m_i}) + (2\frac{m_e}{m_i}\alpha)x - \alpha^2\frac{m_e}{m_i} = 0 \tag{4.13}$$

Figure 4.2 shows all solutions of ω as a function of wave number k, which include one real root $\omega > \alpha$, one real root $\omega < 0$. The other two roots are two real roots $\omega = \omega_{r1}$ and ω_{r2} or two complex conjugates roots $\omega = \omega_r \pm i\omega_i$. The most unstable wave mode occurs near $\alpha \approx 1$ or $kV_0 \approx \omega_{pe}$ as can be seen in lower panel of Figure 4.2. The curve of ω_i in Figure 4.2b is similar to the curve 4 in Figure 9.3.2 in the textbook [N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics,* McGraw-Hill, New York, p.452 (1973)] and in literature [T. E. Stringer, Plasma Physics, *J. Nucl. Energy, C6*, 267 (1964)]. To understand solutions shown in Figure 2a, we can compare them with the solutions of Eq. (4.7) at $m_i \to \infty$ or $\omega_{pi} \to 0$. It can be shown that for $\omega_{pi} \to 0$, the four roots are

$x = 0, 0, \alpha + 1,$ and $\alpha - 1$ or $\omega = 0, 0, kV_0 + \omega_{pe},$ and $kV_0 - \omega_{pe}$.

Similarly, for finite ion mass, we can expect the following four wave modes,

$$x = \omega_{pi}/\omega_{pe}, -\omega_{pi}/\omega_{pe}, \alpha + 1, \text{and } \alpha - 1 \tag{4.14}$$

or

$$\omega = \omega_{pi}, -\omega_{pi}, kV_0 + \omega_{pe}, \text{and } kV_0 - \omega_{pe} \tag{4.15}$$

It can be seen from top panel of Figure 4.2 that the four real roots at short wavelength limit ($kV_0/\omega_{pe} >> 1$) approach to the solutions listed in Eq. (4.14) or (4.15). In long wavelength limit $kV_0/\omega_{pe} < 1$, two real roots approach

$x = \alpha + 1,$ and $\alpha - 1$ or $\omega = kV_0 + \omega_{pe},$ and $kV_0 - \omega_{pe}$.

Wave-mode coupling occurs at the intersection of $x = -\omega_{pi}/\omega_{pe},$ and $x = \alpha - 1$.

Maximum growth rate occurs near intersection of $x = \omega_{pi}/\omega_{pe}$ and $x = \alpha - 1$. Namely, after Doppler shift, the wave mode that is close to ion's plasma frequency becomes electrons' plasma frequency in electrons' moving frame.

