# Chapter 3. Pseudopotential Methods

Nonlinear waves are solutions of a set of nonlinear time-dependent and spatial-dependent partial differential equations. If we consider solutions of nonlinear one-dimensional plane wave ($\nabla = \hat{x}\partial / \partial x$), which propagates at a constant speed, then we can choose a moving frame such that the wave structure becomes stationary (time-independent, i.e., $\partial / \partial t = 0$). Thus, the set of partial differential equations (PDEs) are reduced into a set of ordinary differential equations (ODEs). Since numerical methods for solving ordinary differential equations are well established, one can always obtain nonlinear-wave solutions by solving these ODEs numerically with different boundary conditions. To understand systematic changes on wave characteristics, one may need to examine numerical solutions for many different parameters and initial conditions. Thus, it is very difficult to obtain complete dependence from numerical solutions. However, if the nonlinear wave equations can be reduced into equations similar to equations of motion, one can then use the so-called pseudopotential method to explore solution space analytically. Both electrostatic waves and electromagnetic waves in collisionless plasma can be studied by means of pseudopotential method. A brief summary of pseudopotential methods of these waves is given below.

*Case 1:*

If a function $\Phi(x)$ satisfies

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{d\Psi(\Phi)}{d\Phi},$$

then the solution characteristics of $\Phi(x)$ should be similar to a particle's trajectory $x(t)$, which satisfies $\ddot{x}(t) = -\nabla\Psi(x)$. In this case, we can consider solution of $\Phi(x)$ as a pseudo particle's trajectory moving in the pseudo potential $\Psi(\Phi)$ at a given pseudo time $x$. We can easily sketch profile of $\Phi(x)$, based on the structure of pseudo potential $\Psi(\Phi)$. Examples of nonlinear equations of this type include Korteweg de Vries (KdV) equation and nonlinear equation of ion acoustic waves in two-fluid plasma.

*Case 2:*

Let us consider two functions $B_y(x)$ and $B_z(x)$, which satisfy the following equations

$$c_1 V_x(x)\frac{d}{dx}[V_x(x)\frac{dB_y(x)}{dx}] = -\frac{\partial\Psi(B_y, B_z)}{\partial B_y} - c_2[V_x(x)\frac{dB_z(x)}{dx}]B_{x0}$$

$$c_1 V_x(x)\frac{d}{dx}[V_x(x)\frac{dB_z(x)}{dx}] = -\frac{\partial \Psi(B_y,B_z)}{\partial B_z} + c_2[V_x(x)\frac{dB_y(x)}{dx}]B_{x0}$$

If we define $\tau(x)$, such that $d/d\tau(x) = V_x(x)(d/dx)$ denotes convective time-derivative, we can rewrite the above equations to the following form

$$c_1\frac{d^2B_y(x)}{d\tau(x)^2} = -\frac{\partial \Psi(B_y,B_z)}{\partial B_y} - c_2\frac{dB_z(x)}{d\tau(x)}B_{x0}$$

$$c_1\frac{d^2B_z(x)}{d\tau(x)^2} = -\frac{\partial \Psi(B_y,B_z)}{\partial B_z} + c_2\frac{dB_y(x)}{d\tau(x)}B_{x0}$$

The solution characteristics of $\mathbf{B}_t(x) = \hat{y}B_y(x) + \hat{z}B_z(x)$ should be similar to a particle's trajectory $\mathbf{r}(t) = \hat{y}y(t) + \hat{z}z(t)$, which satisfies the following equations of motion $c_1\ddot{\mathbf{r}}(t) = -\nabla\Psi - c_2\dot{\mathbf{r}}(t)\times\hat{x}B_{x0}$. In this case, we can consider solution of $\mathbf{B}_t$ as a trajectory of a pseudo particle at a given pseudo time $\tau(x)$. The motion of pseudo particle is under the influence of a pseudopotential-gradient force $-\nabla_{\mathbf{B}_t}\Psi(B_y,B_z)$ and a pseudo-velocity-dependent force $-c_2(d\mathbf{B}_t/d\tau)\times\hat{x}B_{x0}$. If $\sqrt{c_1} << c_2B_{x0}$ the motion of pseudo particle can be decomposed into a high-pseudo-frequency ($\tau/\sqrt{c_1} \approx 1 >> \tau/c_2B_{x0}$) gyro motion and a low-pseudo-frequency ( $\tau/c_2B_{x0} \approx 1$ ) drift motion, where the low-pseudo-frequency drift motion is characterized by an average drift trajectory $<\mathbf{B}_t>$ follows closely (not exactly) along a $\Psi = const.$ contour.

**Exercise 3.1.** Read the following articles:

[1] Section 7.15 in Nicholson (1983) [Nicholson, D. R., *Introduction to Plasma Theory,* pp. 171-177, John Wiley & Sons, New York, 1983.]

[2] Section 8.3 in Chen (1984) [Chen, F. F., *Introduction to Plasma Physics and Controlled Fusion, Volume 1: Plasma Physics,* pp. 297-305, Plenum Press, New York, 1984.]

[3] Chapter 6 in Tidman and Krall (1971) [Tidman, D. A., and N. A. Krall, *Shock Waves in Collissionless Plasma,* pp. 99-112, Wiley, New York, 1971.].

[4] Montgomery, D., Nonlinear Alfven waves in a cold ionized gas, *Phys. Fluids, 2,* 585, 1959.

[5] Lyu, L. H., and J. R. Kan, Nonlinear two-fluid hydromagnetic waves in the solar wind: Rotational discontinuity, soliton, and finite-extent Alfven wave train solutions, *J. Geophys. Res., 94,* 6523, 1989.

## 3.1. Solutions of Korteweg-de Vries Equation

Nonlinear wave solutions in a dispersive but non-dissipative medium consist of solitons and wavetrains. A soliton is a solitary wave, which can travel at a constant speed and maintain a constant waveform. Solitons are formed when nonlinear steepening is balanced by dispersion. Soliton solution is first demonstrated by the well-known Korteweg-de Vries (KdV) equation:

$$\frac{\partial V}{\partial t} + (C_0 + V)\frac{\partial V}{\partial x} + \alpha \frac{\partial^3 V}{\partial x^3} = 0 \tag{3.1}$$

To demonstrate the dispersion effect, let us consider linear wave dispersion of the KdV equation in a uniform background and we choose a moving frame such that $V_0 = 0$. Linearizing Eq. (3.1) yields

$$\frac{\partial V_1}{\partial t} + C_0 \frac{\partial V_1}{\partial x} + \alpha \frac{\partial^3 V_1}{\partial x^3} = 0 \tag{3.1a}$$

For plane wave solution, we can assume

$$V_1(x,t) = \tilde{V}_1(k,\omega)\exp[i(kx - \omega t)] \tag{3.1b}$$

Substituting Eq. (3.1b) into Eq. (3.1a) yields

$$\frac{\omega}{k} = C_0 - \alpha k^2 \tag{3.1c}$$

where $\omega$ is wave angular frequency, $k$ is wave number, and $C_0$ is phase speed of the linear wave at long wavelength limit. Dispersion characteristics of the KdV equation depend on the sign of $\alpha$. For $\alpha > 0$, phase speed $\omega / k$ decreases with increasing $k$. For $\alpha < 0$, phase speed increases with increasing $k$.

### 3.1.1. Method of Characteristic Curves

To demonstrate the nonlinear steepening effect, let us consider the simplest case, in which the dispersion term $\alpha(\partial^3 V / \partial x^3)$ in the KdV equation (3.1) vanishes. Namely, let us consider the following equation

$$\frac{\partial V}{\partial t} + (C_0 + V)\frac{\partial V}{\partial x} = 0 \tag{3.2}$$

Eq. (3.2) can be solved based on the concept of characteristic curves.

If we can find functions $\xi = \xi(x,t)$, $\eta = \eta(x,t)$ and inverse functions $x = x(\xi,\eta)$, $t = t(\xi,\eta)$, such that $V(x,t) = \overline{V}(\xi)$. Namely, $\xi(x,t)$ is the characteristic curve of Eq. (3.2), and $\partial V / \partial\eta = 0$. Thus, we have

$$\left.\frac{\partial V[x(\xi,\eta),t(\xi,\eta)]}{\partial\eta}\right|_{\xi} = 0 = \left.\frac{\partial V}{\partial x}\frac{\partial x}{\partial\eta}\right|_{\xi} + \left.\frac{\partial V}{\partial t}\frac{\partial t}{\partial\eta}\right|_{\xi} \tag{3.2a}$$

Comparing Eq. (3.2) and Eq. (3.2a), it yields

$$\left.\frac{\partial t}{\partial\eta}\right|_{\xi} = 1 \tag{3.2b}$$

$$\left.\frac{\partial x}{\partial\eta}\right|_{\xi} = (C_0 + V) \tag{3.2c}$$

Solving Eqs. (3.2b) and (3.2c) yields

$$t(\xi = \text{constant},\ \eta) = \eta \tag{3.2d}$$

$$\begin{aligned} x(\xi = \text{constant},\ \eta) &= (C_0 + V)\eta + \text{constant} \\ &= (C_0 + V)[t(\xi = \text{constant},\ \eta)] + \text{constant} \end{aligned} \tag{3.2e}$$

For convenience, we can choose the last term in Eq. (3.2e) to be $\xi$. Thus, the characteristic curve $\xi(x,t)$ becomes

$$\xi(x,t) = x - (C_0 + V)t \tag{3.3}$$

Solution of $V$ is constant along each characteristic curve. Namely, $V$ is a function of $\xi$. Since $\xi = x$ at $t = 0$, the slope of $\xi = \text{constant}$ contour in the $t - x$ plane depends on the amplitude of $V(x,t=0)$. Figure 3.1 shows characteristic curves of Eq. (3.2) and time evolution of the nonlinear wave $V(x,t)$. According to these characteristic curves, we can obtain profile of $V(x,t)$ at $t > 0$ from a given initial profile $V(x,t=0)$. Nonlinear steepening process can be seen at the leading edge of $V(x,t)$ profiles. It can be seen from Figure 3.1 that characteristic curves started from large-$V$ region will overtake those started from small-$V$ region. Note that when two characteristic curves intersect with each other, we can no longer define functions $\xi = \xi(x,t)$, $\eta = \eta(x,t)$ and inverse functions $x = x(\xi,\eta)$, $t = t(\xi,\eta)$ at these intersection points. Thus, the derivations given in Eqs. (3.2a)~(3.2e) are no longer legitimate.

**Exercise 3.2.**

Varify solution given in Eq. (3.3) by substituting Eq. (3.3) into Eq. (3.2).

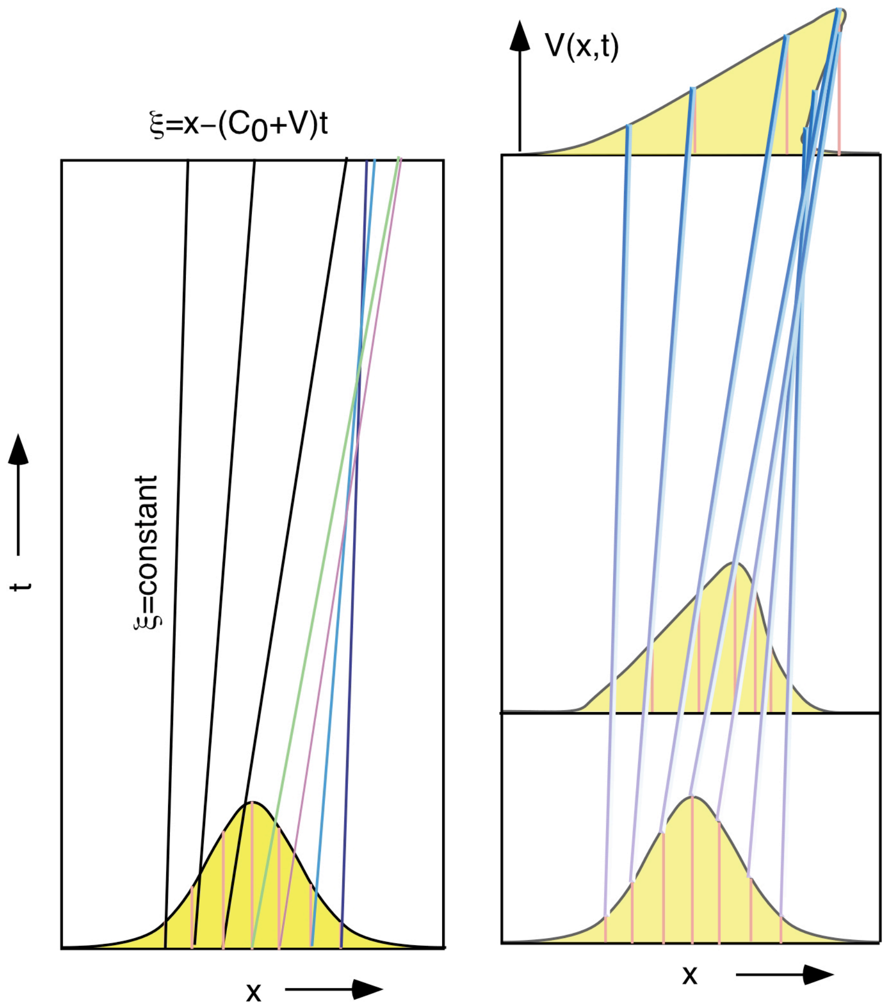


**Figure 3.1** Characteristic curves of Eq. (3.2) and time evolution of the nonlinear wave $V(x,t)$.

Now, let us consider solitary wave solution of Eq. (3.1), in which the nonlinear steepening effect balances the dispersion effect.

