Appendix C: Lagrangians With Velocity-Dependent Forces

小學到中學 12 年，大家都把「學習物理」與「解題」劃上了等號。難怪大家一直希望老師出個題目來練習一下。看看自己懂不懂，會不會做。可是未來遇到的問題，主要是要會寫出描述物理現象的方程式。只要寫得出方程式、初始條件、邊界條件，剩下的就可以請電腦程式設計師，或數學家幫忙算了。（當然，自己會算，更棒！）如果做不出來，通常就是方程式、初始條件、或邊界條件的問題！（例如：該簡化的，沒有簡化。或者簡化錯了，丟錯東西了。或者猜錯了初始條件選。或者選錯了邊界條件。）我的辦公室門上方，有一句我心愛的句子

"To understand is to know to how to calculate." --Dirac

（Dirac 是一位有名的量子物理學家。他描述的量子物理，就是又精簡，又漂亮！）

也就是說，口頭會描述一個現象是不夠的。你必須算得出這種現象（這要有正確的方程式、初始條件、邊界條件），甚至正確的預測它未來的發展，才是真的了解這個物理系統。

這份補充講義，要告訴大家，從某個觀點看來，找個「適合」的 Lagrangian 來描述一個物理系統，可以讓自己「偷懶」少打很多方程式。且讓我在這裡，打個比方：就像古文是一個一個字刻在竹簡上，那時大家都惜字如金，不會「落落長」寫又臭又長的文章。等到有了紙筆以後，文章就變得又臭又長了！另外一個例子，以前電腦草創初期，幾個 KB 就可以讓系統運作。後來 OS 與各種 APP，越寫越大，先是要幾個 MB 才能跑，現在有些要幾個 GB 才能跑！精簡的文章看起來漂亮，精簡的系統用起來也有效率，同樣的，用 Lagrangian 來描述一個物理系統也是又漂亮又有效率！不過，也很抽象，所以許多小朋友們可能一時會反應不過來！這篇補充教材，純粹供大家欣賞，增廣見識。

C.1. Motion Under the Lorentz Force

我們拿「在有外加電磁場時兩個速度大小遠小於光速的帶電粒子的運動所對應的 non-relativistic Lagrangian」當作範例，讓大家看看，用 Lagrangian 來形容這樣一個物理系統，有多精簡！

描述 1：（傳統向量形式）

考慮兩個速度大小遠小於光速的帶電粒子在電磁場中運動。這個系統的 governing equations （支配方程式）are

$$\frac{d\vec{x}_1}{dt} = \vec{v}_1 \tag{1}$$

$$\frac{d\vec{x}_2}{dt} = \vec{v}_2 \tag{2}$$

$$m_1\frac{d\vec{v}_1}{dt} = q_1[\vec{E}(\vec{x}_1,t) + \vec{v}_1 \times \vec{B}(\vec{x}_1,t)] \tag{3}$$

$$m_2\frac{d\vec{v}_2}{dt} = q_2[\vec{E}(\vec{x}_2,t) + \vec{v}_2 \times \vec{B}(\vec{x}_2,t)] \tag{4}$$

$$\nabla\cdot\vec{E} = \frac{\rho_c}{\epsilon_0} \tag{5}$$

$$\nabla\cdot\vec{B} = 0 \tag{6}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{7}$$

$$\nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \tag{8}$$

Equation (6) yields that we can find a vector potential $\vec{A}$ such that

$$\vec{B} = \nabla\times\vec{A} \tag{9}$$

Let $\vec{E} = \vec{E}^{ES} + \vec{E}^{EM}$, where the superscript ES demotes the electrostatic component and the superscript EM denotes the electromagnetic component. For electrostatic waves $\partial\vec{B}/\partial t = 0$, it yields $\nabla\times\vec{E}^{ES} = 0$. Thus, we can find a scalar potential ϕ^{ES} such that

$$\vec{E}^{ES} = -\nabla\phi^{ES} \tag{10}$$

For the electromagnetic waves

$$\nabla\times\vec{E}^{EM} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial(\nabla\times\vec{A})}{\partial t} \tag{11}$$

Equation (11) yields,

$$\vec{E}^{EM} = -\frac{\partial\vec{A}}{\partial t} - \nabla\phi^{EM} \tag{12}$$

Equations (10) & (12) yields

$$\vec{E} = \vec{E}^{ES} + \vec{E}^{EM} = -\nabla(\phi^{ES} + \phi^{EM}) - \frac{\partial\vec{A}}{\partial t} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \tag{13}$$

where $\phi = \phi^{ES} + \phi^{EM}$. Substituting Equation (13) into Equation (5), it yields

$$-\nabla^2\phi - \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = \frac{\rho_c}{\epsilon_0} \tag{14}$$

Substituting Equations (9) and (13) into Equation (8) to eliminate respectively the $\vec{B}$ and $\vec{E}$, it yields

$$-\nabla^2\vec{A} + \nabla(\nabla\cdot\vec{A}) = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right) \tag{15}$$

Note that we can add an arbitrary constant to the scalar potential ϕ without changing the electric field $\vec{E}$. Likewise, we can add an arbitrary curl free function to the vector potential $\vec{A}$ without changing the magnetic field $\vec{B}$. Let us consider the following two well-known gauges.

