Motions in a Rotating System

PART A

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Symon (1960):
Chapter 7: Moving Coordinate Systems

Part 1:

7-1 Moving origin of coordinates

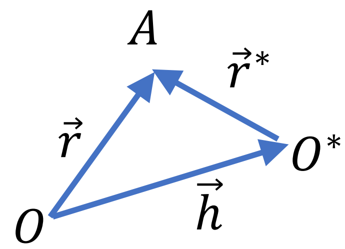
7-2 Rotating coordinate systems

7-3 Laws of motion on the rotating earth

7-4 The Foucault pendulum

7-5 Laimor's theorem (不重要，很少應用，所以不教）

Moving Origin of Coordinates
Fictitious Force



考慮一個**慣性系統**原點為 O ，以及一個沿著一個平面平行移動的系統原點為 O^* 。若 O^* 相對 O 的位置向量為 $\vec{h}(=\overrightarrow{OO^*})$，則以 O 為原點的**參考座標系**，以及以 O^* 為原點的**參考座標系**中的觀測者，看到一個質量為 m 的質點 A 的位置速度、加速度、與感受到的力，有何差異？

[**慣性系統** (inertial frame) **參考座標系**(frame of reference 或 reference frame)]

Reference frame	**慣性系 O**	**平移系 O^***
Position vector	$\vec{r}\ (=\vec{h}+\vec{r}^*)$ $(\overrightarrow{OA}=\overrightarrow{OO^*}+\overrightarrow{O^*A})$	$\vec{r}^*(=\vec{r}-\vec{h})$ $(\overrightarrow{O^*A}=\overrightarrow{OA}-\overrightarrow{OO^*})$
Velocity	$\vec{v}=\dfrac{d\vec{r}}{dt}$	$\vec{v}^*=\dfrac{d\vec{r}^*}{dt}=\dfrac{d\vec{r}}{dt}-\dfrac{d\vec{h}}{dt}=\vec{v}-\vec{v}_h$
Acceleration	$\vec{a}=\dfrac{d^2\vec{r}}{dt^2}$	$\vec{a}^*=\dfrac{d^2\vec{r}^*}{dt^2}=\dfrac{d^2\vec{r}}{dt^2}-\dfrac{d^2\vec{h}}{dt^2}=\vec{a}-\vec{a}_h$
Force	$\vec{F}=m\vec{a}$	$\vec{F}^*=m\vec{a}^*=\vec{F}-m\vec{a}_h$

其中 $-m\vec{a}_h$ 就是所謂的**假力** Fictitious Force。如果 $\vec{a}_h=0$ 則平移系 O^* 也是一個慣性系。

請問：如果你體重過重，擔心逃生繩斷裂，該如何利用逃生繩下降到地面逃生呢？

Rotating Coordinate Systems 原點位置重疊

考慮兩個**參考座標系** frames of reference。
一個是慣性系 O ，有三個方向不變的基底向量$\{\hat{x}, \hat{y}, \hat{z}\}$ 。
一個是旋轉系 O^* ，有三個方向一直在改變的基底向量$\{\hat{x}^*, \hat{y}^*, \hat{z}^*\}$
在此，先假設這兩個系統的原點位置一直重疊著。
若 O^* 以通過原點的一個軸旋轉著，其旋轉角速度為 $\vec{\omega}$ 。
則基底向量 $\hat{x}^*, \hat{y}^*, \hat{z}^*$ 都將繞著此旋轉軸以角速度 $\vec{\omega}$ 打轉。