Let us consider a solitary wave that propagates at a constant speed $V_0$ in the lab frame. The nonlinear wave solution can be written as $V(x,t) = V(x - V_0 t)$. Let $X(x,t) = x - V_0 t$. Eq. (3.1) becomes

$$(-V_0 + C_0 + V)\frac{dV}{dX} + \alpha \frac{d^3V}{dX^3} = 0 \tag{3.4}$$

Eq. (3.4) can be integrated once, which yields

$$\frac{d^2V}{dX^2} = (V_0 - C_0)\frac{V}{\alpha} - \frac{V^2}{2\alpha} \tag{3.5}$$

where we have taken the integration constant to be zero (i.e., we assume that $V = 0$ at $X = 0$). Analytic solution of Eq. (3.5) can be obtained by consider solution of the following differential equation

$$f'' = af + bf^2 \tag{3.6}$$

It can be shown that solution of Eq. (3.6) can be written as

If $b = 0$ and $a < 0$, then $f(x) = C_1 \sin(\sqrt{a}x) + C_2 \cos(\sqrt{a}x)$ (3.7a)

If $b = 0$ and $a > 0$, then $f(x) = C_1 \exp(\sqrt{a}x) + C_2 \exp(-\sqrt{a}x)$ (3.7b)

If $b \neq 0$ and $a < 0$, then $f(x) = -\dfrac{3a}{2b}\sec^2(\sqrt{\dfrac{-a}{4}}x)$ (3.7c)

If $b \neq 0$, and $a > 0$, then $f(x) = -\dfrac{3a}{2b}\mathrm{sech}^2(\sqrt{\dfrac{a}{4}}x)$ (3.7d)

*Proof of Eqs. (3.7a) and (3.7b):*

If $f(x) = \sin(kx)$, then $f'' = -k^2 f$

If $f(x) = \cos(kx)$, then $f'' = -k^2 f$

If $f(x) = \exp(kx)$, then $f'' = k^2 f$

If $f(x) = \exp(-kx)$, then $f'' = k^2 f$

Thus, for $b = 0$, solution of Eq. (3.6) can be written as

If $a < 0$, then $f(x) = C_1 \sin(\sqrt{a}x) + C_2 \cos(\sqrt{a}x)$

If $a > 0$, then $f(x) = C_1 \exp(\sqrt{a}x) + C_2 \exp(-\sqrt{a}x)$

*Proof of Eq. (3.7c):*

Now let us consider $f(x) = \tan(kx) = \sin(kx)/\cos(kx)$, then

$$f' = k[1 + \tan^2(kx)] = k(1 + f^2) = k\sec^2(kx)$$

$$f'' = k(2f\,f')$$

$$f''' = 2k(f')^2 + 2k\,f\,f'' = 2k(f')^2 + 2k\,f\,(2k\,f\,f') = 2k(f')^2 + 4k^2 f^2 f' \tag{3.8}$$

Let $g(x) = \sec^2(kx)$. Eq. (3.8) can be rewritten as

$$f''' = (f')'' = k[\sec^2(kx)]'' = k\,g'' = 2k(f')^2 + 4k^2 f^2 f' = 2k(kg)^2 + 4k^2(g-1)(kg)$$

or

$$g'' = 6k^2 g^2 - 4k^2 g \tag{3.8a}$$

Let $G = Ag = A\sec^2(kx)$, where $A$ is a constant. Eq. (3.8a) can be rewritten as

$$G'' = -4k^2 G + \frac{6k^2}{A} G^2 \tag{3.8b}$$

*Proof of Eq. (3.7d):*

Likewise, let us consider $f(x) = \tanh(kx) = (e^{kx} - e^{-kx})/(e^{kx} + e^{-kx})$, then

$$f' = k[1 - \tanh^2(kx)] = k(1 - f^2) = k\,\mathrm{sech}^2(kx)$$

$$f'' = k(-2f\,f')$$

$$f''' = -2k(f')^2 - 2k\,f\,f'' = -2k(f')^2 - 2kf(-2kf\,f') = -2k(f')^2 + 4k^2 f^2 f' \tag{3.9}$$

Let $g(x) = \mathrm{sech}^2(kx)$

Eq. (3.9) can be rewritten as

$$f''' = (f')'' = k[\mathrm{sech}^2(kx)]'' = k\,g'' = -2k(f')^2 + 4k^2 f^2 f' = -2k(kg)^2 + 4k^2(1-g)(kg)$$

or

$$g'' = -6k^2 g^2 + 4k^2 g \tag{3.9a}$$

Let $G = Ag = A\,\mathrm{sech}^2(kx)$, where $A$ is a constant. Eq. (3.9a) can be rewritten as

$$G'' = +4k^2 G - \frac{6k^2}{A} G^2 \tag{3.9b}$$

Applying Eq. (3.6) and it solutions shown in Eqs. (3.7a)~(3.7d) to Eq. (3.5), a finite amplitude solitary wave solution can be obtained when $(V_0 - C_0)/\alpha > 0$. Namely, for $(V_0 - C_0)/\alpha > 0$,

$$V(X) = 3(V_0 - C_0)\,\mathrm{sech}^2[\sqrt{\frac{(V_0 - C_0)}{4\alpha}} X] \tag{3.10}$$

is a solution of Eq. (3.5). Substituting $X(x,t) = x - V_0 t$ into Eq. (3.10) yields

$$V(x,t) = 3(V_0 - C_0)\,\mathrm{sech}^2[\sqrt{\frac{(V_0 - C_0)}{4\alpha}}(x - V_0 t)] \tag{3.10a}$$

is a solution of Eq. (3.1). Let Mach number $M = V_0/C_0$. Eq. (3.10a) can be written as

$$V(x,t) = 3C_0(M-1)\,\mathrm{sech}^2[\sqrt{\frac{C_0(M-1)}{4\alpha}}(x - V_0 t)] \tag{3.10b}$$

Eq. (3.10b) implies that (1) for $\alpha > 0$, solitary wave solutions can exist only if Mach number $M > 1$, and (2) for $\alpha < 0$ soliton solutions can exist only if $M < 1$.

**Exercise 3.3.**

Varify solution (3.10a) by substituting Eq. (3.10a) into KdV equation (3.1).

[See, Section 7.15 in Nicholson (1983)]

### 3.1.2. Classical Pseudopotential Method

Nonlinear solutions of the KdV equations can also be studied systematically by means of pseudopotential method.

According to the classical pseudopotential method, Eq. (3.5) can be rewritten as

$$\frac{d^2V}{dX^2} = -\frac{d\Psi(V)}{dV} \tag{3.11}$$

where

$$\Psi(V) = \frac{V^3}{6\alpha} - (V_0 - C_0)\frac{V^2}{2\alpha} + \Psi_0 \tag{3.12}$$

is the pseudopotential of the KdV equation. Namely, Eq. (3.11) can be viewed as an "equation of motion" of a "fictitious particle" under the influence of a one-dimensional pseudopotential field $\Psi(V)$ with pseudo-time $X$, pseudo-coordinates $V$, pseudo-velocity $dV/dX$, and pseudo-acceleration $d^2V/dX^2$.

Figure 3.2 sketches dispersion curves and pseudopotential structures of KdV equation, where column (1) is for $\alpha > 0$, and column (2) is for $\alpha < 0$. According to the classical pseudopotential method, a soliton solution can exist if the following two conditions are fulfilled. (i) Pseudopotential $\Psi$ is a local maximum at $X = 0$ and with at least one local minimum next to it. (ii) The curve of the pseudopotential on the other side of the local minimum must raise up and become higher than the pseudopotential at $X = 0$. Pseudopotential structures shown in panels (1a) and (2b) satisfy the above conditions. In these cases, a fictitious particle can leave point $A$ (where $V = 0$) by an infinitesimal displacement toward point $B$ and then falling back through the well to return to point $A$.

Figure 3.3 sketches spatial profiles of these soliton solutions. Spatial profiles of $V(X)$ shown in Figure 3.3(a) and Figure 3.3(b) are the pseudo-time profiles of fictitious particle trajectory moving in pseudopotentials shown in panels (1a) and (2b) of Figure 3.2, respectively. Conditions for soliton to exist shown in Figures 3.2 and 3.3 are consistent with the conditions obtained in Eq. (3.10b).

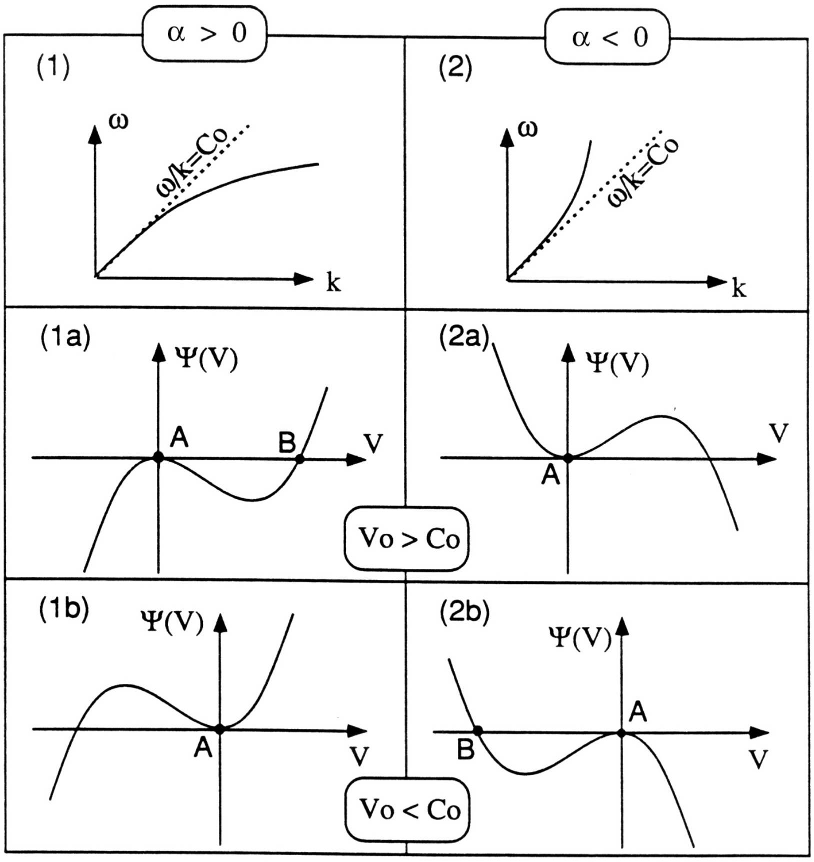


**Figure 3.2** Sketches dispersion curves and pseudopotential structures of Korteweg de Vries equation (3.1), where column (1) is for $\alpha > 0$, and column (2) is for $\alpha < 0$.

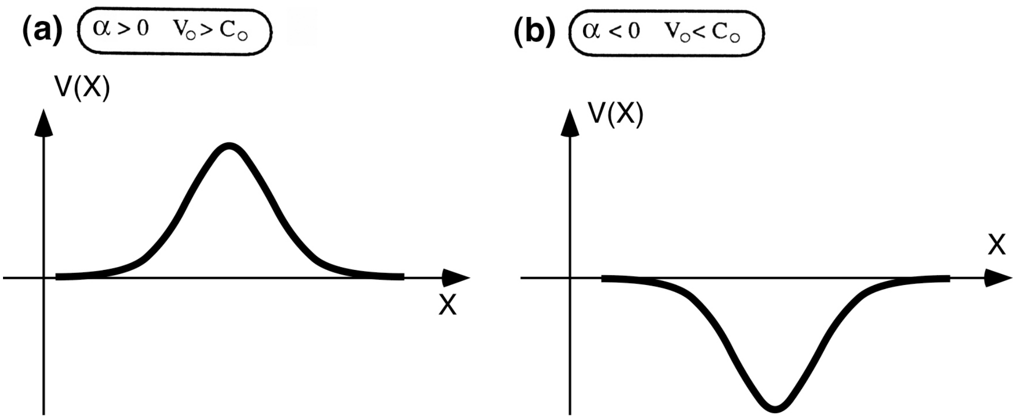


**Figure 3.3** Sketches of spatial profiles of soliton solutions. Spatial profiles of $V(X)$ shown in (a) and (b) are the pseudo-time profiles of fictitious particle trajectory moving in pseudopotentials shown in panels (1a) and (2b) of Figure 3.2, respectively.