Coulomb gauge:

For Coulomb gauge, we choose

$$\nabla\cdot\vec{A} = 0 \tag{16}$$

Equation (14) can be rewritten as

$$\nabla^2\phi = \nabla^2\phi^{ES} = -\frac{\rho_c}{\epsilon_0} \tag{17}$$

Where we have chosen $\phi^{EM} = 0$. Equation (15) can be rewritten as

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\nabla\phi^{ES}}{\partial t} \tag{18}$$

For a given charge density distribution $\rho_c(\vec{x},t)$, we can solve Equation (17) to obtain ϕ^{ES}. Substituting the resulting ϕ^{ES} into Equation (18) and for a given electric current density distribution $\vec{J}(\vec{x},t)$, we can solve the wave equation of the vector potential $\vec{A}$ in Equation (18) with non-zero source term.

For $\rho_c(\vec{x},t) = \vec{J}(\vec{x},t) = 0$, Equations (17) and (18) are reduced to, respectively,

$$\phi^{ES} = 0 \tag{19}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0 \tag{20}$$

Lorentz gauge:

Equation (15) can be rewritten as

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) \tag{21}$$

For Lorentz gauge, we choose

$$\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \tag{22}$$

So that Equation (21) becomes

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \tag{23}$$

Substituting Equation (22) into Equation (14) to eliminate $\nabla\cdot\vec{A}$, it yields

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho_c}{\epsilon_0} \tag{24}$$

For a given charge density distribution $\rho_c(\vec{x},t)$, we can solve the wave equation of the potential ϕ in Equation (24) with non-zero source term. For a given electric current density distribution $\vec{J}(\vec{x},t)$, we can solve the wave equation of the vector potential $\vec{A}$ in Equation (23) with non-zero source term.

For $\rho_c(\vec{x},t)=\vec{J}(\vec{x},t)=0$, Equations (23) and (24) are reduced to, respectively,

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=0 \tag{25}$$

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0 \tag{26}$$

在一個自洽的系統，這兩個帶電粒子在電磁場中運動，會改變 charge density ρ_c and electric current density $\vec{J}$ 的大小與分佈。但是，若考慮兩個帶電粒子是在一個外加的強磁場$\vec{B}(\vec{x},t)$與強電場$\vec{E}(\vec{x},t)$中運動，則帶電粒子運動所產生的擾動電場與磁場相對很弱，可以忽略不計。因此考慮粒子的運動時，就可以將外加場當成已知條件，直接解粒子的運動。也因此可以寫得出一個 Lagrangian 來描述這個系統。否則僅僅知道 Lagrangian 還不夠，還是要將由 Lagrange's equations 求出來的粒子位置與速度，換算出新的電荷與電流的密度分佈，然後帶回 Equations (17)-(18) or (23)-(24)求解新的 $\vec{A}$ and ϕ 。再將更新後的場帶回 Lagrange's equations 求粒子未來的位置與速度。

描述 2：(傳統純量形式)

Governing Equations(1)~(4) 可寫成以下分量的形式

$$\frac{dx_1}{dt}=v_{x1} \tag{27}$$

$$\frac{dy_1}{dt}=v_{y1} \tag{28}$$

$$\frac{dz_1}{dt}=v_{z1} \tag{29}$$

$$\frac{dx_2}{dt}=v_{x2} \tag{30}$$

$$\frac{dy_2}{dt}=v_{y2} \tag{31}$$

$$\frac{dz_2}{dt}=v_{z2} \tag{32}$$

$$m_1\frac{dv_{x1}}{dt}=q_1[E_x(x_1,y_1,z_1,t)+v_{y1}(t)B_z(x_1,y_1,z_1,t)-v_{z1}(t)B_y(x_1,y_1,z_1,t)] \tag{33}$$

$$m_1\frac{dv_{y1}}{dt}=q_1[E_y(x_1,y_1,z_1,t)+v_{z1}(t)B_x(x_1,y_1,z_1,t)-v_{x1}(t)B_z(x_1,y_1,z_1,t)] \tag{34}$$

