# Appendix B. Special Relativity 狹義相對論

## B.1. Relativistic equations of motion

We define the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$

The relativistic momentum is （這樣的定義是讓兩個粒子碰撞後的系統總動量守恆）

$$\vec{p} = \gamma m \vec{v}$$

Relativistic equations of motion are

$$\frac{d\vec{x}}{dt} = \vec{v}$$

$$\frac{d\vec{p}}{dt} = \frac{d(\gamma m \vec{v})}{dt} = \vec{F}$$

where the force can be

$$\vec{F} = m\vec{g} + q\left(\vec{E} + \vec{v} \times \vec{B}\right) + \cdots$$

Since both $\gamma$ and $\vec{v}$ are functions of time, we define a new variable $\vec{u}$.
Let

$$\vec{u} = \gamma \vec{v}$$

Thus, the equations of motion can be rewritten as

$$\frac{d\vec{x}}{dt} = \frac{\vec{u}}{\gamma}$$

$$m\frac{d\vec{u}}{dt} = m\vec{g} + q\left(\vec{E} + \frac{\vec{u}}{\gamma} \times \vec{B}\right) + \cdots$$

It can be easily shown that the Lorentz factor can be rewritten as

$$\gamma = \sqrt{1+\frac{u^2}{c^2}}$$

or

$$\gamma = \sqrt{1+\frac{p^2}{m^2c^2}}$$

Kinetic energy

$$T = (\gamma - 1)mc^2 = \frac{mu^2}{\gamma + 1}$$

For $\gamma \to 1$, $\vec{u} \approx \vec{v}$, we obtain the non-relativistic expression of the kinetic energy

$$T \approx \frac{mv^2}{2}$$

## B.2. The Lorentz transformations [1]

讓我們考慮一個 primed frame 相對一個 unprimed frame 以等速率 $v$ 沿 $x$-軸方向前進。則兩慣性系之間的時空轉換（Lorentz transformations）為

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad (1)$$

$$x' \approx \gamma(x - vt) \quad (2)$$

$$y' \approx y \quad (3)$$

$$z' \approx z \quad (4)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

is the Lorentz factor and $c$ is the light speed.

一個比較對稱的 Lorentz 轉換形式為

$$ct' = \gamma(ct - \beta x) \quad (5)$$

$$x' \approx \gamma(x - \beta ct) \quad (6)$$

$$y' \approx y \quad (7)$$

$$z' \approx z \quad (8)$$

where $\beta = v/c$. 其中 $[ct, x, y, z]$ 是 **Minkowski** 所建議的四度空間的位置座標。[2]

## B.3. Length contraction and time dilation [3]

Table 1 的結果顯示，因為 primed frame 系觀測者（運動中的觀測者），其實是在「不同時間」測量一個 unprimed frame（靜止系）中的物體兩端點 $x_1$ and $x_2$ 的位置，所以「運動中的觀測者」會覺得，沿著移動方向所看到的物體長度發生縮短現象（length contraction）。

Let $L$ be the length measured in the proper frame （物體所在運動座標系） and $L'$ be the length measured in the moving frame, then we have

$$L' = \frac{L}{\gamma} \quad (9)$$

Table 2 的結果顯示，同一位置 $x$ 傳過來的週期$T$ 閃燈，對於 primed frame 系觀測者（運動中的觀測者），會覺得此閃燈的週期增加了 $\gamma$ 倍的時間。這就是時間變慢現象（time dilation）。

Let $T$ be the proper time and $T'$ be the time measured in the moving frame, then we have

$$T' = \gamma T \quad (10)$$

**Table 1.** Length contraction

| Event | Proper frame | *Reasons* | Moving Frame |
|---|---|---|---|
| (1) | $x_1 = 0$ at $t = 0$ | Eqs. (1) & (2) yield | $x_1{}' = 0$ at $t' = 0$ |
| (2) | $x_2 = L$ at $t = 0$ | Eqs. (1) & (2) yield | $t' \neq 0$ |
| | $x_2 - x_1 = L$ at $t = 0$ | | |
| | | For $x_2 = L$ & $t' = 0$, Eq. (5) yields $t = vx_2/c^2 = vL/c^2$ | |
| (3) | $x_2 = L$ at $t = vL/c^2$ | Eqs. (1) & (2) yield | $x_2{}' = L/\gamma$ at $t' = 0$ |
| | | | $L' = x_2' - x_1' = L/\gamma$ at $t' = 0$ |

**Table 2.** Time dilation

| Event | Proper frame | *Reasons* | Moving Frame |
|---|---|---|---|
| (A) | $x = 0$ at $t_1 = 0$ | Eqs. (1) & (2) yield | $t_1' = 0$ & $x' = 0$ |
| (B) | $x = 0$ at $t_2 = T$ | Eqs. (1) & (2) yield | $t_2' = \gamma T$ & $x' = -\gamma vT$ |
| | $t_2 - t_1 = T$ | | $T' = t_2' - t_1' = \gamma T$ |

**References**

[1] https://en.wikipedia.org/wiki/Lorentz_transformation
[2] https://en.wikipedia.org/wiki/Minkowski_space
[3] https://en.wikipedia.org/wiki/Length_contraction

以下為自學參考網頁，可作為期末報告

Transformation of the E, B fields between inertial frames
https://en.wikipedia.org/wiki/Classical_electromagnetism_and_special_relativity

Relativistic Doppler effect
https://en.wikipedia.org/wiki/Relativistic_Doppler_effect

Minkowski space (or Minkowski spacetime)
https://en.wikipedia.org/wiki/Minkowski_space