### 3.1.3. Multiple-Pseudopotential Method

Figure 3.4 summaries numerical simulation results of the Korteweg de Vries equation (3.1) with a given initial profile. To explain these simulation results, we shall introduce a new method, which will be called multiple-pseudopotential method.

The concept of multiple-pseudopotential method is first introduced by Moiseev and Sagdeev (1963), when they try to extent the pseudopotential method to an ion acoustic solitary wave at Mach number $M > 1.6$. Pseudopotential method discussed above is not limited to the Korteweg de Vries equation. Electrostatic ion acoustic soliton solution can also be solved by pseudopotential method. The pseudopotential of ion acoustic wave with cold ions and isothermal electrons is called Sagdeev potential [see Section 8.3 in Chen (1984) and Chapter 6 in Tidman and Krall (1971)]. The model of Sagdeev potential is limited by Mach number $M < M_{\max} \approx 1.6$. At $M \geq M_{\max}$, all the upstream cold ions will be reflected by electrostatic potential of the soliton. Namely, the cold ion assumption is no longer valid. One must take into account the finite thermal spread in the ions' distribution function when the average velocity of these low temperature ions, observed in the soliton rest frame, is reduced to a magnitude closed to their thermal speed. To overcome this difficulty, Moiseev and Sagdeev (1963) introduced the concept of multiple-pseudopotential method [see Chapter 6 in Tidman and Krall (1971), for detail discussion]. As a result, an oscillatory ES shock can be obtained from this multiple-pseudopotential method. In brief, the concept of multiple-pseudopotential method is that the pseudopotential is allowed to be a pseudo-time dependent structure. Namely, at different spatial domain, we can use different pseudopotential to describe the corresponding nonlinear wave structure.

Figure 3.5 shows how to use multiple-pseudopotential method to explain simulation results discussed in Figure 3.4. A fictitious particle moving successively from $\Psi_1$, $\Psi_2$, $\Psi_3$, …, to $\Psi_n$ in Figure 3.5 can form to a series of solitons or wavetrains as those shown in Figure 3.4. Note that $V_0$ decreases successively from $\Psi_1$, $\Psi_2$, $\Psi_3$, …, to $\Psi_n$ in each plot shown of Figure 3.5.

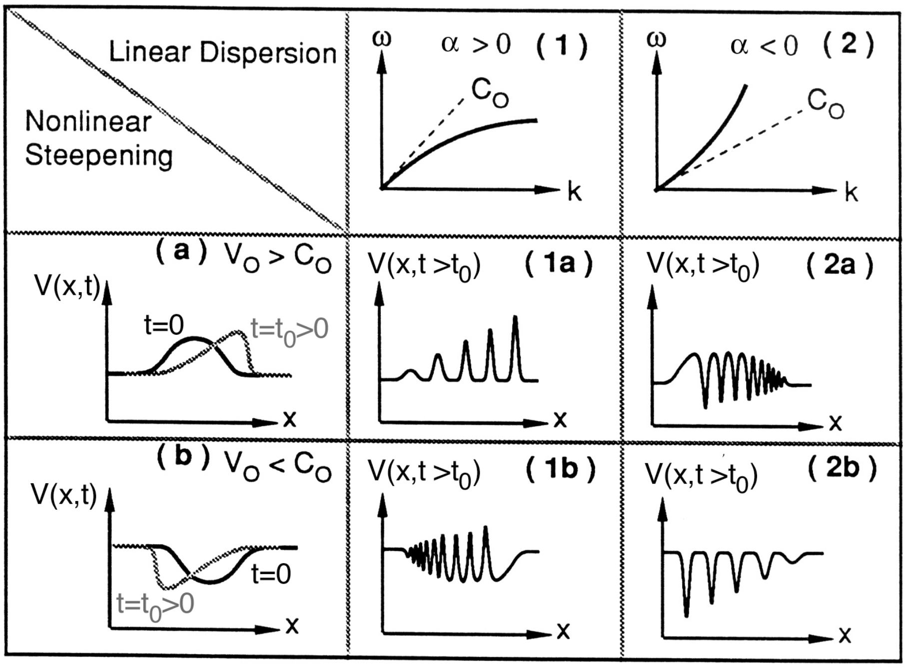


**Figure 3.4** Summary of numerical simulation results of the Korteweg de Vries equation (3.1) with a given initial profile.

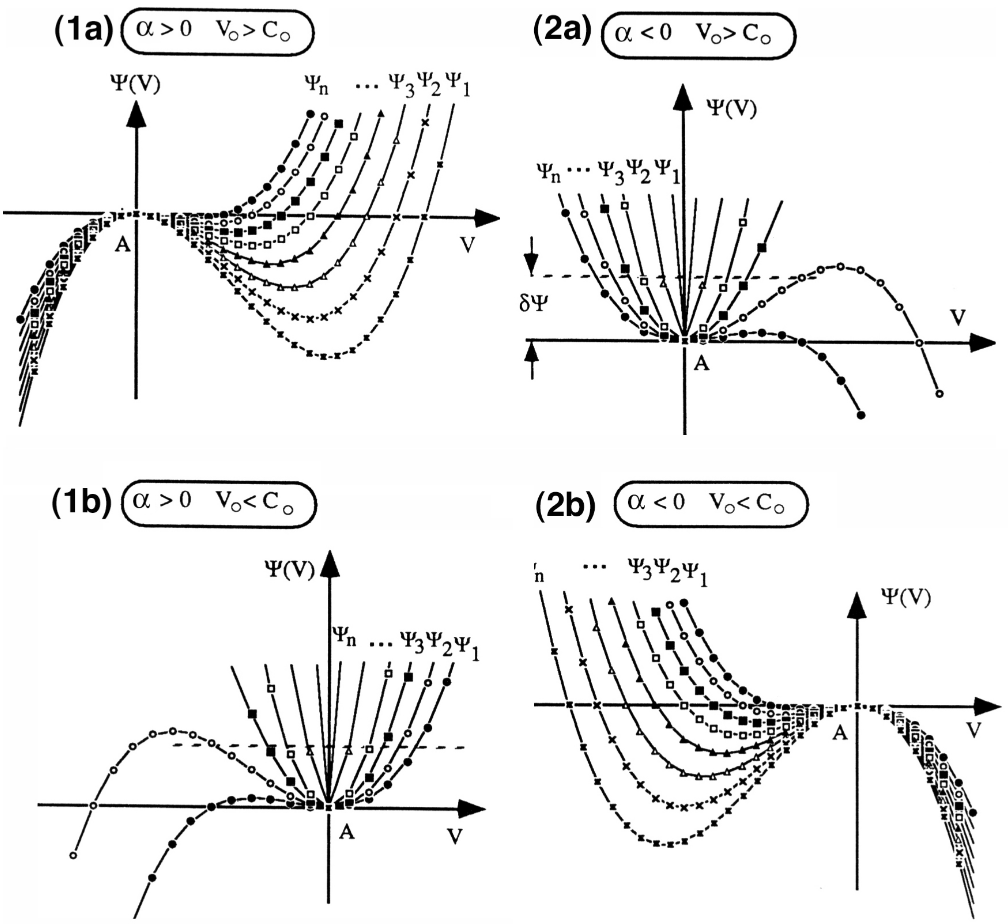


**Figure 3.5** Illustrations of multiple-pseudopotential method for the corresponding simulation results shown in Figure 3.4. See text for detail discussion.

## 3.2. One-Dimensional Electrostatic Ion Acoustic Solitary Waves

Basic equations for electrostatic two-fluid plasma

| | | |
|---|---|---|
| (3.2.1) $\frac{d}{dx}(n_i V_{ix}) = 0$ | $n_i V_{ix} = n_0 V_0$ | $\frac{dV_{ix}}{dx} = -\frac{n_0 V_0}{n_i^2}\frac{dn_i}{dx}$ |
| (3.2.2) $\frac{d}{dx}(n_e V_{ex}) = 0$ | $n_e V_{ex} = n_0 V_0$ | $\frac{dV_{ex}}{dx} = -\frac{n_0 V_0}{n_e^2}\frac{dn_e}{dx}$ |
| (3.2.3) $\frac{d}{dx}(p_i n_i^{-\gamma_i}) = 0$ | $p_i = p_{i0}(\frac{n_i}{n_0})^{\gamma_i}$ | $\frac{dp_i}{dx} = \frac{\gamma_i p_{i0}}{n_0}(\frac{n_i}{n_0})^{\gamma_i - 1}\frac{dn_i}{dx}$ |
| (3.2.4) $\frac{d}{dx}(p_e n_e^{-\gamma_e}) = 0$ | $p_e = p_{e0}(\frac{n_e}{n_0})^{\gamma_e}$ | $\frac{dp_e}{dx} = \frac{\gamma_e p_{e0}}{n_0}(\frac{n_e}{n_0})^{\gamma_e - 1}\frac{dn_e}{dx}$ |
| (3.2.5) $\frac{d}{dx}(m_i n_i V_{ix} V_{ix} + p_i) = e n_i E_x$ | $[-\frac{m_i}{e}\frac{n_0^2 V_0^2}{n_i^3} + \frac{\gamma_i p_{i0}}{e n_0^2}(\frac{n_i}{n_0})^{\gamma_i - 2}]\frac{dn_i}{dx} = E_x$ | |
| (3.2.6) $\frac{d}{dx}(m_i n_e V_{ex} V_{ex} + p_e) = -e n_e E_x$ | $[\frac{m_e}{e}\frac{n_0^2 V_0^2}{n_e^3} - \frac{\gamma_e p_{e0}}{e n_0^2}(\frac{n_e}{n_0})^{\gamma_e - 2}]\frac{dn_e}{dx} = E_x$ | |
| (3.2.7) $\frac{d}{dx}(m_i n_i V_{ix} V_{ix} + m_e n_e V_{ex} V_{ex} + p_i + p_e - \frac{\varepsilon_0 E_x^2}{2}) = 0$ | $m_i \frac{n_0^2 V_0^2}{n_i} + m_e \frac{n_0^2 V_0^2}{n_e} + p_{i0}(\frac{n_i}{n_0})^{\gamma_i} + p_{e0}(\frac{n_e}{n_0})^{\gamma_e} - \frac{\varepsilon_0 E_x^2}{2} = (m_i + m_e) n_0 V_0^2 + p_{i0} + p_{e0}$ | |
| (3.2.8) $\frac{dE_x}{dx} = \frac{e(n_i - n_e)}{\varepsilon_0}$ | $\frac{d^2 E_x}{dx^2} = \frac{e}{\varepsilon_0}(\frac{dn_i}{dx} - \frac{dn_e}{dx})$ | |
| (3.2.9) $E_x = -\frac{d\phi}{dx}$ | $\frac{d^2\phi}{dx^2} = -\frac{e(n_i - n_e)}{\varepsilon_0}$ | |

### 3.2.1. Sagdeev Potential for Electrostatic Ion Acoustic Solitons

If $m_i n_i V_{ix} V_{ix} >> p_i$ and $m_e n_e V_{ex} V_{ex} << p_e$, we can ignore the ion thermal pressure term and the electron inertial term in the ions' and electrons' momentum equations, respectively. Substituting equation (3.2.9) into ions' momentum equation (3.2.5), it yields

$$\frac{d}{dx}(\frac{1}{2} m_i V_{ix}^2 + e\phi) = 0 \quad (3.2.10)$$

Integrating equation (3.2.10) once, and substituting equation (3.2.1) to eliminate $V_{ix}$, it yields

$$\frac{1}{2} m_i V_0^2 \frac{n_0^2}{n_i^2} + e\phi = \frac{1}{2} m_i V_0^2$$

or

$$n_i = n_0 (1 - \frac{2e\phi}{m_i V_0^2})^{-1} \tag{3.2.11}$$

Substituting equation (3.2.9) into electrons' momentum equation (3.2.6), it yields

$$\frac{d}{dx}(p_e) = \frac{d}{dx}(n_e k_B T_e) = en_e \frac{d\phi}{dx} \tag{3.2.12}$$

