Motion of a Particle in a Two- or Three-Dimensional System

PART-B

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Symon (1960): Chapter 3: Motion of a Particle in Two or Three Dimensions

“二元”或 “三元”「聯立的」 微分方程式 (System ODEs)

Part 3:

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Momentum & Energy Theorems

- **The (differential) momentum theorem:**

$$\frac{d\vec{p}}{dt} = \vec{F}$$

$$\Rightarrow \begin{cases} \dot{p}_x(t) = m\dot{v}_x(t) = F_x\,[x(t), y(t), z(t), v_x(t), v_y(t), v_z(t), t] \\ \dot{p}_y(t) = m\dot{v}_y(t) = F_y[x(t), y(t), z(t), v_x(t), v_y(t), v_z(t), t] \\ \dot{p}_z(t) = m\dot{v}_z(t) = F_z[x(t), y(t), z(t), v_x(t), v_y(t), v_z(t), t] \end{cases}$$

- **The (differential) Energy theorem:**

$$\frac{dT}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \vec{F} = v_x F_x + v_y F_y + v_z F_z$$

where $\vec{v} \cdot \vec{F}$ is the power

Plane and vector angular momentum theorems

Let $\vec{p}$ be the linear momentum of a particle and $\vec{F}$ be the force acting on the particle

$$\vec{p} = m\vec{v}$$

$$\frac{d\vec{p}}{dt} = \vec{F}$$

Let us consider the particle is gyrating around an axis. The distance between the particle and the axis is r . then the angular momentum of the particle with respect to the axis is

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

where $\vec{r} = \hat{r}r$ and $\hat{r}$ is the unit vector perpendicular to the axis and point from the axis toward the particle.

If there is a force $\vec{F}$ acting on the particle, then we have

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times \vec{p} + \vec{r} \times \vec{F}$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}$$

$$= \vec{r} \times \vec{F} = \vec{\tau}$$

where $\vec{\tau}$ is the torque that acting on the particle with respect to the axis

Angular momentum theorems (物理概念 中文補充說明–1)

當一個質點 m 繞著一個轉軸 $\hat{z}$ 以角速度 $\vec{\omega} = \omega\hat{z}$ 旋轉。則此質點瞬間的切線速度為 $\vec{v} = \vec{\omega}\times\vec{r}$ ，其中$\vec{r}$ 為質點到旋轉軸上任一點的距離。因為旋轉軸會軸通過支點，所以若取支點為原點，則$\vec{r}$ 的大小可以是質點的柱面座標的位置向量 $r\hat{r}_{柱} + z\hat{z}$ 中的 r ，也可以是質點的球面座標的位置向量 $r\hat{r}_{球}$中的 r 。所以，不管把支點設在哪裡，只要位在轉軸上，外積的結果都相同。

若 $\vec{p}$ 表示質點 m 的線動量 linear momentum

$$\vec{p} = m\vec{v} = m\vec{\omega}\times\vec{r}$$

但是當我們考慮質點 m 的角動量 時

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times m\vec{v}$$

這裡的 $\vec{r}$ 的選取就是一門藝術了！也就是說，選擇不同的原點（支點），算出來的角動量 $\vec{L}$ 大小與方向都不相同！

因為角動量隨時間的變化為力矩 torque（參考上頁證明），

$$\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F} = \vec{\tau}$$

因此，通常考慮一個質點的旋轉運動，原點通常選在讓質點進行旋轉運動，改變方向的施力源區。這樣，$\vec{r} \parallel \vec{F}$，因此 $\vec{r}\times\vec{F} = 0$ 。所以質點相對於施力源的角動量守恆 the angular momentum is conserved.

注意：如果旋轉軸的方向不變，則切線速度會一直改變方向，因此線動量也會一直改變方向，但是轉動角動量卻是一個守衡量。因此研究這種運動，考慮 角動量 angular momentum，比考慮 線動量 linear momentum 更為方便。

Angular Momentum & Rotational Inertia (or moment of inertia) (物理概念 中文補充說明-2)

角動量的問題，更多應用於一個由很多質點緊密結合的剛體的運動。如果我們可以把這樣一個剛體的運動，想成是 N 個質點以相同角速度繞著一個旋轉軸打轉的運動。則總線動量為

$$\vec{p} = \sum_{k=1}^{N} \vec{p}_k = \sum_{k=1}^{N} m_k \vec{v}_k = \vec{\omega} \times \sum_{k=1}^{N} m_k \vec{r}_k$$