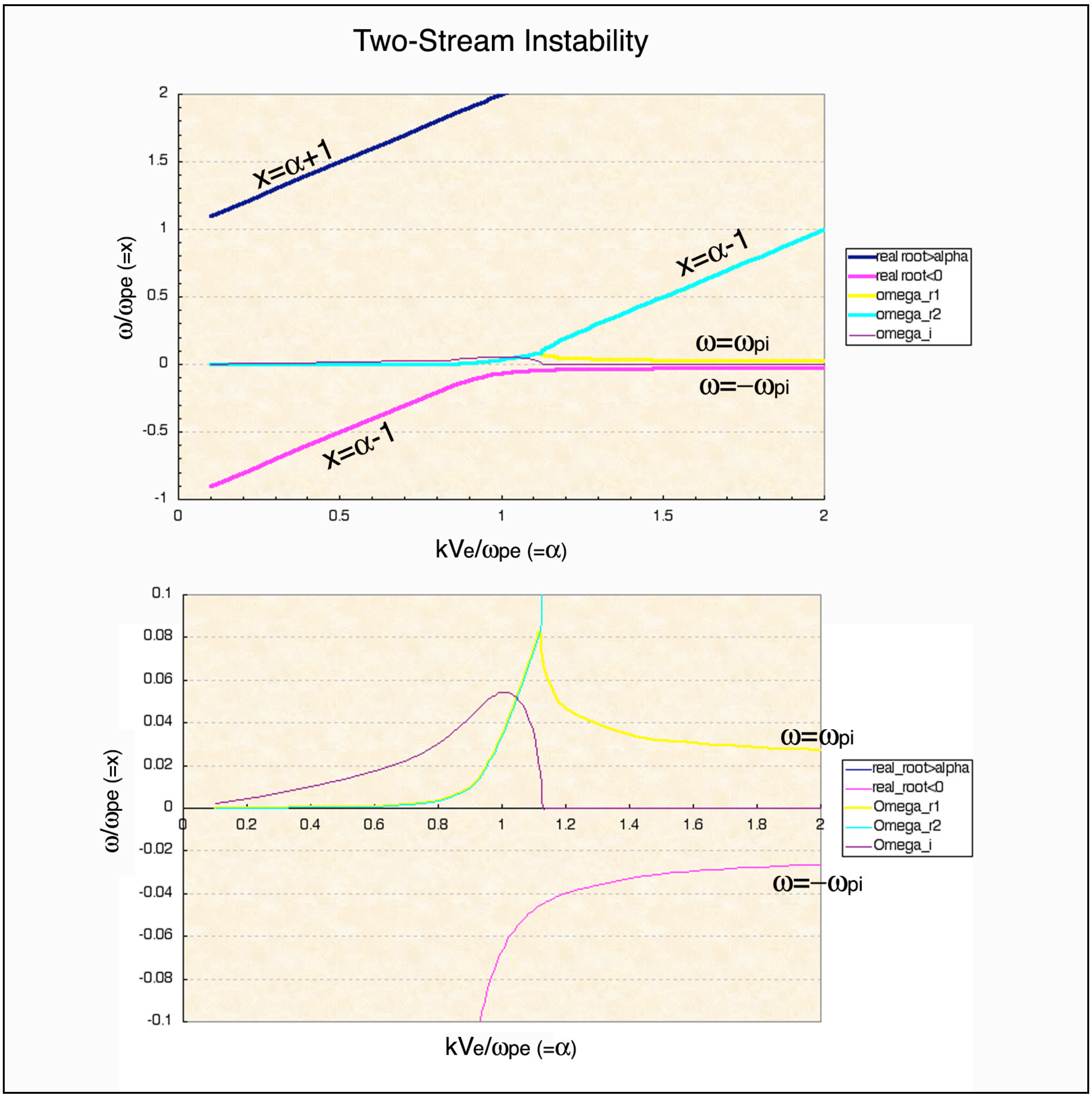
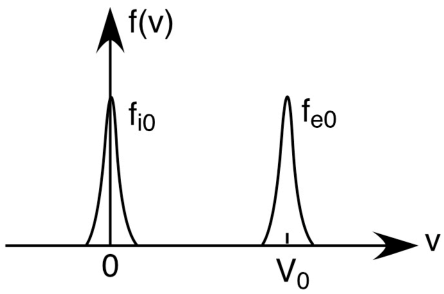


Figure 4.2 Solutions of Eq. (4.13) plotted with x as a function of α, (or ω as a function of wave number k). Solutions include one real root $\omega > \alpha$, one real root $\omega < 0$. The other two roots are two real roots $\omega = \omega_{r1}$ and ω_{r2} or two complex conjugates roots $\omega = \omega_r \pm i\omega_i$. The most unstable wave mode occurs near $\alpha \approx 1$ or $kV_0 \approx \omega_{pe}$ (lower panel). Four real roots at short wavelength limit ($kV_0/\omega_{pe} >> 1$) approach to $x = \omega_{pi}/\omega_{pe}, -\omega_{pi}/\omega_{pe}, \alpha + 1,$ and $\alpha - 1$. Two real roots in long wavelength limit ($kV_0/\omega_{pe} < 1$) approach $x = \alpha + 1,$ and $\alpha - 1$. Wave-mode coupling occurs at the intersection of $x = -\omega_{pi}/\omega_{pe}$, and $x = \alpha - 1$. Maximum growth rate occurs near intersection of $x = \omega_{pi}/\omega_{pe}$ and $x = \alpha - 1$.