慣性系 O 的觀測者描述	旋轉系 O^* 的觀測者描述
$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$	$\vec{r}^*(t) = x^*(t)\hat{x}^*(t) + y^*(t)\hat{y}^*(t) + z^*(t)\hat{z}^*(t)$
$\vec{v} = \frac{dx(t)}{dt}\hat{x} + \frac{dy(t)}{dt}\hat{y} + \frac{dz(t)}{dt}\hat{z}$	$\vec{v}^* = v_x^*(t)\hat{x}^*(t) + v_y^*(t)\hat{y}^*(t) + v_z^*(t)\hat{z}^*(t) = \frac{dx^*(t)}{dt}\hat{x}^*(t) + \frac{dy^*(t)}{dt}\hat{y}^*(t) + \frac{dz^*(t)}{dt}\hat{z}^*(t)$
$\vec{a} = \frac{d^2x(t)}{dt^2}\hat{x} + \frac{d^2y(t)}{dt^2}\hat{y} + \frac{d^2z(t)}{dt^2}\hat{z}$	$\vec{a}^* = a_x^*(t)\hat{x}^*(t) + a_y^*(t)\hat{y}^*(t) + a_z^*(t)\hat{z}^*(t) = \frac{d^2x^*(t)}{dt^2}\hat{x}^*(t) + \frac{d^2y^*(t)}{dt^2}\hat{y}^*(t) + \frac{d^2z^*(t)}{dt^2}\hat{z}^*(t)$

由上表可知，慣性系 O 與旋轉系 O^* 的原點相同，所以 $\vec{r}(t) = \vec{r}^*(t)$。但是因為旋轉系 O^* 的觀測者沒有考慮自己的基底座標隨時間在打轉，所以 $\vec{v}(t) \neq \vec{v}^*(t)$ 且 $\vec{a}(t) \neq \vec{a}^*(t)$ 。**學習目的**：設法找出 $\vec{v}(t), \vec{v}^*(t), \vec{a}(t), \vec{a}^*(t)$ 之間的關係。

Rotating Coordinate Systems 之基底向量 $\hat{x}^*, \hat{y}^*, \hat{z}^*$對時間的微分

現在先考慮一個通過原點的向量 $\vec{B}$ 繞著此旋轉軸以角速度 $\vec{\omega}$ 打轉

相對慣性系 O中的觀測者，向量 $\vec{B}$ 對時間的微分值，由課本 Fig. 7-3 幾何證明 。證明過程，很像球面座標中 因為 ϕ 隨時間改變，導致 $\hat{r}$ 隨時間改變。球面座標中，微分結果的方向沿著 $\hat{\phi}$ 方向。在 Fig. 7-3 中 $\vec{B}$ 對時間微分的方向沿著 $\vec{\omega}\times\vec{B}$ 的方向。剛好微分的大小也是 $\omega B\sin\theta$ 。因此可知，相對慣性系 O中的觀測者， $\vec{B}$ 對時間的微分 就是等於$\vec{\omega}\times\vec{B}$ 。

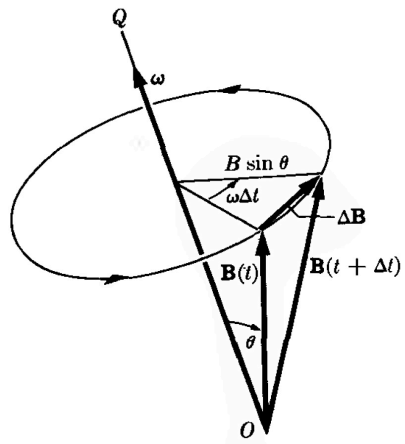


FIG. 7-3. Time derivative of a rotating vector.

$$\frac{d\vec{B}}{dt} = \lim_{\Delta t\to 0}\frac{\vec{B}(t+\Delta t)-\vec{B}(t)}{\Delta t} = \vec{\omega}\times\vec{B}$$

相對旋轉系 O^*中的觀測者， $\vec{B}$ 對時間的微分卻是零：

$$\frac{d^*\vec{B}}{dt} = 0$$

同理可證，相對 慣性系 O 中的觀測者，

基底向量 $\hat{x}^*, \hat{y}^*, \hat{z}^*$對時間微分的方向，分別為

$$\frac{d\hat{x}^*}{dt} = \vec{\omega}\times\hat{x}^* \qquad \frac{d\hat{y}^*}{dt} = \vec{\omega}\times\hat{y}^* \qquad \frac{d\hat{z}^*}{dt} = \vec{\omega}\times\hat{z}^*$$

相對 旋轉系 O^* 中的觀測者， 基底向量 $\hat{x}^*, \hat{y}^*, \hat{z}^*$對時間微分都是零。

$$\frac{d^*\hat{x}^*}{dt} = \frac{d^*\hat{y}^*}{dt} = \frac{d^*\hat{z}^*}{dt} = 0$$