$$m_1\frac{dv_{z1}}{dt}=q_1[E_z(x_1,y_1,z_1,t)+v_{x1}(t)B_y(x_1,y_1,z_1,t)-v_{y1}(t)B_x(x_1,y_1,z_1,t)] \tag{35}$$

$$m_2\frac{dv_{x2}}{dt}=q_2[E_x(x_2,y_2,z_2,t)+v_{y2}(t)B_z(x_2,y_2,z_2,t) \\ -v_{z2}(t)B_y(x_2,y_2,z_2,t)] \tag{36}$$

$$m_2\frac{dv_{y2}}{dt}=q_2[E_y(x_2,y_2,z_2,t)+v_{z2}(t)B_x(x_2,y_2,z_2,t)-v_{x2}(t)B_z(x_2,y_2,z_2,t)] \quad (37)$$

$$m_2\frac{dv_{z2}}{dt}=q_2[E_z(x_2,y_2,z_2,t)+v_{x2}(t)B_y(x_2,y_2,z_2,t)-v_{y2}(t)B_x(x_2,y_2,z_2,t)] \quad (38)$$

如果已知 vector potential $\vec{A}$ and scalar potential ϕ 則 Equations (33)~(38)中，磁場分量可由 Equation (9)得

$$B_x=\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z} \quad (39)$$

$$B_y=\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x} \quad (40)$$

$$B_z=\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y} \quad (41)$$

電場場分量可由 Equation (13) 得

$$E_x=-\frac{\partial \phi}{\partial x}-\frac{\partial A_x}{\partial t} \quad (42)$$

$$E_y=-\frac{\partial \phi}{\partial y}-\frac{\partial A_y}{\partial t} \quad (43)$$

$$E_z=-\frac{\partial \phi}{\partial z}-\frac{\partial A_z}{\partial t} \quad (44)$$

如果已知電場與磁場，就可以直接解 Equations (27)~(38).

描述 3：（Lagrangian 純量形式）

先回顧一下，根據上星期的補充講義，一個$\gamma=1$的帶電粒子在電磁場中運動，它的 Lagrangian 可寫為（忽略靜止能量mc^2那一項）

$$L=T-q\phi+q\vec{A}\cdot\vec{v}$$

現在我們要考慮兩個粒子（可推廣到 N 個粒子）：

考慮兩個速度大小遠小於光速的帶電粒子在電磁場中運動。的帶電粒子在電磁場中運動，若已知 vector potential $\vec{A}$ and scalar potential ϕ 則系統的 Lagrangian 可寫為

$$\begin{aligned}
L(x_1,y_1,z_1,x_2,y_2,z_2,\dot{x}_1,\dot{y}_1,\dot{z}_1,\dot{x}_2,\dot{y}_2,\dot{z}_2,t)
&=\frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2+\dot{z}_1^2)+\frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2+\dot{z}_2^2)\\
&-q_1\phi(x_1,y_1,z_1,t)-q_2\phi(x_2,y_2,z_2,t)\\
&+q_1\big[\dot{x}_1A_x(x_1,y_1,z_1,t)+\dot{y}_1A_y(x_1,y_1,z_1,t)\\
&+\dot{z}_1A_z(x_1,y_1,z_1,t)\big]\\
&+q_1\big[\dot{x}_2A_x(x_2,y_2,z_2,t)+\dot{y}_2A_y(x_2,y_2,z_2,t)\\
&+\dot{z}_2A_z(x_2,y_2,z_2,t)\big]
\end{aligned} \quad (45)$$

一但得到 Lagrangian L，接下來寫出 Lagrange's equations 就是固定的格式，所以在文章中描述此系統時，只要打一個式(45)就夠了，可以少打很多方程式。

我們可以證明，「描述 2」與「描述 3」其實在描述同樣的物理系統。顯然，同樣是純量形式的「描述 3」要比「描述 2」精簡多了！當然向量形式的「描述 1」，也相當精簡（只有四個方程式 Equations (1)~(4)）。所以只要學會正確的標示向量符號，也可以偷懶少打一些字。

HomeWork: 請利用 Lagrange's equations 證明，「描述 2」與「描述 3」其實在描述同樣一組運動方程式

補充說明： Substituting Equations (9) and (13) into a momentum equation

$$m\frac{d\vec{v}}{dt} = q\left[\vec{E} + \vec{v} \times \vec{B}\right]$$

it yields

$$m\frac{d\vec{v}}{dt} = q\left[\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right) + \vec{v} \times \left(\nabla \times \vec{A}\right)\right] = q\left[-\nabla\phi - \frac{\partial\vec{A}}{\partial t} - \vec{v}\cdot\nabla\vec{A} + \nabla\left(\vec{v}\cdot\vec{A}\right)\right]$$

or

$$\frac{d}{dt}(m\vec{v} + q\vec{A}) = -\nabla(q\phi - q\vec{v}\cdot\vec{A}) \tag{46}$$

Thus, the generalized momentum and potential are respectively $m\vec{v} + q\vec{A}$ and $q\phi - q\vec{v}\cdot\vec{A}$.