For isothermal electrons, the equation (3.2.12) yields

$$n_e = n_0 \exp(e\phi / k_B T_e) \tag{3.2.13}$$

Substituting the equations (3.2.9), (3.2.11) and (3.2.13) into the Poisson's equation (3.2.8), it yields

$$\frac{d^2\phi}{dx^2} = -\frac{e(n_i - n_e)}{\varepsilon_0} = -\frac{en_0}{\varepsilon_0}[\frac{1}{1 - \frac{2e\phi}{m_i V_0^2}} - \exp(\frac{e\phi}{k_B T_e})] = -\frac{d\psi(\phi)}{d\phi} \tag{3.2.14}$$

where the pseudo potential $\psi(\phi)$ in the above equation is called the Sagdeev potential. For convenience, we define the sonic Mach number based on the ion acoustic wave speed $C_S = \sqrt{k_B T_e / m_i}$, i.e.,

$$M_S = V_0 / \sqrt{k_B T_e / m_i} \tag{3.2.15}$$

The pseudo particle equation of motion (3.2.14) can be rewritten as

$$\frac{d^2\phi}{dx^2} = -\frac{d\psi(\phi)}{d\phi} = -\frac{en_0}{\varepsilon_0}[\frac{1}{1 - \frac{2e\phi}{k_B T_e}\frac{1}{M_S^2}} - \exp(\frac{e\phi}{k_B T_e})] \tag{3.2.16}$$

Integrating (3.2.16) once yields

$$\psi(\phi) = \frac{k_B T_e n_0}{\varepsilon_0}[-\frac{M_S^2}{2}\ln(1 - \frac{e\phi}{k_B T_e}\frac{2}{M_S^2}) - \exp(\frac{e\phi}{k_B T_e})] + \psi_0 \tag{3.2.17}$$

Let $\psi = 0$ at $\phi = 0$, it yields $\psi_0 = k_B T_e n_0 / \varepsilon_0$. Let $\phi_0 = k_B T_e / e$, $\phi^* = \phi / \phi_0$, and $\psi^* = \psi / \psi_0$. The normalized pseudopotential becomes

$$\psi^*(\phi^*) = 1 - \frac{M_S^2}{2}\ln(1 - \phi^* \frac{2}{M_S^2}) - \exp(\phi^*) \tag{3.2.17a}$$

Likewise, let $x_0 = (\varepsilon_0 k_B T_e / e^2 n_0)^{1/2} = C_S / \omega_{pi0}$ and $x^* = x / x_0$. The normalized pseudo equation of motion becomes

$$\frac{d^2\phi^*}{d(x^*)^2} = -\frac{d\psi^*(\phi^*)}{d\phi^*} = -[\frac{1}{1 - \phi^* \frac{2}{M_S^2}} - \exp(\phi^*)] \tag{3.2.16a}$$

Solution space of $\phi^*(x^*)$ can be classified based on the function behavior of the normalized pseudopotential function $\psi^*(\phi^*)$.

Solution of $\phi^*(x^*)$ can be obtained by solving the following system ODEs

$dy_2 / dx^* = f(y_1)$

$dy_1 / dx^* = y_2$

where $y_1 = \phi^*(x^*)$, $y_2 = d\phi^*(x^*)/dx^*$, and

$$f(y_1) = -[\frac{1}{1 - y_1 \frac{2}{M_S^2}} - \exp(y_1)]$$

**Exercise 3.4**

(a) Plot the Sagdeev potential at different Mach number $M_S$. (see References [2] and [3] in Exercise 3.1)

(b) Show that soliton solutions obtained from Sagdeev potential is limited by Mach number $M < M_{\max} \approx 1.6$.

(c) Show that for $m_i n_0 V_0^2 < 2e\phi$, the ions' thermal pressure becomes important and cannot be ignored.

(d) Plot characteristic curves of ions and electrons in phase space for a given static soliton potential (electrostatic potential). Show that there must be trapped electrons in the soliton transition region in order to satisfy Boltzmann relation, in which $n_e$ increases with increasing $\Phi$. Explain how these electrons being trapped initially.

### 3.2.2. Pseudo Potential for Electrostatic Ion Acoustic Solitary Waves

The Poisson's equation (3.2.8) yields

$$n_e = n_i - \frac{\varepsilon_0}{e}\frac{dE_x}{dx} \tag{3.2.19}$$

Substituting equation (3.2.19) into equation (3.2.7) yields

$$\boxed{\begin{aligned} & m_i n_0 V_0^2 \{\frac{n_0}{n_i}[1 + \frac{m_e}{m_i}(1 - \frac{\varepsilon_0}{e n_i}\frac{dE_x}{dx})^{-1}] - (1 + \frac{m_e}{m_i})\} \\ & + p_{i0}\{(\frac{n_i}{n_0})^{\gamma_i}[1 + \frac{p_{e0}}{p_{i0}}(1 - \frac{\varepsilon_0}{e n_i}\frac{dE_x}{dx})^{\gamma_e}] - (1 + \frac{p_{e0}}{p_{i0}})\} - \frac{\varepsilon_0 E_x^2}{2} = 0 \end{aligned}} \tag{3.2.20}$$

which yields

$$n_i = n_i(E_x, \frac{dE_x}{dx}) \qquad (3.2.20a)$$

Substituting the left column equations (3.2.5) and (3.2.6) into (3.2.8) yields

$$\frac{d^2E_x}{dx^2} = E_x \frac{e}{\varepsilon_0}\{[-\frac{m_i}{e}\frac{n_0^2V_0^2}{n_i^3} + \frac{\gamma_i p_{i0}}{en_0^2}(\frac{n_i}{n_0})^{\gamma_i - 2}]^{-1} - [\frac{m_e}{e}\frac{n_0^2V_0^2}{n_e^3} - \frac{\gamma_i p_{e0}}{en_0^2}(\frac{n_e}{n_0})^{\gamma_e - 2}]^{-1}\} \qquad (3.2.21)$$

Substituting equation (3.2.19) into (3.2.21) and then substituting (3.2.20a) into the resulting equation it yields

$$\frac{d^2E_x}{dx^2} = F_{pseudo}(E_x, \frac{dE_x}{dx}) \qquad (3.2.22)$$

Solution space of $E_x(x)$ can be classified based on the function behavior of the normalized pseudo force $F_{pseudo}(E_x, \frac{dE_x}{dx})$.

Let $C_{S0} = [(\gamma_e k_B T_{e0} + \gamma_i k_B T_{i0})/m_i]^{1/2}$ , $C_{i0} = \sqrt{k_B T_{i0}/m_i}$ , $C_{e0} = \sqrt{k_B T_{e0}/m_e}$ , and $\omega_{pi0} = [e^2 n_0/\varepsilon_0 m_i]^{1/2}$. Define $x_0 = C_{S0}/\omega_{pi0}$, $v_0 = C_{S0}$, $t_0 = 1/\omega_{pi0}$, and $m_0 = m_i$. It yields $E_0 = m_0 v_0^2/ex_0 = m_i C_{S0}\omega_{pi0}/e$. The normalized variables include the Mach number $M_{S0} = V_0/C_{S0}$, and $C_{i0}^* = C_{i0}/C_{S0}$, $C_{e0}^* = C_{e0}/C_{S0}$, $x^* = x/x_0$, $n_i^* = n_i/n_0$, $n_e^* = n_e/n_0$, $E_x^* = E_x/E_0$.

Solution $E_x^*(x^*)$ can also be solved numerically by solving the following system equations, which are first order ordinary differential equations of $E_x^*(x^*)$, $n_i^*(x^*)$, and $n_e^*(x^*)$.

$$\frac{dn_i^*}{dx^*} = \frac{E_x^* n_i^{*3}}{-M_{S0}^2 + \gamma_i C_{i0}^{*2}(n_i^*)^{\gamma_i + 1}}$$

$$\frac{dn_e^*}{dx^*} = \frac{E_x^* n_e^{*3}}{M_{S0}^2 - \gamma_e C_{e0}^{*2}(n_e^*)^{\gamma_e + 1}}$$

$$\frac{dE_x^*}{dx^*} = n_i^* - n_e^*$$

Or one can discuss the solution space characteristics of $E_x^*(x^*)$ based on the following normalized equation of motion is

$$\frac{d^2E_x^*}{d(x^*)^2} = E_x^*[\frac{n_i^{*3}}{-M_{S0}^2 + \gamma_i C_{i0}^{*2}(n_i^*)^{\gamma_i + 1}} - \frac{n_e^{*3}}{M_{S0}^2 - \gamma_e C_{e0}^{*2}(n_e^*)^{\gamma_e + 1}}]$$

where

$$n_e^* = n_i^* - \frac{dE_x^*}{dx^*}$$

and

$$M_{S0}^2(\frac{1}{n_i^*} - 1) + m_e^* M_{S0}^2(\frac{1}{n_e^*} - 1) + C_{i0}^{*2}(n_i^{*\gamma_i} - 1) + m_e^* C_{e0}^{*2}(n_e^{*\gamma_e} - 1) - \frac{E_x^{*2}}{2} = 0$$

## 3.3. Nonlinear Solutions of Low-Frequency Electromagnetic Waves in Two-Fluid Plasma

Let us consider static ($\partial/\partial t = 0$) one-dimensional ($\nabla = \hat{x} d/dt$) nonlinear waves in ion-electron two-fluid plasma, in which each species has isotropic pressure and follows adiabatic process. It can be shown that, if a species has isotropic pressure and follows adiabatic process, Eqs. ($1.8_\alpha$) can be rewritten as

$$\frac{3}{2p_\alpha}(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)p_\alpha - \frac{5}{2\rho_\alpha}(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\rho_\alpha = 0 \qquad (3.13_\alpha)$$

where $\rho_\alpha = n_\alpha m_\alpha$.

**Exercise 3.6**

Show that for isotropic pressure and adiabatic process, Eqs. ($1.6_\alpha$)~($1.8_\alpha$) lead to adiabatic equation of state

$$\frac{3}{2}(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\ln(p_\alpha \rho_\alpha^{-5/3}) = 0 \qquad (3.13_\alpha)$$

For static one-dimensional nonlinear wave, Eqs. ($1.6_\alpha$)~(1.15') can be rewritten as

$$\frac{d}{dx}(n_\alpha V_{\alpha x}) = 0 \qquad (3.14_\alpha)$$

$$V_{\alpha x}\frac{d}{dx}\mathbf{V}_\alpha = -\hat{x}\frac{1}{m_\alpha n_\alpha}\frac{d}{dx}p_\alpha + \frac{e_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}) \qquad (3.15_\alpha)$$

$$\frac{1}{p_\alpha}\frac{dp_\alpha}{dx} - \frac{5}{3n_\alpha}\frac{dn_\alpha}{dx} = 0 \qquad (3.16_\alpha)$$

$$\frac{dE_x}{dx} = \frac{e}{\varepsilon_0}(n_i - n_e) \qquad (3.17)$$

$$\frac{dB_x}{dx} = 0 \qquad (3.18)$$

$$\hat{x}\frac{d}{dx}\times\mathbf{E} = 0 \qquad (3.19)$$

$$\hat{x}\frac{d}{dx}\times\mathbf{B} = \mu_0 e(n_i\mathbf{V}_i - n_e\mathbf{V}_e) \qquad (3.20)$$

$$\frac{d}{dx}[m_i n_i V_{ix}\mathbf{V}_i + m_e n_e V_{ex}\mathbf{V}_e + \hat{x}(p_i + p_e + \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}) - \varepsilon_0 E_x \mathbf{E} - \frac{B_x\mathbf{B}}{\mu_0}] = 0 \qquad (3.21)$$

$$\frac{d}{dx}[(\frac{m_i n_i V_i^2}{2}+\frac{5p_i}{2})V_{ix}+(\frac{m_e n_e V_e^2}{2}+\frac{5p_e}{2})V_{ex}+\hat{x}\cdot\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}]=0 \tag{3.22}$$

If we assume uniform boundary condition at upstream side $(x\rightarrow-\infty)$, Eq. (3.17) yields

$$n_i(x\rightarrow-\infty)=n_e(x\rightarrow-\infty)=n_0 \tag{3.23}$$

$$E_x(x\rightarrow-\infty)=E_{x0}=0 \tag{3.24}$$

Substituting (3.23) into (3.20) yields

$$V_{iy}(x\rightarrow-\infty)=V_{ey}(x\rightarrow-\infty)=V_{y0} \tag{3.25}$$

$$V_{iz}(x\rightarrow-\infty)=V_{ez}(x\rightarrow-\infty)=V_{z0} \tag{3.26}$$

Therefore, one can choose a moving frame such that

$$V_{y0}=V_{z0}=0 \tag{3.27}$$

For simplicity, one can choose a coordinate system such that

$$\mathbf{B}(x\rightarrow-\infty)=\hat{x}B_{x0}+\hat{y}B_{y0} \tag{3.28}$$

From Eqs. (3.24)~(3.27), the $x$ component of Eq. $(3.15_\alpha)$ at upstream boundary is automatically fulfilled. Applying uniform boundary condition and Eq. (3.28) to the $y,z$ components of Eq. $(3.15_\alpha)$ yields

$$V_{ix}(x\rightarrow-\infty)=V_{ex}(x\rightarrow-\infty)=V_{x0} \tag{3.29}$$

$$E_y(x\rightarrow-\infty)=E_{y0}=V_{x0}B_z(x\rightarrow-\infty)=0 \tag{3.30}$$

$$E_z(x\rightarrow-\infty)=E_{z0}=-V_{x0}B_y(x\rightarrow-\infty)=-V_{x0}B_{y0} \tag{3.31}$$