Rotating Coordinate Systems 原點位置重疊

已知 $\frac{d\hat{x}^*}{dt}=\vec{\omega}\times\hat{x}^*$ $\frac{d\hat{y}^*}{dt}=\vec{\omega}\times\hat{y}^*$ $\frac{d\hat{z}^*}{dt}=\vec{\omega}\times\hat{z}^*$

慣性系 O 的觀測者描述	旋轉系 O^* 的觀測者描述
$\vec{r}(t)=x(t)\hat{x}+y(t)\hat{y}+z(t)\hat{z}$	$\vec{r}^*(t)=x^*(t)\hat{x}^*(t)+y^*(t)\hat{y}^*(t)+z^*(t)\hat{z}^*(t)$
$\vec{v}(t)=\frac{dx(t)}{dt}\hat{x}+\frac{dy(t)}{dt}\hat{y}+\frac{dz(t)}{dt}\hat{z}$	$\vec{v}^*(t)=v_x^*(t)\hat{x}^*(t)+v_y^*(t)\hat{y}^*(t)+v_z^*(t)\hat{z}^*(t)$ $=\frac{dx^*(t)}{dt}\hat{x}^*(t)+\frac{dy^*(t)}{dt}\hat{y}^*(t)+\frac{dz^*(t)}{dt}\hat{z}^*(t)$
$\vec{a}(t)=\frac{d^2x(t)}{dt^2}\hat{x}+\frac{d^2y(t)}{dt^2}\hat{y}+\frac{d^2z(t)}{dt^2}\hat{z}$	$\vec{a}^*(t)=a_x^*(t)\hat{x}^*(t)+a_y^*(t)\hat{y}^*(t)+a_z^*(t)\hat{z}^*(t)$ $=\frac{d^2x^*(t)}{dt^2}\hat{x}^*(t)+\frac{d^2y^*(t)}{dt^2}\hat{y}^*(t)+\frac{d^2z^*(t)}{dt^2}\hat{z}^*(t)$

請設法找出 $\vec{v}(t)$, $\vec{v}^*(t)$, $\vec{a}(t)$, $\vec{a}^*(t)$ 之間的關係。

$$\vec{v}(t)=\frac{d\vec{r}(t)}{dt}=\frac{d\vec{r}^*(t)}{dt}$$

$$=\left[\frac{dx^*(t)}{dt}\hat{x}^*(t)+\frac{dy^*(t)}{dt}\hat{y}^*(t)+\frac{dz^*(t)}{dt}\hat{z}^*(t)\right]+\left[x^*(t)\frac{d\hat{x}^*(t)}{dt}+y^*(t)\frac{d\hat{y}^*(t)}{dt}+z^*(t)\frac{d\hat{z}^*(t)}{dt}\right]$$

$$=[\vec{v}^*(t)]+[x^*(t)(\vec{\omega}\times\hat{x}^*)+y^*(t)(\vec{\omega}\times\hat{y}^*)+z^*(t)(\vec{\omega}\times\hat{z}^*)]$$

$$=\vec{v}^*(t)+\vec{\omega}\times\vec{r}^*=\vec{v}^*(t)+\vec{\omega}\times\vec{r}$$

Rotating Coordinate Systems 原點位置重疊

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}\{\vec{v}^*(t) + \vec{\omega}\times\vec{r}\} = \left\{\frac{d\vec{v}^*(t)}{dt}\right\} + \left\{\frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}(t)}{dt}\right\}$$