C.2. 有阻尼力的自由落體運動

考慮一個有阻尼力的自由落體牛頓力學方程式

$$m\ddot{z} = -b\dot{z} - mg \tag{1}$$

嘗試找 一個 Lagrangian 來描述這個系統

嘗試一：

$$T = \frac{1}{2}m\dot{z}^2 \tag{2}$$

$$V = mgz + bz\dot{z} \tag{3}$$

則 Lagrangian 為

$$L = T - V = \frac{1}{2}m\dot{z}^2 - mgz - bz\dot{z} \tag{4}$$

則 Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

可寫成

$$\frac{d}{dt}(m\dot{z} - bz) - (-mg - b\dot{z}) = m\ddot{z} - b\dot{z} + mg + b\dot{z} = m\ddot{z} + mg = 0 \tag{5}$$

因為 Equation (5) 與 Equation (1) 不相同，所以 Equation (4) 這個 Lagrangian 是錯的。

嘗試二：

令 Lagrangian 為

$$L = \left(\frac{1}{2}m\dot{z}^2 - mgz\right)\exp\left(\frac{b}{m}t\right) \tag{6}$$

則 Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

可寫成

$$\begin{aligned} &\frac{d}{dt}\left[m\dot{z}\exp\left(\frac{b}{m}t\right)\right] - \left[-mg\exp\left(\frac{b}{m}t\right)\right] \\ &= m\ddot{z}\exp\left(\frac{b}{m}t\right) + m\dot{z}\left(\frac{b}{m}\right)\exp\left(\frac{b}{m}t\right) + mg\exp\left(\frac{b}{m}t\right) \\ &= (m\ddot{z} + b\dot{z} + mg)\exp\left(\frac{b}{m}t\right) = 0 \end{aligned} \tag{7}$$

Since $\exp(bt/m) \neq 0$, it yields

$$m\ddot{z} + b\dot{z} + mg = 0 \tag{8}$$

因為 Equation (8) 與 Equation (1) 相同，所以 Equation (6) 這個 Lagrangian 是對的。

C.3. Relativistic Lagrangian Mechanics [1]

The relativistic action S is a Lorentz scalar [2]

$$S = \int L^{*} d\tau = \int L dt = \int L\gamma d\tau$$

For $V = 0$, let L^{*} also be a Lorentz scalar. It yields $L^{*} = -mc^2 = \text{constant}$ [2]

Thus, we can define a relativistic Lagrangian

$$L = \frac{L^{*}}{\gamma} = -\frac{mc^2}{\gamma} - V$$

Since the relativistic kinetic energy is

$$T = (\gamma - 1)mc^2$$

it yields

$$\frac{-mc^2}{\gamma} = \frac{T - \gamma mc^2}{\gamma} = \frac{T}{\gamma} - mc^2$$

Thus, the relativistic Lagrangian can be rewritten as

$$L = -mc^2 + \frac{T}{\gamma} - V$$

For $\gamma \to 1$, it yields $L = -mc^2 + T - V$

（減去一個常數mc^2，不影響 Lagrange equation ）(See Non-uniqueness in [3])

Since

$$\frac{\partial L}{\partial \dot{q}_x} = p_x$$

for $V = 0$, it yields

$$\frac{\partial L}{\partial \dot{q}_x} = \frac{\partial}{\partial \dot{q}_x}\left(\frac{T}{\gamma}\right) = \frac{\partial}{\partial \dot{x}}\left(\frac{\gamma - 1}{\gamma} mc^2\right) = \frac{\partial}{\partial \dot{x}}\left(1 - \frac{1}{\gamma}\right) mc^2 = \frac{\partial}{\partial \dot{x}}\left(-\frac{1}{\gamma}\right) mc^2 = -mc^2 \frac{\partial}{\partial \dot{x}}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

$$= -\frac{1}{2} mc^2 \frac{-\frac{2\dot{x}}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \gamma m\dot{x} = \gamma m v_x = p_x$$