Integrating Eqs. $(3.14_\alpha)$, $(3.16_\alpha)$, (3.18), (3.19), (3.21), (3.22) once and making use of Eqs. (3.23)~(3.31) yields

$$n_i V_{ix}=n_0 V_{x0} \tag{3.32}$$

$$n_e V_{ex}=n_0 V_{x0} \tag{3.33}$$

$$p_i n_i^{-5/3}=p_{i0}n_0^{-5/3} \tag{3.34}$$

$$p_e n_e^{-5/3}=p_{e0}n_0^{-5/3} \tag{3.35}$$

$$B_x=B_{x0} \tag{3.36}$$

$$E_y=E_{y0}=0 \tag{3.37}$$

$$E_z=E_{z0}=-V_{x0}B_{y0} \tag{3.38}$$

$$m_i n_i V_{ix}^2+m_e n_e V_{ex}^2+p_i+p_e-\frac{\varepsilon_0 E_x^2}{2}+\frac{B_y^2+B_z^2}{2\mu_0}=(m_i+m_e)n_0 V_{x0}^2+p_{i0}+p_{e0}+\frac{B_{y0}^2}{2\mu_0} \tag{3.39}$$

$$m_i n_i V_{ix}V_{iy}+m_e n_e V_{ex}V_{ey}-\frac{B_{x0}B_y}{\mu_0}=-\frac{B_{x0}B_{y0}}{\mu_0} \tag{3.40}$$

$$m_i n_i V_{ix} V_{iz} + m_e n_e V_{ex} V_{ez} + \varepsilon_0 E_x V_{x0} B_{y0} - \frac{B_{x0} B_z}{\mu_0} = 0 \tag{3.41}$$

$$\left(\frac{m_i n_i V_i^2}{2} + \frac{5p_i}{2}\right) V_{ix} + \left(\frac{m_e n_e V_e^2}{2} + \frac{5p_e}{2}\right) V_{ex} + \frac{V_{x0} B_{y0} B_y}{\mu_0}$$
$$= \left[\frac{m_i n_0 V_{i0}^2 + m_e n_0 V_{e0}^2}{2} + \frac{5(p_{i0} + p_{e0})}{2} + \frac{B_{y0}^2}{\mu_0}\right] V_{x0} \tag{3.42}$$

where Eqs. (3.36)~(3.38) have been used to obtain Eqs. (3.39)~(3.42). In addition to these conservation equations, a system of ODEs can be obtained from Eqs. ($3.15_\alpha$), (3.17), and (3.20). From Eq. ($3.15_\alpha$), we have

$$n_i V_{ix} \frac{d}{dx} V_{ix} = n_0 V_{x0} \frac{d}{dx} V_{ix} = -\frac{1}{m_i} \frac{d}{dx} p_i + \frac{e n_i}{m_i} (E_x + V_{iy} B_z - V_{iz} B_y) \tag{3.43}$$

$$n_e V_{ex} \frac{d}{dx} V_{ex} = n_0 V_{x0} \frac{d}{dx} V_{ex} = -\frac{1}{m_e} \frac{d}{dx} p_e - \frac{e n_e}{m_e} (E_x + V_{ey} B_z - V_{ez} B_y) \tag{3.44}$$

$$n_i V_{ix} \frac{d}{dx} V_{iy} = n_0 V_{x0} \frac{d}{dx} V_{iy} = +\frac{e n_i}{m_i} (0 + V_{iz} B_{x0} - V_{ix} B_z) \tag{3.45}$$

$$n_e V_{ex} \frac{d}{dx} V_{ey} = n_0 V_{x0} \frac{d}{dx} V_{ey} = -\frac{e n_e}{m_e} (0 + V_{ez} B_{x0} - V_{ex} B_z) \tag{3.46}$$

$$n_i V_{ix} \frac{d}{dx} V_{iz} = n_0 V_{x0} \frac{d}{dx} V_{iz} = +\frac{e n_i}{m_i} (-V_{x0} B_{y0} + V_{ix} B_y - V_{iy} B_{x0}) \tag{3.47}$$

$$n_e V_{ex} \frac{d}{dx} V_{ez} = n_0 V_{x0} \frac{d}{dx} V_{ez} = -\frac{e n_e}{m_e} (-V_{x0} B_{y0} + V_{ex} B_y - V_{ey} B_{x0}) \tag{3.48}$$

From Eq. (3.20), we have

$$\frac{dB_y}{dx} = \mu_0 e (n_i V_{iz} - n_e V_{ez}) \tag{3.49}$$

$$\frac{dB_z}{dx} = -\mu_0 e (n_i V_{iy} - n_e V_{ey}) \tag{3.50}$$

The above ODEs can be simplified by multiplying each equation by $V_{ix}$. For convenience, we define a convective time $\tau = \int [V_{ix}(x)]^{-1} dx$, such that the convective time derivative can be written as $V_{ix} dA/dx = dA/d\tau = \dot{A}$. Thus, multiplying Eqs. (3.17), (3.43)~(3.50) by $V_{ix}$, yields

$$\dot{E}_x = \frac{e}{\varepsilon_0} n_0 V_{x0} \left(1 - \frac{n_e}{n_i}\right) \tag{3.51}$$

or

$$\frac{n_e}{n_i} = 1 - \frac{\varepsilon_0 \dot{E}_x}{e n_0 V_{x0}} = 1 - \varepsilon(x) \tag{3.52}$$

$$\dot{V}_{ix} = -\frac{\dot{p}_i}{m_i n_0 V_{x0}} + \frac{e}{m_i}(E_x + V_{iy}B_z - V_{iz}B_y) \tag{3.53}$$

$$\dot{V}_{ex} = -\frac{\dot{p}_e}{m_e n_0 V_{x0}} - \frac{en_e}{m_e n_i}(E_x + V_{ey}B_z - V_{ez}B_y) \tag{3.54}$$

$$\dot{V}_{iy} = +\frac{e}{m_i}(V_{iz}B_{x0} - V_{ix}B_z) \tag{3.55}$$

$$\dot{V}_{ey} = -\frac{en_e}{m_e n_i}(V_{ez}B_{x0} - V_{ex}B_z) = -\frac{e}{m_e}(1 - \frac{\varepsilon_0 \dot{E}_x}{en_0 V_{x0}})(V_{ez}B_{x0} - V_{ex}B_z) \tag{3.56}$$

$$\dot{V}_{iz} = +\frac{e}{m_i}(-V_{x0}B_{y0} + V_{ix}B_y - V_{iy}B_{x0}) \tag{3.57}$$

$$\dot{V}_{ez} = -\frac{e}{m_e}(1 - \frac{\varepsilon_0 \dot{E}_x}{en_0 V_{x0}})(-V_{x0}B_{y0} + V_{ex}B_y - V_{ey}B_{x0}) \tag{3.58}$$

$$\dot{B}_y = \mu_0 en_0 V_{x0}(V_{iz} - \frac{n_e}{n_i}V_{ez}) = \mu_0 en_0 V_{x0}[V_{iz} - (1 - \frac{\varepsilon_0 \dot{E}_x}{en_0 V_{x0}})V_{ez}] \tag{3.59}$$

$$\dot{B}_z = -\mu_0 en_0 V_{x0}(V_{iy} - \frac{n_e}{n_i}V_{ey}) = -\mu_0 en_0 V_{x0}[V_{iy} - (1 - \frac{\varepsilon_0 \dot{E}_x}{en_0 V_{x0}})V_{ey}] \tag{3.60}$$

Solving (3.40) and (3.60) for $(V_{iy}, V_{ey})$ yields

$$\frac{V_{iy}}{V_{x0}} = \frac{(1-\varepsilon)(\frac{B_{x0}}{B_0}\frac{B_y - B_{y0}}{B_0}) - \frac{1}{\Omega_{e0}}\frac{\dot{B}_z}{B_0}}{M_{A0}^2[1 - \frac{m_i}{m_i + m_e}\varepsilon]} \tag{3.61}$$

$$\frac{V_{ey}}{V_{x0}} = \frac{\frac{B_{x0}}{B_0}\frac{B_y - B_{y0}}{B_0} + \frac{1}{\Omega_{i0}}\frac{\dot{B}_z}{B_0}}{M_{A0}^2[1 - \frac{m_i}{m_i + m_e}\varepsilon]} \tag{3.62}$$

Likewise, solving (3.41) and (3.59) for $(V_{iz}, V_{ez})$ yields

$$\frac{V_{iz}}{V_{x0}} = \frac{(1-\varepsilon)(\frac{B_{x0}}{B_0}\frac{B_z}{B_0} - \frac{E_x V_{x0}}{B_0 c^2}\frac{B_{y0}}{B_0}) + \frac{1}{\Omega_{e0}}\frac{\dot{B}_y}{B_0}}{M_{A0}^2[1 - \frac{m_i}{m_i + m_e}\varepsilon]} \tag{3.63}$$

$$\frac{V_{ez}}{V_{x0}} = \frac{\frac{B_{x0}}{B_0}\frac{B_z}{B_0} - \frac{E_x V_{x0}}{B_0 c^2}\frac{B_{y0}}{B_0} - \frac{1}{\Omega_{i0}}\frac{\dot{B}_y}{B_0}}{M_{A0}^2[1 - \frac{m_i}{m_i + m_e}\varepsilon]} \tag{3.64}$$

where

$\varepsilon = \varepsilon_0 \dot{E}_x / en_0 V_{x0} = (n_e / n_i) - 1,$

$\Omega_{i0} = eB_0 / m_i$ is the upstream ion cyclotron frequency,

$\Omega_{e0} = eB_0 / m_e$ is the upstream electron cyclotron frequency,

$V_{A0} = B_0 / \sqrt{\mu_0 n_0 (m_i + m_e)}$ is the upstream Alfven speed, and

$M_{A0} = V_{x0} / V_{A0} = \sqrt{\mu_0 n_0 (m_i + m_e)} V_{x0} / B_0$ is the upstream Alfven Mach number.

Taking convective time derivative of Eqs. (3.59) and (3.60) yields

$$\ddot{B}_y = \mu_0 e n_0 V_{x0} [\dot{V}_{iz} - (1 - \frac{\varepsilon_0 \dot{E}_x}{e n_0 V_{x0}}) \dot{V}_{ez} + \frac{\varepsilon_0 \ddot{E}_x}{e n_0 V_{x0}} V_{ez}] \tag{3.65}$$

$$\ddot{B}_z = -\mu_0 e n_0 V_{x0} [\dot{V}_{iy} - (1 - \frac{\varepsilon_0 \dot{E}_x}{e n_0 V_{x0}}) \dot{V}_{ey} + \frac{\varepsilon_0 \ddot{E}_x}{e n_0 V_{x0}} V_{ey}] \tag{3.66}$$

Substituting (3.55)~(3.58) into Eqs. (3.65) and (3.66) to eliminate $(\dot{V}_{iy}, \dot{V}_{ey}, \dot{V}_{iz}, \dot{V}_{ez})$, and then substituting (3.61)~(3.64) into the resulting equations to eliminate $(V_{iy}, V_{ey}, V_{iz}, V_{ez})$ yields

$$\begin{aligned} \frac{\ddot{B}_y / B_0}{\Omega_{i0}\Omega_{e0} M_{A0}^2} = & \frac{\Omega_{i0} + \Omega_{e0}(1-\varepsilon)^2}{\Omega_{i0} + \Omega_{e0}} (-\frac{B_{y0}}{B_0}) \\ & + \frac{\Omega_{i0} + \Omega_{e0}(1-\varepsilon)}{\Omega_{i0} + \Omega_{e0}} (\frac{B_y}{B_0} \frac{V_{ix}}{V_{x0}}) \\ & - (1-\varepsilon) \frac{1}{M_{A0}^2} \frac{B_{x0}^2}{B_0^2} \frac{B_y - B_{y0}}{B_0} \\ & - (\frac{1-\varepsilon}{\Omega_{i0}} - \frac{1}{\Omega_{e0}}) \frac{1}{M_{A0}^2} \frac{B_{x0}}{B_0} \frac{\dot{B}_z}{B_0} \\ & - \frac{1}{\Omega_{i0} + \Omega_{e0}(1-\varepsilon)} \dot{\varepsilon} \frac{1}{M_{A0}^2} (\frac{B_{x0}}{B_0} \frac{B_z}{B_0} - \frac{E_x V_{x0}}{B_0 c^2} \frac{B_{y0}}{B_0} - \frac{1}{\Omega_{i0}} \frac{\dot{B}_y}{B_0}) \end{aligned} \tag{3.67}$$

$$\begin{aligned} \frac{\ddot{B}_z / B_0}{\Omega_{i0}\Omega_{e0} M_{A0}^2} = & \frac{\Omega_{i0} + \Omega_{e0}(1-\varepsilon)}{\Omega_{i0} + \Omega_{e0}} (\frac{B_z}{B_0} \frac{V_{ix}}{V_{x0}}) \\ & - (1-\varepsilon) \frac{1}{M_{A0}^2} \frac{B_{x0}}{B_0} (\frac{B_{x0}}{B_0} \frac{B_z}{B_0} - \frac{E_x V_{x0}}{B_0 c^2} \frac{B_{y0}}{B_0}) \\ & + (\frac{1-\varepsilon}{\Omega_{i0}} - \frac{1}{\Omega_{e0}}) \frac{1}{M_{A0}^2} \frac{B_{x0}}{B_0} \frac{\dot{B}_y}{B_0} \\ & - \frac{1}{\Omega_{i0} + \Omega_{e0}(1-\varepsilon)} \dot{\varepsilon} \frac{1}{M_{A0}^2} [\frac{B_{x0}}{B_0} \frac{(B_y - B_{y0})}{B_0} + \frac{1}{\Omega_{i0}} \frac{\dot{B}_z}{B_0}] \end{aligned} \tag{3.68}$$

where $\dot{\varepsilon} = \varepsilon_0 \ddot{E}_x / e n_0 V_{x0}$ and $1 - \frac{m_i}{m_i + m_e} \varepsilon = \frac{\Omega_{i0} + \Omega_{e0}(1-\varepsilon)}{\Omega_{i0} + \Omega_{e0}}$ has been used to obtain Eqs. (3.67) and (3.68). The last term in Eqs. (3.67) and (3.68) can be important when $B_{x0} \to 0$.