其中

$$\frac{d\vec{v}^*(t)}{dt} = \frac{d}{dt}\left\{\frac{dx^*(t)}{dt}\hat{x}^*(t) + \frac{dy^*(t)}{dt}\hat{y}^*(t) + \frac{dz^*(t)}{dt}\hat{z}^*(t)\right\}$$

$$= \left[\frac{d^2x^*(t)}{dt^2}\hat{x}^*(t) + \frac{d^2y^*(t)}{dt^2}\hat{y}^*(t) + \frac{d^2z^*(t)}{dt^2}\hat{z}^*(t)\right]$$

$$+ \left[\frac{dx^*(t)}{dt}\frac{d\hat{x}^*(t)}{dt} + \frac{dy^*(t)}{dt}\frac{d\hat{y}^*(t)}{dt} + \frac{dz^*(t)}{dt}\frac{d\hat{z}^*(t)}{dt}\right]$$

$$= \vec{a}^*(t) + \left[v_x^*(t)(\vec{\omega}\times\hat{x}^*) + v_y^*(t)(\vec{\omega}\times\hat{y}^*) + v_z^*(t)(\vec{\omega}\times\hat{z}^*)\right]$$

$$= \vec{a}^*(t) + \vec{\omega}\times\vec{v}^*$$

and

$$\frac{d\vec{r}(t)}{dt} = \vec{v}(t) = \vec{v}^*(t) + \vec{\omega}\times\vec{r}$$

Thus

$$\vec{a}(t) = \{\vec{a}^*(t) + \vec{\omega}\times\vec{v}^*\} + \left\{\frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times[\vec{v}^*(t) + \vec{\omega}\times\vec{r}]\right\}$$

$$= \vec{a}^*(t) + 2\vec{\omega}\times\vec{v}^* + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{d\vec{\omega}}{dt}\times\vec{r}$$

總結 Rotating Coordinate Systems 原點位置重疊

已知 $\frac{d\hat{x}^*}{dt} = \vec{\omega}\times\hat{x}^*$ $\frac{d\hat{y}^*}{dt} = \vec{\omega}\times\hat{y}^*$ $\frac{d\hat{z}^*}{dt} = \vec{\omega}\times\hat{z}^*$

已知：慣性系 O 的觀測者描述	已知：旋轉系 O^* 的觀測者描述
$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$	$\vec{r}^*(t) = x^*(t)\hat{x}^*(t) + y^*(t)\hat{y}^*(t) + z^*(t)\hat{z}^*(t)$
$\vec{v}(t) = \frac{dx(t)}{dt}\hat{x} + \frac{dy(t)}{dt}\hat{y} + \frac{dz(t)}{dt}\hat{z}$	$\vec{v}^*(t) = v_x^*(t)\hat{x}^*(t) + v_y^*(t)\hat{y}^*(t) + v_z^*(t)\hat{z}^*(t)$ $= \frac{dx^*(t)}{dt}\hat{x}^*(t) + \frac{dy^*(t)}{dt}\hat{y}^*(t) + \frac{dz^*(t)}{dt}\hat{z}^*(t)$
$\vec{a}(t) = \frac{d^2x(t)}{dt^2}\hat{x} + \frac{d^2y(t)}{dt^2}\hat{y} + \frac{d^2z(t)}{dt^2}\hat{z}$	$\vec{a}^*(t) = a_x^*(t)\hat{x}^*(t) + a_y^*(t)\hat{y}^*(t) + a_z^*(t)\hat{z}^*(t)$ $= \frac{d^2x^*(t)}{dt^2}\hat{x}^*(t) + \frac{d^2y^*(t)}{dt^2}\hat{y}^*(t) + \frac{d^2z^*(t)}{dt^2}\hat{z}^*(t)$
求出 $\vec{v}(t)$, $\vec{v}^*(t)$, $\vec{a}(t)$, $\vec{a}^*(t)$ 之間的關係	求出 $\vec{v}(t)$, $\vec{v}^*(t)$, $\vec{a}(t)$, $\vec{a}^*(t)$ 之間的關係
$\vec{v}(t) = \vec{v}^*(t) + \vec{\omega}\times\vec{r}$	$\vec{v}^*(t) = \vec{v}(t) - \vec{\omega}\times\vec{r}$
$\vec{a}(t) = \vec{a}^*(t) + 2\vec{\omega}\times\vec{v}^* + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{d\vec{\omega}}{dt}\times\vec{r}$	$\vec{a}^*(t) = \vec{a}(t) - 2\vec{\omega}\times\vec{v}^* - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \frac{d\vec{\omega}}{dt}\times\vec{r}$
$\vec{F}(t) = m\vec{a}(t)$	$m\vec{a}^*(t)$ $= \vec{F}(t) - 2m\vec{\omega}\times\vec{v}^* - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\frac{d\vec{\omega}}{dt}\times\vec{r}$ 科氏力 離心力 ＝真實力＋Coriolis force ＋ Centrifugal force ＋另一種假力

