Motion of a Particle in One Dimension

Example 1.

Let us consider a spring at rest with a spring constant k. Let us load an object with mass m_1 on the left end of the spring and load an object with mass m_2 on the right end of the spring. Slightly stretch the two ends of the spring and then release the external stretch at time $t = 0$. Discuss the motions of these two objects. Check the potential energy and the kinetic energy of the system. Let us consider the following two cases. **Case 1:** $m_1 \gg m_2$. **Case 2:** $m_1 = m_2 = m$. **Case 3:** $m_1 \neq m_2$

Case 1: $m_1 \gg m_2$

For $m_1 \gg m_2$, we can assume that the object with mass m_1 is at rest (motionless). Let the displacement of the object with mass m_2 with respect to its equilibrium position be x_2. Hooke's law yields

$$m_2 \ddot{x}_2 = -k x_2 \tag{2.1}$$

The solution of $x_2(t)$ can be written as

$$x_2(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{2.2}$$

where $\omega = \sqrt{k/m_2}$. For $x_2(t=0) = x_0$ and $\dot{x}_2(t=0) = 0$, it yields

$$x_2(t) = x_0 \cos(\omega t) \tag{2.3}$$

The potential energy of this system is

$$V = \frac{1}{2} k x_2^2 \tag{2.4}$$

The kinetic energy of this system is

$$T = \frac{1}{2} m \dot{x}_2^2 \tag{2.5}$$

Multiplying $\dot{x}_2$ to Equation (2.1), it yields

$$\frac{d}{dt}(T + V) = 0 \tag{2.6}$$

Thus, the sum of the potential energy plus the kinetic energy is conserved (is constant with time). This is true as long as the force is a conservative force.

Case 2: $m_1 = m_2 = m$

Let the displacement of the object with mass m_1 with respect to its equilibrium position be x_1. Let the displacement of the object with mass m_2 with respect to its equilibrium position be x_2. Hooke's law yields

$$m_1 \ddot{x}_1 = -k(x_1 - x_2) \tag{2.7}$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1) \tag{2.8}$$

For $m_1 = m_2 = m$, we can assume that $x_1 = -x_2$, $\dot{x}_1 = -\dot{x}_2$, and $\ddot{x}_1 = -\ddot{x}_2$. It yields

$$m\ddot{x}_2 = -2kx_2 \tag{2.9}$$

The solution of $x_2(t)$ can be written as

$$x_2(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{2.10}$$

where $\omega = \sqrt{2k/m}$. For $x_2(t=0) = x_0$ and $\dot{x}_2(t=0) = 0$, it yields

$$x_2(t) = x_0 \cos(\omega t) \tag{2.11}$$

The potential energy of this system is

$$V = \frac{1}{2}k(x_2 - x_1)^2 = 2kx_2^2 \tag{2.12}$$

The kinetic energy of this system is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = m\dot{x}_2^2 \tag{2.13}$$

Multiplying $\dot{x}_2$ to Equation (2.9), it yields

$$\frac{d}{dt}(T + V) = 0 \tag{2.14}$$

Thus, the sum of the potential energy plus the kinetic energy is conserved (is constant with time).

Case 3: $m_1 \neq m_2$

Let the displacement of the object with mass m_1 with respect to its equilibrium position be x_1. Let the displacement of the object with mass m_2 with respect to its equilibrium position be x_2. Hooke's law yields

$$m_1\ddot{x}_1 = -k(x_1 - x_2) \tag{2.15}$$

$$m_2\ddot{x}_2 = -k(x_2 - x_1) \tag{2.16}$$

It yields

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = 0 \tag{2.17}$$

$$m_1\dot{x}_1 + m_2\dot{x}_2 = 0 \tag{2.18}$$

$$m_1x_1 + m_2x_2 = 0 \tag{2.19}$$

Thus, Equation (2.16) can be rewritten as

$$m_2\ddot{x}_2 = -k\left(1 + \frac{m_2}{m_1}\right)x_2 \tag{2.20}$$

The solution of $x_2(t)$ can be written as

$$x_2(t) = C_1\cos(\omega t) + C_2\sin(\omega t) \tag{2.21}$$

where

$$\omega = \sqrt{k(\frac{1}{m_1} + \frac{1}{m_2})} = \sqrt{\frac{k(m_1 + m_2)}{m_1m_2}} \tag{2.22}$$

For $x_2(t = 0) = x_0$ and $\dot{x}_2(t = 0) = 0$, it yields

$$x_2(t) = x_0\cos(\omega t) \tag{2.23}$$

and

$$x_1(t) = -\frac{m_2}{m_1}x_0\cos(\omega t) \tag{2.24}$$

The potential energy of this system is

$$V = \frac{1}{2}k(x_2 - x_1)^2 = \frac{1}{2}k\left(1 + \frac{m_2}{m_1}\right)^2 x_2^2 \tag{2.25}$$

The kinetic energy of this system is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = (\frac{m_2}{m_1} + 1)\frac{1}{2}m_2\dot{x}_2^2 \tag{2.26}$$

Multiplying $\left(1+\frac{m_2}{m_1}\right)\dot{x}_2$ to Equation (2.20), it yields

$$\frac{d}{dt}(T+V)=0 \tag{2.27}$$

Thus, the sum of the potential energy plus the kinetic energy is conserved (is constant with time).