We still need to know $V_{ix}$ and $E_x$ before we can solve $B_y$ and $B_z$. Substituting Eqs. (3.32)~(3.35) into (3.39), yields

$$[\frac{1}{1-\varepsilon}\frac{\Omega_{i0}+\Omega_{e0}(1-\varepsilon)}{\Omega_{i0}+\Omega_{e0}}\frac{V_{ix}}{V_{x0}}-1]+\frac{\beta_0}{2M_{A0}^2}[\frac{p_{i0}+p_{e0}(1-\varepsilon)^{5/3}}{p_{i0}+p_{e0}}(\frac{V_{ix}}{V_{x0}})^{-5/3}-1]$$

$$+\frac{1}{M_{A0}^2}[\frac{B_y^2+B_z^2}{2B_0^2}-\frac{B_{y0}^2}{2B_0^2}]-\frac{1}{2}\frac{E_x^2}{B_0^2}\frac{1}{c^2}\frac{1}{M_{A0}^2}=0 \qquad (3.69)$$

where $\beta_0=(p_{i0}+p_{e0})2\mu_0/B_0^2$ and $c=1/\sqrt{\mu_0\varepsilon_0}$ is the speed of light. The last term in Eq. (3.69), which can also be written as $(E_x^2/2B_0^2V_{x0}^2)(V_{A0}^2/c^2)$, becomes an important term when $\beta_0 c^2<<1$ or $V_{A0}^2\to c^2$. This is the condition for presence of inertial Alfven wave, which may be an important mechanism for auroral arc formation. For simplicity, we can classify nonlinear wave solutions into two types. One of them is for finite $\beta_0$, $V_{A0}^2<<c^2$, and finite $B_{x0}$. The other is for $\beta_0 c^2<<1$ and $B_{x0}\to 0$.

*Case 1*

For finite $\beta_0$, $V_{A0}^2<<c^2$, and finite $B_{x0}$, we can ignore the last term in Eq. (3.69) and make quasi-neutrality assumption, i.e., $\varepsilon=1-(n_e/n_i)\to 0$, and $\dot{\varepsilon}\to 0$. The quasi-neutrality assumption yields $n_i=n_e=n$, $V_{ix}=V_{ex}=V_x$, and $E_xV_{x0}/B_0c^2\to 0$. Thus, equations (3.67)~(3.69) can be rewritten as

$$\frac{\ddot{B}_y/B_0}{\Omega_{i0}\Omega_{e0}M_{A0}^2}=-\frac{B_{y0}}{B_0}+\frac{B_y}{B_0}\frac{V_x}{V_{x0}}-\frac{1}{M_{A0}^2}\frac{B_{x0}^2}{B_0^2}\frac{B_y-B_{y0}}{B_0}-(\frac{1}{\Omega_{i0}}-\frac{1}{\Omega_{e0}})\frac{1}{M_{A0}^2}\frac{B_{x0}}{B_0}\frac{\dot{B}_z}{B_0} \qquad (3.67')$$

$$\frac{\ddot{B}_z/B_0}{\Omega_{i0}\Omega_{e0}M_{A0}^2}=\frac{B_z}{B_0}\frac{V_x}{V_{x0}}-\frac{1}{M_{A0}^2}\frac{B_{x0}^2}{B_0^2}\frac{B_z}{B_0}+(\frac{1}{\Omega_{i0}}-\frac{1}{\Omega_{e0}})\frac{1}{M_{A0}^2}\frac{B_{x0}}{B_0}\frac{\dot{B}_y}{B_0} \qquad (3.68')$$

$$[\frac{V_x}{V_{x0}}-1]+\frac{\beta_0}{2M_{A0}^2}[(\frac{V_x}{V_{x0}})^{-5/3}-1]+\frac{1}{M_{A0}^2}[\frac{B_y^2+B_z^2}{2B_0^2}-\frac{B_{y0}^2}{2B_0^2}]=0 \qquad (3.69')$$

We can solve Eq. (3.69') to obtain $V_x=V_x(B_y,B_z)$. Substituting $V_x=V_x(B_y,B_z)$ into (3.67') and (3.68') yields

$$\frac{\ddot{B}_y/B_0}{\Omega_{i0}\Omega_{e0}M_{A0}^2}=-\frac{\partial\Psi(B_y,B_z)}{\partial B_y}-(\frac{1}{\Omega_{i0}}-\frac{1}{\Omega_{e0}})\frac{1}{M_{A0}^2}\frac{B_{x0}}{B_0}\frac{\dot{B}_z}{B_0} \qquad (3.67'')$$

$$\frac{\ddot{B}_z/B_0}{\Omega_{i0}\Omega_{e0}M_{A0}^2}=-\frac{\partial\Psi(B_y,B_z)}{\partial B_z}+(\frac{1}{\Omega_{i0}}-\frac{1}{\Omega_{e0}})\frac{1}{M_{A0}^2}\frac{B_{x0}}{B_0}\frac{\dot{B}_y}{B_0} \qquad (3.68'')$$

where

$$\frac{\partial\Psi(B_y,B_z)}{\partial B_y}=[\frac{1}{M_{A0}^2}\frac{B_{x0}^2}{B_0^2}-\frac{V_x(B_y,B_z)}{V_{x0}}]\frac{B_y}{B_0}-(\frac{1}{M_{A0}^2}\frac{B_{x0}^2}{B_0^2}-1)\frac{B_{y0}}{B_0} \tag{3.70}$$

$$\frac{\partial\Psi(B_y,B_z)}{\partial B_z}=[\frac{1}{M_{A0}^2}\frac{B_{x0}^2}{B_0^2}-\frac{V_x(B_y,B_z)}{V_{x0}}]\frac{B_z}{B_0} \tag{3.71}$$

Let $\mathbf{B}_t = \hat{y}B_y + \hat{z}B_z$, the Eqs. (3.67") and (3.68") can be rewritten in the following vector form

$$\frac{\ddot{\mathbf{B}}_t/B_0}{\Omega_{i0}\Omega_{e0}M_{A0}^2}=-\nabla_{\mathbf{B}_t}\Psi(B_y,B_z)-(\frac{1}{\Omega_{i0}}-\frac{1}{\Omega_{e0}})\frac{1}{M_{A0}^2}\frac{\dot{\mathbf{B}}_t}{B_0}\times\frac{\hat{x}B_{x0}}{B_0} \tag{3.72}$$

Solution $\mathbf{B}_t(x)=\hat{y}B_y(x)+\hat{z}B_z(x)$ of Eq. (3.72) is similar to a particle's trajectory $\mathbf{r}(t)=\hat{y}y(t)+\hat{z}z(t)$, which satisfies the following equations of motion $c_1\ddot{\mathbf{r}}(t)=-\nabla\Psi-c_2\dot{\mathbf{r}}(t)\times\hat{x}B_{x0}$. In this case, we can consider solution $\mathbf{B}_t$ as a trajectory of a pseudo particle at a given pseudo time $\tau(x)$. The motion of pseudo particle is under the influence of a pseudopotential-gradient force $-\nabla_{\mathbf{B}_t}\Psi(B_y,B_z)$ and a pseudo-velocity-dependent force as the last term in Eq. (3.72).

For $\sqrt{\Omega_{i0}\Omega_{e0}} >> \Omega_{i0}(B_0/B_{x0})$ the motion of pseudo particle can be decomposed into a high-pseudo-frequency ( $\sqrt{\Omega_{i0}\Omega_{e0}}\tau\approx 1 >> \Omega_{i0}(B_0/B_{x0})\tau$ ) gyro motion and a low-pseudo-frequency ($\Omega_{i0}(B_0/B_{x0})\tau\approx 1$) drift motion, where the low-pseudo-frequency drift motion is characterized by an average drift trajectory $<\mathbf{B}_t>$ follows closely (not exactly) along a $\Psi = const.$ contour.

Previous studies (References [4] and [5] in Exercise 3.1) show that pseudopotential $\Psi(B_y,B_z)$ can be obtained analytically and indirectly from conservation of energy flux.

From $(\dot{\mathbf{B}}_t/B_0)\cdot(2.72)$ yields

$$\frac{1}{\Omega_{i0}\Omega_{e0}M_{A0}^2}\frac{1}{2}\frac{d}{d\tau}\frac{\dot{\mathbf{B}}_t^2}{B_0^2}=-\frac{d}{d\tau}\Psi(B_y,B_z) \tag{3.73}$$

Integrating (3.73) once, yields

$$\frac{1}{\Omega_{i0}\Omega_{e0}M_{A0}^2}\frac{1}{2}\frac{\dot{B}_y^2+\dot{B}_z^2}{B_0^2}+\Psi(B_y,B_z)=0 \tag{3.74}$$

Substituting Eqs. (3.61)~(3.64) and (3.69) into (3.42), then applying quasi-neutrality condition to the resulting equation, yields,

$$\frac{M_{A0}^2}{2}\left(\frac{V_x}{V_{x0}}\right)^2+\frac{5\beta_0}{4}\left(\frac{V_x}{V_{x0}}\right)^{1-\gamma}+\frac{B_{y0}}{B_0}\frac{B_y}{B_0}+\frac{1}{2M_{A0}^2}\frac{B_{x0}^2}{B_0^2}\left[\left(\frac{B_y-B_{y0}}{B_0}\right)^2+\left(\frac{B_z}{B_0}\right)^2\right]$$
$$+\frac{1}{\Omega_{i0}\Omega_{e0}M_{A0}^2}\frac{1}{2}\frac{\dot{B}_y^2+\dot{B}_z^2}{B_0^2}=\frac{M_{A0}^2}{2}+\frac{5\beta_0}{4}+\frac{B_{y0}^2}{B_0^2} \quad (3.75)$$

where $\gamma=5/3$. Comparing (3.74) and (3.75) yields

$$\Psi(B_y,B_z)=\frac{M_{A0}^2}{2}\left\{\left[\frac{V_x(B_y,B_z)}{V_{x0}}\right]^2-1\right\}+\frac{5\beta_0}{4}\left\{\left[\frac{V_x(B_y,B_z)}{V_{x0}}\right]^{1-\gamma}-1\right\}$$
$$+\frac{B_{y0}}{B_0}\frac{B_y-B_{y0}}{B_0}+\frac{1}{2M_{A0}^2}\frac{B_{x0}^2}{B_0^2}\left[\left(\frac{B_y-B_{y0}}{B_0}\right)^2+\left(\frac{B_z}{B_0}\right)^2\right] \quad (3.76)$$

As we can see from Eq. (3.74) that the average drift trajectory of the pseudo particle $<\mathbf{B}_t>$ slightly deviates from $\Psi=const.$ contour. The deviation from $\Psi=const.$ contour increases with increasing pseudo kinetic energy.

**Exercise 3.7**

Follow reference [5] in Exercise 3.1 to obtain structures of $V_x=V_x(B_y,B_z)$ and pseudopotential $\Psi(B_y,B_z)$ for a given set of upstream conditions.