Rotating Coordinate Systems 原點位置不重疊

$$\mathbf{r} = \mathbf{r}^* + \mathbf{h}, \tag{7–38}$$

$$\frac{d\mathbf{r}}{dt} = \frac{d^*\mathbf{r}^*}{dt} + \boldsymbol{\omega} \times \mathbf{r}^* + \frac{d\mathbf{h}}{dt}, \tag{7–39}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^{*2}\mathbf{r}^*}{dt^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^*) + 2\boldsymbol{\omega} \times \frac{d^*\mathbf{r}^*}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}^* + \frac{d^2\mathbf{h}}{dt^2}. \tag{7–40}$$

$$\vec{r}^* = \vec{r} - \vec{h}$$

$$\frac{d^*\vec{r}^*}{dt} = \frac{d\vec{r}}{dt} - \vec{\omega}\times\vec{r}^* - \frac{d\vec{h}}{dt}$$

$$\vec{v}^* = \vec{v} - \vec{\omega}\times\vec{r}^* - \vec{v}_h$$

$$\frac{d^{*2}\vec{r}^*}{dt^2} = \frac{d^2\vec{r}}{dt^2} - \vec{\omega}\times(\vec{\omega}\times\vec{r}^*) - 2\vec{\omega}\times\frac{d^*\vec{r}^*}{dt} - \frac{d\vec{\omega}}{dt}\times\vec{r}^* - \frac{d^2\vec{h}}{dt^2}$$

$$\vec{a}^* = \vec{a} - -\vec{\omega}\times(\vec{\omega}\times\vec{r}^*) - 2\vec{\omega}\times\vec{v}^* - \frac{d\vec{\omega}}{dt}\times\vec{r}^* - \vec{a}_h$$

7–3 Laws of motion on the rotating earth. We write the equation of motion, relative to a coordinate system fixed in space, for a particle of mass m subject to a gravitational force $m\mathbf{g}$ and any other nongravitational forces $\mathbf{F}$:

$$m \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} + m\mathbf{g}. \tag{7–41}$$

Now if we refer the motion of the particle to a coordinate system at rest relative to the earth, which rotates with constant angular velocity $\boldsymbol{\omega}$, and if we measure the position vector $\mathbf{r}$ from the center of the earth, we have, by Eq. (7–34):

$$\begin{aligned} m \frac{d^2\mathbf{r}}{dt^2} &= \mathbf{F} + m\mathbf{g} \\ &= m \frac{d^{*2}\mathbf{r}}{dt^2} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\boldsymbol{\omega} \times \frac{d^*\mathbf{r}}{dt}, \end{aligned} \tag{7–42}$$

which can be rearranged in the form

$$m \frac{d^{*2}\mathbf{r}}{dt^2} = \mathbf{F} + m[\mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] - 2m\boldsymbol{\omega} \times \frac{d^*\mathbf{r}}{dt}. \tag{7–43}$$

This equation has the same form as Newton's equation of motion. We have combined the gravitational and centrifugal force terms because both are proportional to the mass of the particle and both depend only on the position of the particle; in their mechanical effects these two forces are indistinguishable. We may define the effective gravitational acceleration $\mathbf{g}_e$ at any point on the earth's surface by:

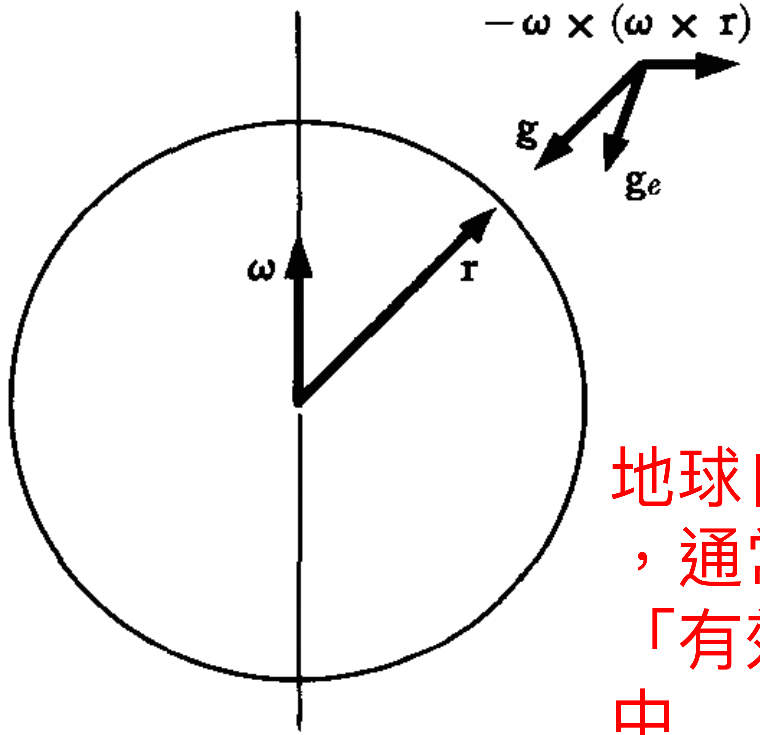
$$\mathbf{g}_e(\mathbf{r}) = \mathbf{g}(\mathbf{r}) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \tag{7–44}$$

The gravitational force which we measure experimentally on a body of mass m at rest† on the earth's surface is $m\mathbf{g}_e$. Since $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points radially outward from the earth's axis, $\mathbf{g}_e$ at every point north of the equator will point slightly to the south of the earth's center, as can be seen from Fig. 7–5. A body released near the earth's surface will begin to fall in the direction of $\mathbf{g}_e$, the direction determined by a plumb line is that of $\mathbf{g}_e$, and a liquid will come to equilibrium with its surface perpendicular to $\mathbf{g}_e$. This is why the earth has settled into equilibrium in the form of an oblate ellipsoid, flattened at the poles. The degree of flattening is just such as to make the earth's surface at every point perpendicular to $\mathbf{g}_e$ (ignoring local irregularities).

Equation (7–43) can now be written

$$m\frac{d^{*2}\mathbf{r}}{dt^2} = \mathbf{F} + m\mathbf{g}_e - 2m\boldsymbol{\omega} \times \frac{d^*\mathbf{r}}{dt}. \tag{7–45}$$

The velocity and acceleration which appear in this equation are unaffected if we relocate our origin of coordinates at any convenient point at the surface of the earth; hence this equation applies to the motion of a particle of mass m at the surface of the earth relative to a local coordinate system at rest on the earth's surface. The only unfamiliar term is the coriolis force



地球自轉的離心加速度，通常已經被納入「有效的重力加速度」中

FIG. 7–5. Effective acceleration of gravity on the rotating earth.

How ”large” is it?

The coriolis force is of major importance in the motion of large air masses, and is responsible for the fact that in the northern hemisphere ~~tornados~~ and cyclones circle in the direction south to east to north to west. In the northern hemisphere, the coriolis force acts to deflect a moving object toward the right. As the winds blow toward a low pressure area, they are deflected to the right, so that they circle the low pressure area in a counterclockwise direction. An air mass circling in this way will have a low pressure on its left, and a higher pressure on its right. This is just what is needed to balance the coriolis force urging it to the right. An air mass can move steadily in one direction only if there is a high pressure to the right of it to balance the coriolis force. Conversely, a pressure gradient over the surface of the earth tends to develop winds moving at right angles to it. The prevailing westerly winds in the northern temperate zone indicate that the atmospheric pressure toward the equator is greater than toward the poles, at least near the earth's surface. The easterly trade winds in the equatorial zone are due to the fact that any air mass moving toward the equator will acquire a velocity toward the west due to the coriolis force acting on it. The trade winds are maintained by high pressure areas on either side of the equatorial zone.