Thus, we have

$$p_x = \gamma m v_x$$

Likewise, we have

$$p_y = \gamma m v_y$$

and

$$p_z = \gamma m v_z$$

For $V = 0$, the relativistic Hamiltonian is

$$H = \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L = \sum_k p_k \dot{q}_k + \frac{mc^2}{\gamma} = \frac{p^2}{\gamma m} + \frac{mc^2}{\gamma} = \frac{p^2c^2 + m^2c^4}{\gamma mc^2} = \gamma mc^2 = mc^2 + T$$

（加上一個常數mc^2，不會影響 Hamiltonian equations ）

where the kinetic energy can be written as

$$T = (\gamma - 1)mc^2 = \left[\left(1 + \frac{p^2}{m^2c^2}\right)^{\frac{1}{2}} - 1\right] mc^2$$

One of the Hamiltonian equations yields

$$\frac{\partial H}{\partial p_x} = \dot{q}_x = \dot{x}$$

$$\frac{\partial H}{\partial p_x} = \frac{\partial T}{\partial p_x} = \frac{1}{2}\frac{\frac{2p_x}{m^2c^2}}{\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}}mc^2 = \frac{p_x}{\gamma m} = \dot{x} = v_x$$

Thus, we have

$$p_x = \gamma m v_x$$

Likewise, we have

$$p_y = \gamma m v_y$$

and

$$p_z = \gamma m v_z$$

C.4. Relativistic charged particle in an electromagnetic field [2, 4]

(example where $V \neq 0$)

在電磁場中(in Gaussian unit)[2]

$$\mathcal{L} = L/\gamma = -m_0 c^2 \sqrt{1-\beta^2} - q\phi + \frac{q}{c} \vec{A} \cdot \vec{v}$$

in the non-relat. limit $\simeq -m_0 c^2 + \frac{m_0}{2} v^2 - q\phi + \frac{q}{c} \vec{A} \cdot \vec{v}$

在電磁場中(in SI unit)[3]

$$L = -mc^2 + \frac{T}{\gamma} - V = -mc^2 + \frac{T}{\gamma} - q\phi + q\vec{A} \cdot \vec{v} = -mc^2 + \frac{(\gamma - 1)mc^2}{\gamma} - q\phi + q\vec{A} \cdot \vec{v}$$

$$= -\frac{mc^2}{\gamma} - q\phi + q\vec{A} \cdot \vec{v}$$

where

$$\frac{T}{\gamma} = \frac{\frac{m\gamma^2 v^2}{\gamma + 1}}{\gamma} = \frac{m\gamma v^2}{\gamma + 1}$$

For $\gamma \to 1$, it yields $T/\gamma = (mv^2)/2$ and

$$L = -mc^2 + \frac{T}{\gamma} - V = -mc^2 + \frac{mv^2}{2} - q\phi + q\vec{A} \cdot \vec{v}$$

For $\gamma > 1$

$$L = -\frac{mc^2}{\gamma} - q\phi + q\vec{A} \cdot \vec{v}$$

The particle's canonical (total) momentum is

$$\frac{\partial L}{\partial \dot{x}} = \gamma m\dot{x} + qA_x = P_x$$

$$\frac{\partial L}{\partial \dot{y}} = \gamma m\dot{y} + qA_y = P_y$$

$$\frac{\partial L}{\partial \dot{z}} = \gamma m\dot{z} + qA_z = P_z$$

The relativistic Hamiltonian is [4]

$$H(\vec{P}, \vec{x}, t) = \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L = \sum_k P_k \dot{q}_k - L = \left(\frac{p^2}{\gamma m} + q\vec{A} \cdot \vec{v}\right) + \left(\frac{mc^2}{\gamma} + q\phi - q\vec{A} \cdot \vec{v}\right)$$

$$= \frac{p^2 c^2 + m^2 c^4}{\gamma mc^2} + q\phi = \gamma mc^2 + q\phi = mc^2 \sqrt{1 + \frac{(\vec{P} - q\vec{A}) \cdot (\vec{P} - q\vec{A})}{m^2 c^2}} + q\phi$$

更多範例，可參考 Ref. [3].

References

[1] https://en.wikipedia.org/wiki/Relativistic_Lagrangian_mechanics
[2] https://itp.tugraz.at/~arrigoni/vorlesungen/elektrodynamik/scripts-elektro/actual/lagrangian-relativistic-invariant-all.pdf
[3] https://en.wikipedia.org/wiki/Lagrangian_mechanics
[4] https://en.wikipedia.org/wiki/Hamiltonian_mechanics

以下為自學參考網頁，許多範例可作為期末報告

https://en.wikipedia.org/wiki/Lagrangian_mechanics

Pendulum on a movable support
Two-body central force problem
Electromagnetism

https://en.wikipedia.org/wiki/Hamiltonian_mechanics