For a given set of $(B_y,B_z)$, two roots of $V_x$ can be obtained from Eq. (3.69'). One of them is greater than the local sound speed. The other is less than the local sound speed. Since $V_x$ is a double value function of $(B_y,B_z)$, according to Eq. (3.76), $\Psi(B_y,B_z)$ should also be a double value function of $(B_y,B_z)$. Figure 3.6 shows structures $\Psi(B_y,B_z)$ for different upstream parameters, where $b_y=B_y-B_{y0}$, $b_z=B_z$, superscripts '*' denotes normalized variables. Namely, $b_y^*=b_y/B_0$, $b_z^*=b_z/B_0$, $B_{y0}^*=B_{y0}/B_0$, … etc. Under the uniform upstream boundary condition, the nonlinear solution must start from point A, where $b_y=B_y$ or $B_y=B_{y0}$. Thus only if an equal pseudopotential contour passing through point A with a finite length can be considered as a nonlinear solution. However, if we remove the uniform upstream boundary conditions, then any equal pseudopotential contour can be a nonlinear wave solution (for $B_{x0}/B_0>>\sqrt{m_e/m_i}$), but definition of constants $B_{x0},B_{y0},B_0,V_{x0},p_{i0},p_{e0},n_0$ need to be redefined.

It can be seen from Figure 3.6 that the supersonic surface and subsonic surface of the pseudopotential structure meet at a sonic circle. Equal pseudopotential contour may intersect with the sonic circle. It can be shown that only points on half of the sonic circle can be attractors in a pseudo particle's trajectory. Namely, an acceptable nonlinear solution can be a trajectory of a pseudo particle, which starts from point A or a sonic emitter, then follows closely along an equal pseudopotential contour to a sonic attractor or back to point A. We can also apply the multiple-pseudopotential method to this problem. Figure 3.7 shows how to use two pseudopotential surfaces with slightly different $M_{A0}$ to explain the hook-shaped magnetic hodogram obtained from a numerical simulation study of a rotational discontinuity. Isentropic nonlinear solutions obtained in this section are particularly useful in modeling isentropic nonlinear waves propagated near Alfven mode speed.

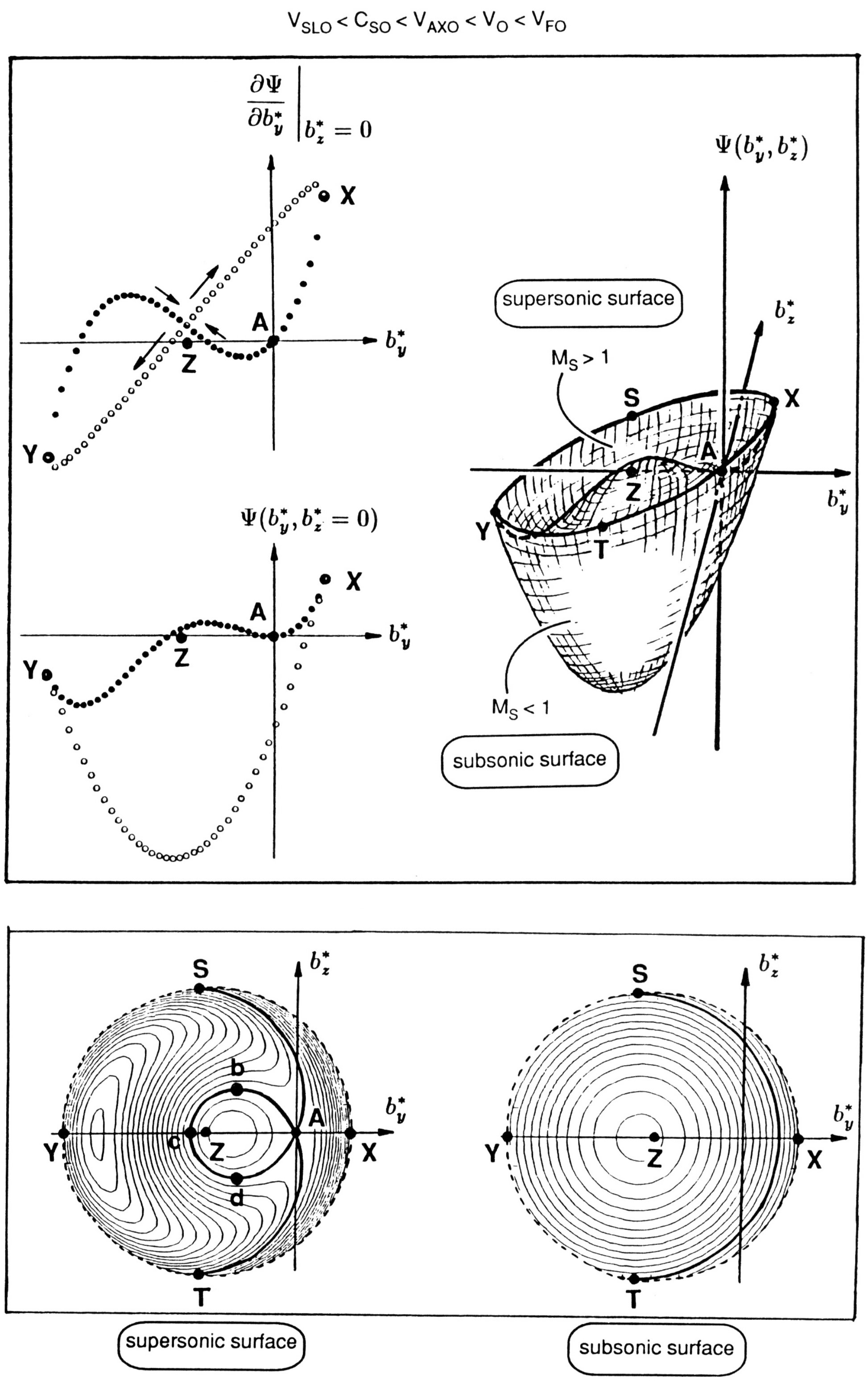


**Figure 3.6** Structures of $\Psi(B_y, B_z)$ for different upstream parameters, where $b_y = B_y - B_{y0}$, $b_z = B_z$, superscripts '*' denotes normalized variables. Point A is located at $(B_y, B_z) = (B_{y0}, 0)$. Point is located at $(B_y, B_z) = (0,0)$. Notations $V_{SL0}, V_{AX0}, V_{F0}$ are the upstream MHD slow mode, Alfven mode, and fast mode speed, respectively. $C_{S0}$ is the upstream sound speed. See text for detail discussion.

*Case 2*

For $\beta_0 c^2 << 1$ and $B_{x0} \to 0$, we can ignore changes on magnetic field and the thermal pressure terms in the x-component momentum equations as well as in the equations of conservations of momentum and energy fluxes. We are looking for nonlinear electrostatic waves in a magnetize plasma with wave normal direction nearly perpendicular to the local magnetic field ($B_{x0} / B_0 < \sqrt{m_e / m_i}$). Studies of these types of nonlinear waves are on going research topics. Students are encouraged to solve the general problem without simplifications. Governing equations of a general problem include Eqs. (3.42)~(3.44), (3.61)~(3.64), and (3.67)~(3.69).

**Exercise 3.8**

Obtain pseudopotential structure and nonlinear static wave solutions of Case 2 with $\beta_0 c^2 << 1$ and $B_{x0} / B_0 << \sqrt{m_e / m_i}$.