Exercise

1. 請分析地表赤道地區的觀測者，觀測以下質量為m的衛星所受到的柯氏力Coriolis force 與離心力 Centrifugal force 的方向與大小

a) 在慣性座標系的觀測者看此衛星，以24小時的週期，繞行地球赤道一圈

b) 在慣性座標系的觀測者看此衛星，以12 小時的週期， 繞行地球赤道一圈

c) 在慣性座標系的觀測者看此衛星，以48 小時的週期， 繞行地球赤道一圈

2. 請分析地表赤道地區的觀測者，觀測觀測無窮遠恆星所受到的柯氏力Coriolis force 與離心力 Centrifugal force 的方向與大小

佛科擺 Foucault pendulum 的擺面進動問題

7–4 The Foucault pendulum. An interesting application of the theory of rotating coordinate systems is the problem of the Foucault pendulum. The Foucault pendulum has a bob hanging from a string arranged to swing freely in any vertical plane. The pendulum is started swinging in a definite vertical plane and it is observed that the plane of swinging gradually precesses about the vertical axis during a period of several hours. The bob must be made heavy, the string very long, and the support nearly frictionless, in order that the pendulum can continue to swing freely for long periods of time. If we choose the origin of coordinates directly below the point of support, at the point of equilibrium of the pendulum bob of mass m, then the vector $\mathbf{r}$ will be nearly horizontal, for small amplitudes of oscillation of the pendulum. In the northern hemisphere, $\boldsymbol{\omega}$ points in the general direction indicated in Fig. 7–6, relative to the vertical. Writing $\boldsymbol{\tau}$ for the tension in the string, we have as the equation of motion of the bob, according to Eq. (7–45):

$$m\frac{d^{*2}\mathbf{r}}{dt^2} = \boldsymbol{\tau} + m\mathbf{g}_e - 2m\boldsymbol{\omega} \times \frac{d^*\mathbf{r}}{dt}. \qquad (7\text{–}46)$$

If the coriolis force were not present, this would be the equation for a simple pendulum on a nonrotating earth. The coriolis force is very small, less than 0.1% of the gravitational force if the velocity is 5 mi/hr or less, and its vertical component is therefore negligible in comparison with the gravitational force. (It is the vertical force which determines the magnitude of the tension in the string.) However, the horizontal component of the coriolis force is perpendicular to the velocity $d^*\mathbf{r}/dt$, and as there are no other forces in this direction when the pendulum swings to and fro, it can change the nature of the motion. Any force with a horizontal component perpendicular to $d^*\mathbf{r}/dt$ will make it impossible for the pendulum to continue to swing in a fixed vertical plane. In order to solve the problem including

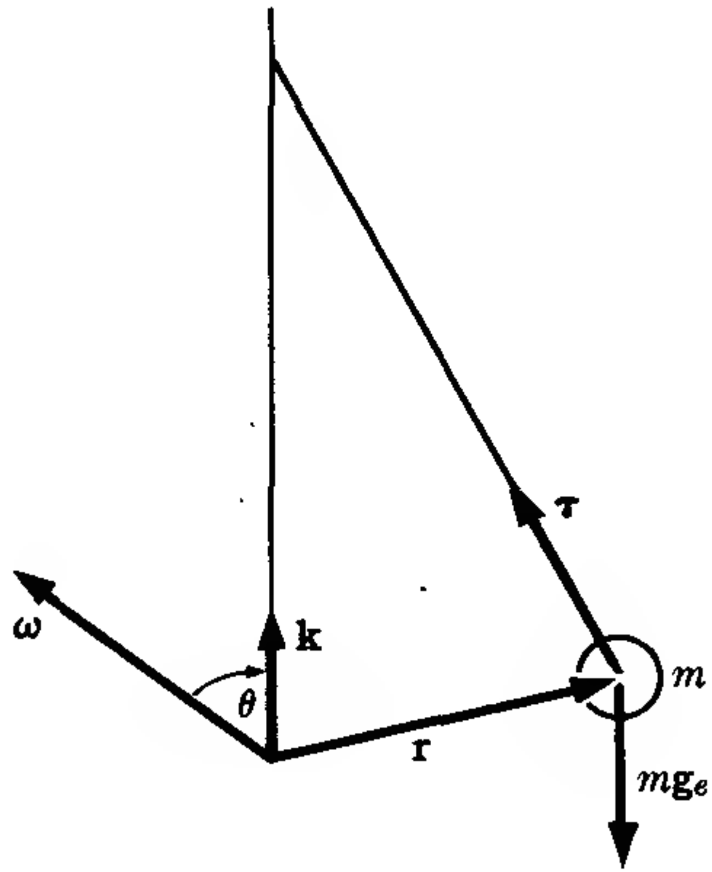


FIG. 7–6. The Foucault pendulum.