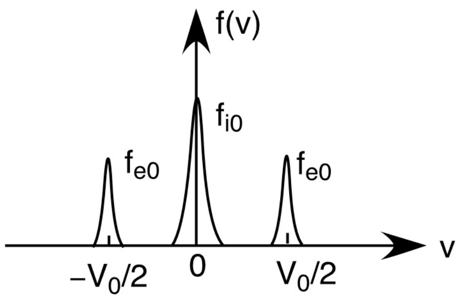
Exercise 4.1

Consider a field-free plasma system, which consists of a cold ion fluid at rest with density n_0, and a finite temperature electron fluid with velocity $\mathbf{V}_{e0} = V_0\hat{x}$, number density n_0, and temperature T_{e0}. Assume the thermal speed of the electron fluid is much smaller then the electron drift speed V_0. Determine linear dispersion relations of this system. Find solutions of ω as a function of wave number k and plot you results in $\omega - k$ space.



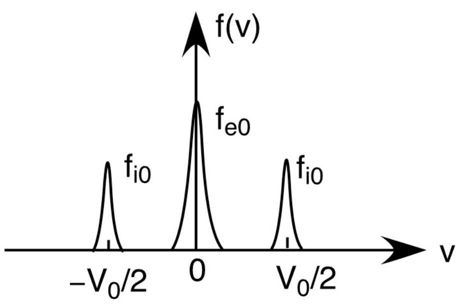
Exercise 4.2

Consider a field-free plasma system, which consists of a cold ion fluid at rest with density n_0, and two counter-streaming electron fluids. One of them is characterized by density $n_0/2$, and velocity $(V_0/2)\hat{x}$. The other one is characterized by density $n_0/2$, and velocity $-(V_0/2)\hat{x}$. Determine linear dispersion relations of this system. Find solutions of ω as a function of wave number k and plot you results in $\omega - k$ space.



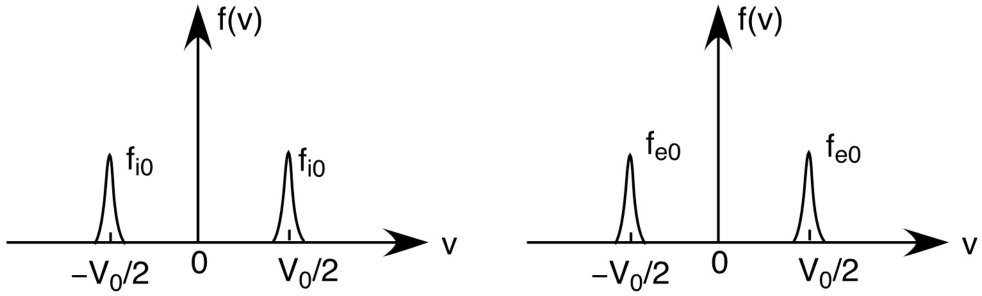
Exercise 4.3

Consider a field-free plasma system, which consists of a cold electron fluid at rest with density n_0, and two counter-streaming ion fluids. One of them is characterized by density $n_0/2$, and velocity $(V_0/2)\hat{x}$. The other one is characterized by density $n_0/2$, and velocity $-(V_0/2)\hat{x}$. Determine linear dispersion relations of this system. Find solutions of ω as a function of wave number k and plot you results in $\omega - k$ space.



Exercise 4.4

Consider a field-free plasma system, which consists of two counter-streaming ion fluids and two counter-streaming electron fluids. One of the electron fluids and one of the ion fluids are characterized by density $n_0/2$, and velocity $(V_0/2)\hat{x}$. The other electron fluid and ion fluid are characterized by density $n_0/2$, and velocity $\boxed{-(V_0/2)\hat{x}}$. Determine linear dispersion relations of this system and plot linear wave modes and growth rate in a $\omega - k$ diagram.



Exercise 4.5

(a) Find a positive feedback effect that can lead to nonlinear amplification of two-stream instability.

(b) Find an electrostatic kinetic wave-particle interaction process that can lead to nonlinear saturation of two-stream instability.

(c) Discuss long-term evolution of two-stream instability obtained in Exercises 4.1–4.4, if there is a uniform but weak background magnetic field along the x-direction.

(d) Perform one-dimensional full particle code simulations to verify your results.