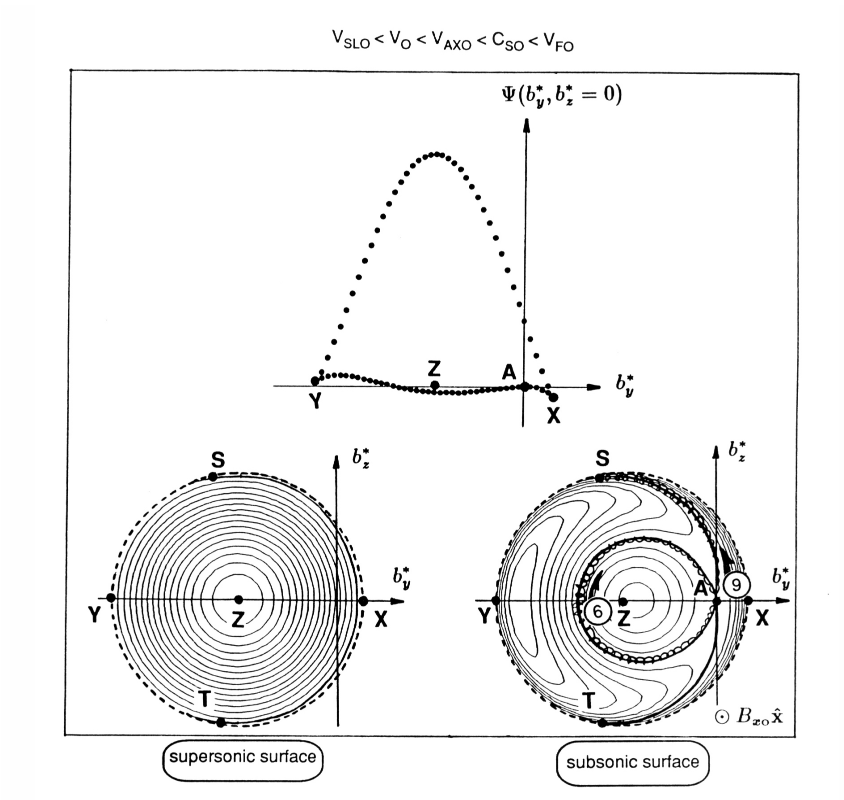


**Figure 3.6** (Continued) Structures of $\Psi(B_y, B_z)$ for different upstream parameters.

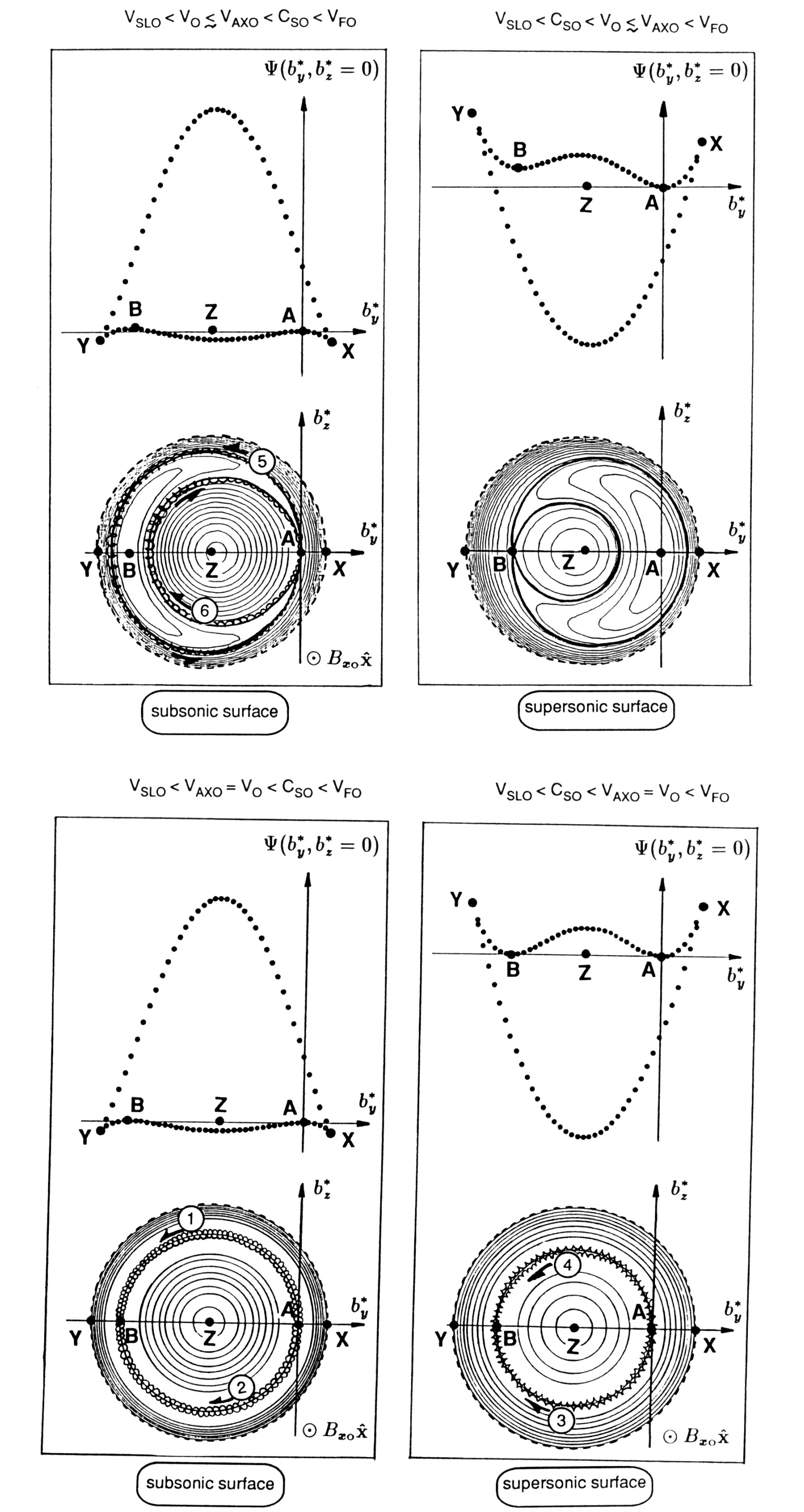


**Figure 3.6** (Continued) Structures of $\Psi(B_y, B_z)$ for different upstream parameters.

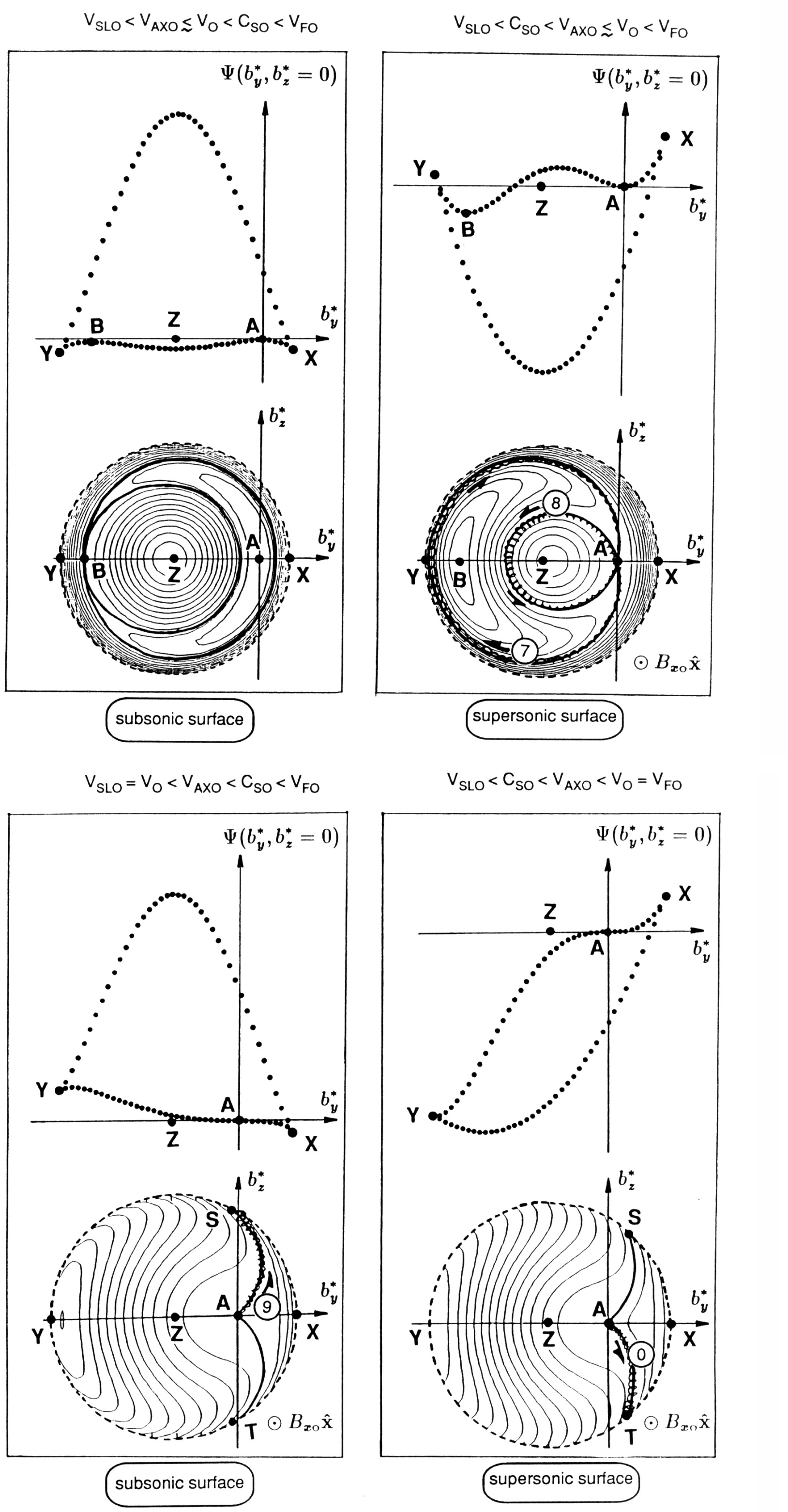


**Figure 3.6** (Continued) Structures of $\Psi(B_y, B_z)$ for different upstream parameters.

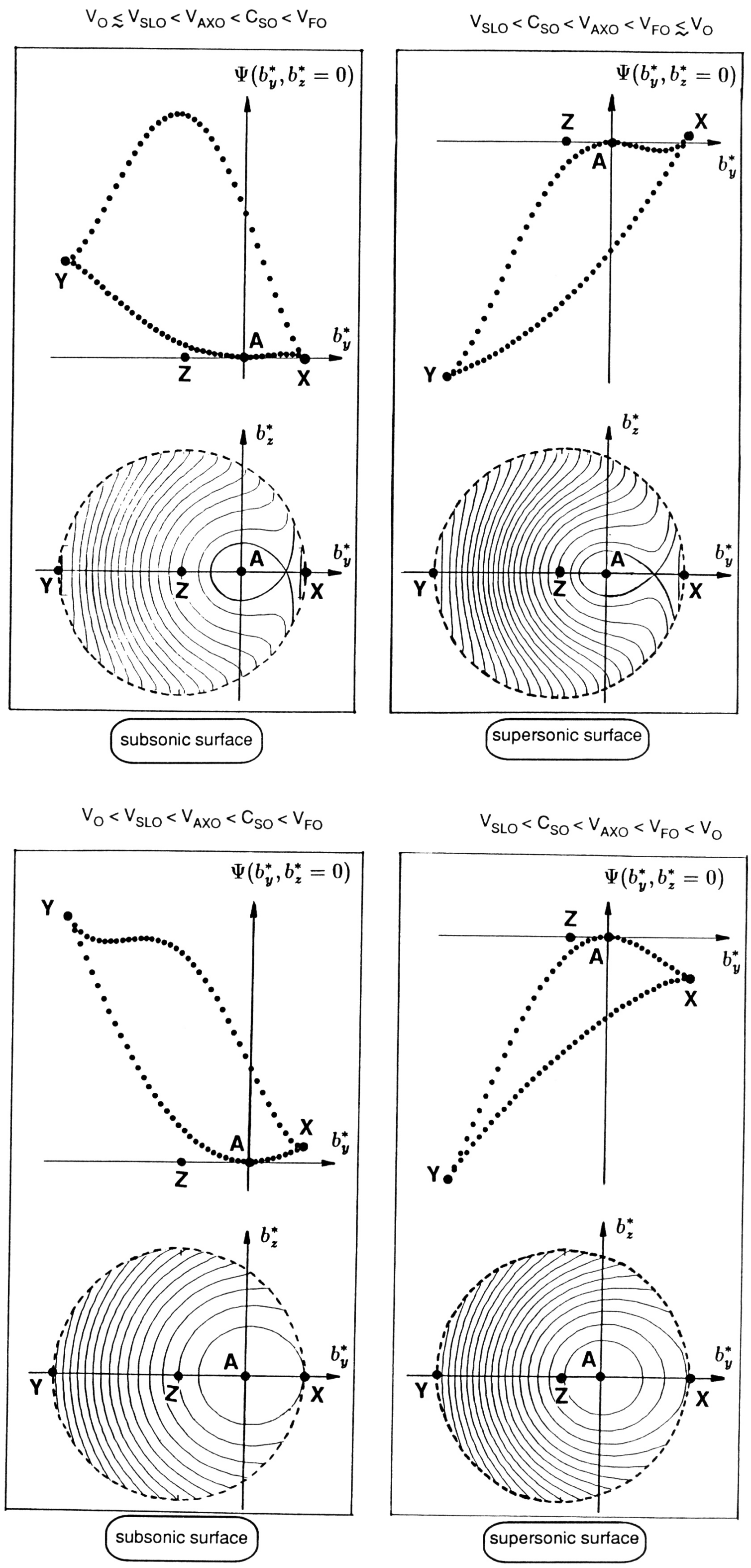


**Figure 3.6** (Continued) Structures of $\Psi(B_y, B_z)$ for different upstream parameters.

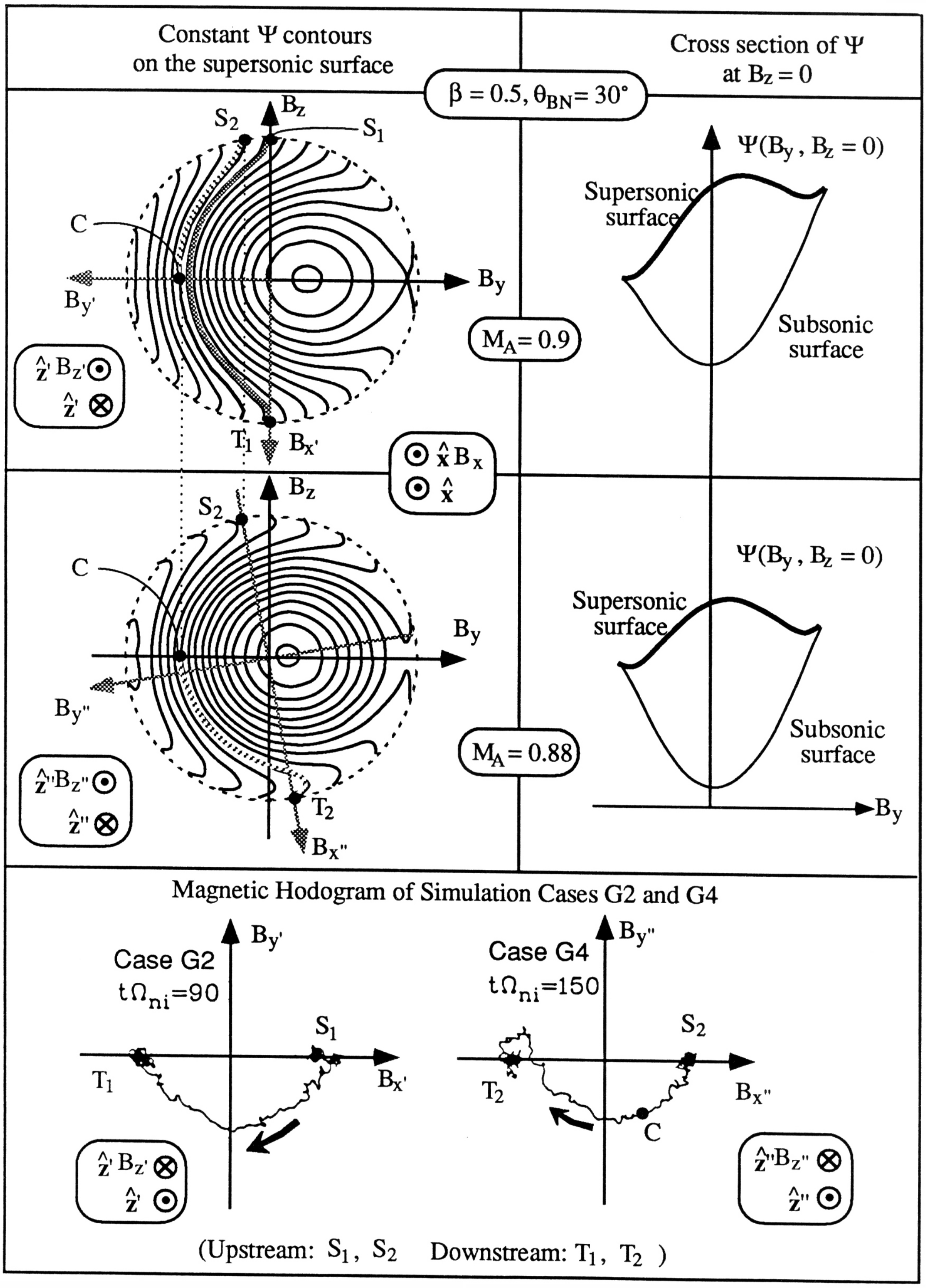


**Figure 3.7** An illustration on how to use one or two pseudopotential surfaces to explain magnetic hodograms obtained from numerical simulations of rotational discontinuities. The two pseudopotential surfaces are obtained for slightly different $M_{A0}$, which can explain the hook-shaped magnetic hodogram obtained in the simulation.