swing in a fixed vertical plane. In order to solve the problem including the coriolis term, we use the experimental result as a clue, and try to find a new coordinate system rotating about the vertical axis through the point of support at such an angular velocity that in this system the coriolis terms, or at least their horizontal components, are missing. Let us introduce a new coordinate system rotating about the vertical axis with constant angular velocity $\mathbf{k}\Omega$, where $\mathbf{k}$ is a vertical unit vector. We shall call this precessing coordinate system the primed coordinate system, and denote the time derivative with respect to this system by d'/dt. Then we shall have, by Eqs. (7–33) and (7–34):

找一個旋轉系，讓擺錘的水平受力為零，則此旋轉系的旋轉角速度，就是「佛科擺」擺面的進動角速度。

$$\frac{d^*\mathbf{r}}{dt} = \frac{d'\mathbf{r}}{dt} + \Omega\mathbf{k} \times \mathbf{r}, \tag{7–47}$$

$$\frac{d^{*2}\mathbf{r}}{dt^2} = \frac{d'^2\mathbf{r}}{dt^2} + \Omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{r}) + 2\Omega\mathbf{k} \times \frac{d'\mathbf{r}}{dt}. \tag{7–48}$$

Equation (7–46) becomes

$$\begin{aligned} m\frac{d'^2\mathbf{r}}{dt^2} &= \boldsymbol{\tau} + m\mathbf{g}_e - 2m\boldsymbol{\omega} \times \left(\frac{d'\mathbf{r}}{dt} + \Omega\mathbf{k} \times \mathbf{r}\right) \\ &\quad - m\Omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{r}) - 2m\Omega\mathbf{k} \times \frac{d'\mathbf{r}}{dt} \\ &= \boldsymbol{\tau} + m\mathbf{g}_e - 2m\Omega\boldsymbol{\omega} \times (\mathbf{k} \times \mathbf{r}) - m\Omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{r}) \\ &\quad - 2m(\boldsymbol{\omega} + \mathbf{k}\Omega) \times \frac{d'\mathbf{r}}{dt}. \end{aligned} \tag{7–49}$$

We expand the triple products by means of Eq. (3–35):

$$m \frac{d'^2\mathbf{r}}{dt^2} = \boldsymbol{\tau} + m\mathbf{g}_e - m(2\Omega\boldsymbol{\omega}\cdot\mathbf{r} + \Omega^2\mathbf{k}\cdot\mathbf{r})\mathbf{k}$$

$$+ m(2\Omega\mathbf{k}\cdot\boldsymbol{\omega} + \Omega^2)\mathbf{r} - 2m(\boldsymbol{\omega} + \mathbf{k}\Omega) \times \frac{d'\mathbf{r}}{dt}. \quad (7\text{–}50)$$

Every vector on the right side of Eq. (7–50) lies in the vertical plane containing the pendulum, except the last term. Since, for small oscillations, $d'\mathbf{r}/dt$ is practically horizontal, we can make the last term lie in this vertical plane also by making $(\boldsymbol{\omega} + \mathbf{k}\Omega)$ horizontal. We therefore require that

$$\mathbf{k}\cdot(\boldsymbol{\omega} + \mathbf{k}\Omega) = 0. \quad (7\text{–}51)$$

This determines Ω:

$$\Omega = -\omega \cos\theta, \quad (7\text{–}52)$$

where ω is the angular velocity of the rotating earth, Ω is the angular velocity of the precessing coordinate system relative to the earth, and θ is the angle between the vertical and the earth's axis, as indicated in Fig. 7–6.

Note that we have been able to give a fairly complete discussion of the Foucault pendulum, by using Coriolis' theorem twice, without actually solving the equations of motion at all.