Example 2.

Let us consider a harmonic oscillation with velocity-dependent damping force term, such that

$$m\ddot{x}=-kx-b\dot{x} \tag{2.28}$$

The solution of $x(t)$ can be written as

$$x(t)=\sum_{i=1,2} C_i\exp(\gamma_i t) \tag{2.29}$$

where $\gamma_1\neq\gamma_2$. Equation (2.29) yields

$$m\left[\sum_{i=1,2}\gamma_i^2 C_i\exp(\gamma_i t)\right]=-k\left[\sum_{i=1,2}C_i\exp(\gamma_i t)\right]-b\left[\sum_{i=1,2}\gamma_i C_i\exp(\gamma_i t)\right]$$

or

$$\left[\sum_{i=1,2}(m\gamma_i^2+b\gamma_i+k)C_i\exp(\gamma_i t)\right]=0 \tag{2.30}$$

For non-zero $C_i\exp(\gamma_i t)$, it yields

$$m\gamma_i^2+b\gamma_i+k=0 \tag{2.31}$$

or

$$\gamma_{\frac{1}{2}}=-\frac{b}{2m}\pm\sqrt{\left(\frac{b}{2m}\right)^2-\frac{k}{m}} \tag{2.32}$$

For $b^2-4mk=0$, Equation (2.32) yields $\gamma_1=\gamma_2=-b/2m$. Thus, we need consider another type of solutions for the second-order ODE. Let

$$x(t) = (C_0 + D_0 t)\exp(\gamma_1 t) \tag{2.33}$$

Substituting Equation (2.33) into Equation (2.28), it yields

$$(C_0 + D_0 t)(m\gamma_1^2 + b\gamma_1 + k)\exp(\gamma_1 t) + D_0(2m\gamma_1 + b)\exp(\gamma_1 t) = 0 \tag{2.34}$$

For $b^2 - 4mk = 0$, and $\gamma_1 = -b/2m$. Thus, Equation (2.34) is true for all C_0 and D_0.

Let $\gamma = b/2m$ and $\omega_0 = \sqrt{k/m}$. For $\omega_0 \neq \gamma$, Equation (2.32) can be rewritten as

$$\gamma_{\frac{1}{2}} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{2.35}$$

For $\omega_0 = \gamma$, Equation (2.33) can be rewritten as

$$x(t) = (C_0 + D_0 t)\exp(-\gamma t) \tag{2.36}$$

Let us consider the following three cases:

Case A: $\omega_0 > \gamma$, **Case B:** $\omega_0 < \gamma$, and **Case C:** $\omega_0 = \gamma$.

Case A: $\omega_0 > \gamma$ (**underdamping**)

For $\omega_0 > \gamma$, Equation (2.35) yields

$$\gamma_{\frac{1}{2}} = -\gamma \pm i\omega \tag{2.37}$$

where

$$\omega = \omega_0\sqrt{1 - \left(\frac{\gamma}{\omega_0}\right)^2} = \omega_0\left[1 - \frac{1}{2}\left(\frac{\gamma}{\omega_0}\right)^2 + \cdots\right]$$

It yields

$$x(t) = C_1\exp(-\gamma t)\,[\cos(\omega t) + i\sin(\omega t)] + C_2\exp(-\gamma t)\,[\cos(\omega t) - i\sin(\omega t)]$$

or

$$x(t) = \exp(-\gamma t)\,[D_1\cos(\omega t) + D_2\sin(\omega t)] \tag{2.38}$$

For $x(t=0) = x_0$ and $\dot{x}(t=0) = 0$, it yields $D_1 = x_0$ and $D_2 = (\gamma/\omega)x_0$. It yields

$$x(t) = \exp(-\gamma t)\,[x_0\cos(\omega t) + \frac{\gamma}{\omega}x_0\sin(\omega t)] \tag{2.39}$$

Case B: $\omega_0 < \gamma$ (**overdamping**)