總角動量為

$$\vec{L} = \sum_{k=1}^{N} \vec{L}_k = \sum_{k=1}^{N} \vec{r}_k \times (m_k \vec{v}_k)$$

$$= \sum_{k=1}^{N} \vec{r}_k \times (m_k \vec{\omega} \times \vec{r}_k)$$

$$= \vec{\omega} \cdot \sum_{k=1}^{N} m_k \left[(\vec{r}_k \cdot \vec{r}_k) \overleftrightarrow{1} - \vec{r}_k \vec{r}_k \right] = \overleftrightarrow{I} \cdot \vec{\omega}$$

其中 轉動慣量 rotational inertia $\overleftrightarrow{I}$ is a symmetric second-rank tensor.(所以 $\vec{\omega} \cdot \overleftrightarrow{I} = \overleftrightarrow{I} \cdot \vec{\omega}$)， $\overleftrightarrow{1}$ is the unit second-rank tensor. ($\overleftrightarrow{1} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$)

這裡的 $\vec{r}_k$ 的選取仍舊就是一門學問！也就是說，選擇不同的原點（支點），算出來的角動量 $\vec{L}_k$ 大小與方向都不相同！算出來的 轉動慣量 $\overleftrightarrow{I}$ 也會不同。為了要讓同一個剛體有相同的某一種轉動慣量，科學家定義以剛體重心為原點所算出來的轉動慣量為 $\overleftrightarrow{I}_G$ 。這樣做的好處是，當剛體自由落下時，受力點是重心，所以力臂長度為零，故力矩為零，因此又適用角動量守恆（**第11章介紹**）。但是如果剛體不是自由落下，而是有支點呢？

Angular momentum theorems
(物理概念 中文補充說明-3)

如果剛體不是自由落下，而是有支點時，但只要轉軸通過支點，也通過重心，則 $\vec{\vec{I}} = \vec{\vec{I}}_G$ (詳見下一頁的推導：若 $\vec{R} = R\hat{x}$ 則$M(R^2\vec{\vec{1}} - \vec{R}\vec{R})$ 不含$\hat{x}\hat{x}$分量)。但是此時整個系統受力與受力矩的情形，就比較複雜了。例如一個陀螺在打轉。支點所受的 力的力臂長度為零，因此力矩為零。但是如果轉軸稍微傾斜，則重力作用在重心上，距離支點，有一個長度不為零的力臂，就有一個力矩。所以角動量就會發生進動。

https://www.youtube.com/watch?v=ty9QSiVC2g0 Gyroscopic Precession	非常簡潔有趣的影片，簡單明瞭，但黑板上寫的 torque 的定義，寫錯了（應該是 $\vec{\tau} = \vec{r}\times\vec{F}$ 誤寫成 $\vec{\tau} = \vec{F}\times\vec{r}$ ）非常可惜。雖然講者說的都沒錯！
https://www.youtube.com/watch?v=XPUuF_dECVI	MIT 的一堂力學課 8.01x - Lect 24 - Rolling Motion, Gyroscopes
https://www.youtube.com/watch?v=p9zhP9Bnx-k	Gyroscope Tricks and Physics Stunts ~ Incredible Science 我們也會做的實驗
https://www.youtube.com/watch?v=3vPb3fwd9dQ	Prática: rotação, torque e momento angular 傳統物理課的實驗(2020–10)

Angular momentum theorems
(物理概念 中文補充說明-4)

如果剛體不是自由落下，而是有支點時 ，且轉軸通過支點，但是不通過重心時，則要考慮平行軸定理，重新計算一個等效的轉動慣量。若重心距離轉軸的距離為 $\vec{R}$ ，剛體上第 k 點相對重心的位置向量為 $\vec{r}_k^*$ ，則

$$\vec{\vec{I}} = \sum_{k=1}^{N} m_k \left[(\vec{r}_k \cdot \vec{r}_k)\vec{\vec{1}} - \vec{r}_k\vec{r}_k\right] = \sum_{k=1}^{N} m_k \left[(\vec{R} + \vec{r}_k^*) \cdot (\vec{R} + \vec{r}_k^*)\vec{\vec{1}} - (\vec{R} + \vec{r}_k^*)(\vec{R} + \vec{r}_k^*)\right]$$

$$= \left[\left(\sum_{k=1}^{N} m_k\right) R^2 + 2\left(\sum_{k=1}^{N} m_k \vec{r}_k^*\right) \cdot \vec{R}\right]\vec{\vec{1}} - \left(\sum_{k=1}^{N} m_k\right)\vec{R}\vec{R} - \left(\sum_{k=1}^{N} m_k \vec{r}_k^*\right)\vec{R}$$