For $\omega_0 < \gamma$, Equation (2.35) yields

$$\gamma_{\frac{1}{2}} = -\gamma\left(1 \mp \sqrt{1 - \frac{\omega_0^2}{\gamma^2}}\right) \tag{2.40}$$

or

$$\gamma_1 = -\gamma\left(1 - 1 + \frac{1}{2}\frac{\omega_0^2}{\gamma^2} + \cdots\right) = -\gamma(\frac{1}{2}\frac{\omega_0^2}{\gamma^2} + \cdots)$$

$$\gamma_2 = -\gamma\left(1 + 1 - \frac{1}{2}\frac{\omega_0^2}{\gamma^2} + \cdots\right) = -2\gamma + \gamma(\frac{1}{2}\frac{\omega_0^2}{\gamma^2} + \cdots)$$

and

$$x(t) = C_1 \exp(-\gamma_1 t) + C_2 \exp(-\gamma_2 t) \tag{2.41}$$

For $x(t = 0) = x_0$ and $\dot{x}(t = 0) = 0$, Equation (2.41) yields

$$C_1 + C_2 = x_0 \tag{2.42}$$

$$-\gamma_1 C_1 - \gamma_2 C_2 = 0 \tag{2.43}$$

Solving Equations (2.42) and (2.43), it yields

$$C_1 = \frac{\gamma_2 x_0}{\gamma_2 - \gamma_1} \tag{2.44}$$

$$C_2 = \frac{-\gamma_1 x_0}{\gamma_2 - \gamma_1} \tag{2.45}$$

Substituting Equations (2.44) and (2.45) into Equation (2.41), it yields

$$x(t) = x_0\left[\frac{\gamma_2}{\gamma_2 - \gamma_1}\exp(-\gamma_1 t) - \frac{\gamma_1}{\gamma_2 - \gamma_1}\exp(-\gamma_2 t)\right] \tag{2.46}$$

where the second term is a fast damping term. The first term becomes dominate at $\gamma_2 t \gg 1$

Case C: $\omega_0 = \gamma$ (**critical damping**)

For $\omega_0 = \gamma$, the solution of $x(t)$ is given by Equation (2.36), i.e.,

$$x(t) = (C_0 + D_0 t)\exp(-\gamma t)$$

For $x(t = 0) = x_0$ and $\dot{x}(t = 0) = 0$, it yields

$$C_0 = x_0 \qquad (2.47)$$

$$D_0 = \gamma C_0 = \gamma x_0 \qquad (2.48)$$

Thus, Equation (2.36) becomes

$$x(t) = x_0(1 + \gamma t)\exp(-\gamma t) \qquad (2.49)$$

For $\gamma t \ll 1$, the initial solution is approximately equal to

$$x(t) \approx x_0(1 + \gamma t)(1 - \gamma t) = x_0(1 - \gamma^2 t^2) \qquad (2.50)$$

In the critical damping event, the restoring force can help the oscillator back to the equilibrium position faster than the oscillator in the overdamping event.

Results of the three cases are sketched in Figure 2-5 on page 50 in the textbook,

Applications of Example 2. (RLC circuit & LC circuit)

If the oscillator in the example 2 is subject to an additional impressed force $F(t)$, then the equation of motion of the oscillator becomes

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Now, let us consider an RLC circuit.

The potential jump across a resistor with resistance R is

$$\delta V_R = IR = R\frac{dQ}{dt}$$

The potential jump across an inductance with inductance L is

$$\delta V_I = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$$

The potential jump across a capacitor with capacitance C is

$$\delta V_C = \frac{Q}{C}$$

where Q is the charge, and I is the electric current. If we apply an electromotive potential $V(t)$, it yields

$$\delta V_I + \delta V_R + \delta V_C = V(t)$$

i.e.,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V(t)$$

The circuit equation is similar to the oscillator equation given above. We can use the RLC circuit to study the motion of oscillator, or we can use the harmonic oscillator to study the RLC circuit.

For $R = 0$ and $V(t > 0) = 0$, the circuit become an LC oscillator with oscillating frequency $1/\sqrt{LC}$.

Summary

Sections 2-1, 2-2, 2-5, 2-7, 2-8, 2-9 教完了！

Section 2-3 說明為何電磁波射入電離層，可能會被反射。這部分內容等三年級電漿物理導論 再教。你們知道嗎：本書作者跟我們同行，是一位電漿物理學家。不像其他力學課本作者，大多是工程力學家。

Section 2-4 很簡單自己看

Section 2-6 很簡單自己看。可以得由 **終端速度** 反推 阻尼力的**阻尼係數**。你知道嗎：用中壢 VHF radar 可以觀測到 雨滴的終端速度 ！

Section 2-10 能力強的同學，請自己看。整個推導有點繁瑣，實際上，最好用**數值模擬**來求解。

Section 2-11 等到 應用數學 教了 Fourier Transform 我們的力學再教這個主題。這樣可以**事半功倍**。下學期力學 II 教第十二章的時候，一定會教 Fourier Transform。敬請期待。有些學校與系所，大三才教力學，所以本書作者會在第二章的時候就提到這些進階的數學。等三年級電漿物理導論 的時候，還會再教一次 Fourier Transform！

Sections 2-4, 2-6, 2-10 看了之後有問題，可以找老師討論。