$$- \vec{R}\left(\sum_{k=1}^{N} m_k \vec{r}_k^*\right) + \sum_{k=1}^{N} m_k \left[(\vec{r}_k^* \cdot \vec{r}_k^*)\vec{\vec{1}} - \vec{r}_k^*\vec{r}_k^*\right] = M(R^2\vec{\vec{1}} - \vec{R}\vec{R}) + \vec{\vec{I}}_G$$

where $\left(\sum_{k=1}^{N} m_k \vec{r}_k^*\right) = 0$ and $\left(\sum_{k=1}^{N} m_k\right) = M$. Let $\vec{R} = R\hat{x}$. It yields

$$\vec{\vec{I}} = M\left(R^2\vec{\vec{1}} - \vec{R}\vec{R}\right) + \vec{\vec{I}}_G = MR^2(\hat{y}\hat{y} + \hat{z}\hat{z}) + \vec{\vec{I}}_G$$

也就是說當角速度 $\vec{\omega}$ 有 $\hat{y}$ 或 $\hat{z}$ 分量時，且轉軸不通過重心，則

$$\vec{L} = \vec{\vec{I}} \cdot \vec{\omega} = MR^2\left(\omega_y\hat{y} + \omega_z\hat{z}\right) + \vec{\vec{I}}_G \cdot \vec{\omega}$$

The harmonic oscillator in two and three dimensions

Let us consider an object with mass m, which is connected to three springs in perpendicular directions. The spring constants of these springs in the x, y, and z-directions are k_1, k_2, and k_3, respectively. The equation of motion of the object can be written as

$$m\frac{d^2}{dt^2}(\hat{x}x + \hat{y}y + \hat{z}z)$$
$$= \hat{x}(-k_1 x) + \hat{y}(-k_2 y) + \hat{z}(-k_3 z)$$

Then we have

$$\ddot{x} = -(k_1/m)x$$
$$\ddot{y} = -(k_2/m)y$$
$$\ddot{z} = -(k_3/m)z$$

The solution of $x(t), y(t), z(t)$ can be written as

$$x(t) = A_1 \sin(\omega_1 t + \theta_1)$$
$$y(t) = A_2 \sin(\omega_2 t + \theta_2)$$
$$z(t) = A_3 \sin(\omega_3 t + \theta_3)$$

where

$$\omega_1^2 = k_1/m$$
$$\omega_1^2 = k_1/m$$
$$\omega_1^2 = k_1/m$$

Exercise: (Lissajous curves)

Plot $x(t), y(t)$ on the x-t, y-t, and x-y diagrams choose $A_1 = A_2 = 1, \omega_1 : \omega_2 = 1:1, \theta_1 = 0,$

$$\theta_2 = (a)0, (b)\frac{\pi}{4}, (c)\frac{\pi}{2}, (d)\frac{3\pi}{4}, (e)\frac{3\pi}{2}, (f)\frac{5\pi}{4}$$

or other combinations of these parameters.

$\mathrm{e.g.}, \omega_1 : \omega_2 = 1:1.01$

Projectiles 拋射體
Ballistic trajectory 彈道

Ignoring the air resistance

$$m\frac{d\vec{v}}{dt} = mg\hat{z}$$

Including the velocity dependent air resistance

$$m\frac{d\vec{v}}{dt} = mg\hat{z} - b\vec{v}$$

Given initial conditions of the projectile:

$$x = y = z = 0$$
$$v_x = v_0 \cos\theta$$
$$v_y = 0$$
$$v_z = v_0 \sin\theta$$

Find the solution of $x(t), z(t), v_x(t), v_z(t),$ and the trajectory $z(x)$ under different combinations of parameters: bv/mg and initial shooting angle θ

Potential Energy

If the curl of a force $\vec{F}$ is equal to zero, that is

$$\nabla \times \vec{F} = 0$$

Then, the force can be written in the following form

$$\vec{F} = -\nabla \Phi$$

Because $\nabla \times \nabla f = 0.$ (有兩種以上的方法，可以證明此式，請「試證之」)

The work done by the force should be independent of the path, and is equal to

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} -\nabla \Phi \cdot d\vec{r} = \Phi(\vec{r}_1) - \Phi(\vec{r}_2)$$

where Φ is the corresponding potential energy of the force.