Motion of a Particle in a Two- or Three-Dimensional System

PART-A

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Symon (1960): Chapter 3: Motion of a Particle in Two or Three Dimensions

“二元”或 “三元”「聯立的」 微分方程式 (System ODEs)

Part 1:

3-1 Vector algebra

3-3 Differentiation and integration of vectors

3-2 Applications to a set of forces acting on a particle

3-17 Motion of a particle in an electromagnetic field

帶電粒子在靜電場中的運動

帶電粒子在磁場中的迴旋運動，

帶電粒子在靜電場與磁場中的運動（如何求解：聯立ODEs)

（如何由運動軌跡，猜可能的答案）

Vector algebra

- 向量相加
- 向量相減
- 向量相乘
  - 向量內積
  - 向量外積
  - 向量平行積
- 張量
  - 純量（零階張量）
  - 向量（ 一階張量）
  - 二階張量
  - …

Differentiation of Vectors

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t + \Delta t) - \vec{A}(t)}{\Delta t}$$

$$\frac{d}{dt}\left(\vec{A} + \vec{B}\right) = \frac{d}{dt}\left(\vec{A}\right) + \frac{d}{dt}\left(\vec{B}\right)$$

$$\frac{d}{dt}\left(f\vec{A}\right) = \frac{df}{dt}\vec{A} + f\frac{d}{dt}\left(\vec{A}\right)$$

$$\frac{d}{dt}\left(\vec{A} \cdot \vec{B}\right) = \frac{d\vec{A}}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}\left(\vec{A} \times \vec{B}\right) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}\left(\vec{A}\vec{B}\right) = \frac{d\vec{A}}{dt}\vec{B} + \vec{A}\frac{d\vec{B}}{dt}$$

Integration of Vectors

$$W = \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \left(\frac{d\vec{r}}{ds}\right) ds$$

For

$$\vec{F} = \hat{x}F_x + \hat{y}F_y + \hat{z}F_z$$

$$\frac{d\vec{r}}{ds} = \hat{x}\frac{dx}{ds} + \hat{y}\frac{dy}{ds} + \hat{z}\frac{dz}{ds}$$

It yields, for a given trajectory $[x(s), y(s), z(s)]$

$$W = \int \vec{F} \cdot d\vec{r} = \int \left[F_x\frac{dx}{ds} + F_y\frac{dy}{ds} + F_z\frac{dz}{ds}\right] ds$$

Integration of Vectors

$$W = \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \left(\frac{d\vec{r}}{dt}\right) dt = \int (\vec{F} \cdot \vec{v}) dt$$

Thus

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v} = \text{Power}$$

Non-relativistic Particle Motion

$$\frac{d\vec{x}}{dt} = \vec{v}$$

$$\frac{d\vec{v}}{dt} = \vec{a}$$

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

$$\vec{p} = m\vec{v}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v} \cdot \vec{v}$$

Motion of a Charged Particle

- Let us consider a charge particle with mass m and charge q.
- Let us assume that the Lorentz force is much stronger than the the gravitational force, we shall ignore the gravitational force in the following discussion.
- 帶電粒子在均勻的靜電場 $\vec{E} = \hat{y}E_y$ 中的運動

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\vec{E} \Rightarrow \dot{v}_y = \left(\frac{q}{m}\right)E_y$$

- 帶電粒子在均勻的磁場 $\vec{B} = \hat{z}B_z$ 中的迴旋運動

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\vec{v}\times\vec{B} \Rightarrow \begin{cases} \dot{v}_x = \left(\frac{q}{m}\right)v_y B_z \\ \dot{v}_y = -\left(\frac{q}{m}\right)v_x B_z \end{cases}$$

- 帶電粒子在均勻的靜電場 $\vec{E} = \hat{y}E_y$ 與均勻的磁場 $\vec{B} = \hat{z}B_z$ 中的運動

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\left(\vec{E} + \vec{v}\times\vec{B}\right) \Rightarrow \begin{cases} \dot{v}_x = \left(\frac{q}{m}\right)v_y B_z \\ \dot{v}_y = \left(\frac{q}{m}\right)[E_y - v_x B_z] \end{cases}$$

Motion of a Charged Particle

- Let us consider a charge particle with mass m and charge q.
- Let us assume that the Lorentz force is much stronger than the the gravitational force, we shall ignore the gravitational force in the following discussion.
- 帶電粒子在均勻的靜電場 $\vec{E}$ 與均勻的磁場 $\vec{B}$ 中的運動 & $E/B \ll c$

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\left(\vec{E} + \vec{v}\times\vec{B}\right) \Rightarrow \frac{d^2\vec{v}}{dt^2} = \frac{q}{m}\left(\frac{d\vec{v}}{dt}\times\vec{B}\right)$$

$$\Rightarrow \frac{d^2\vec{v}}{dt^2} = \frac{q}{m}\left[\frac{q}{m}\left(\vec{E} + \vec{v}\times\vec{B}\right)\times\vec{B}\right] = \frac{q^2}{m^2}\vec{E}\times\vec{B} + \frac{q^2}{m^2}\left(\vec{v}\times\vec{B}\right)\times\vec{B}$$

$$\Rightarrow \frac{d^2\vec{v}}{dt^2} = \frac{q^2B^2}{m^2}\left[\frac{\vec{E}\times\vec{B}}{B^2} + \vec{v}\cdot\hat{B}\hat{B} - \vec{v}\right] = -\frac{q^2B^2}{m^2}\left[\vec{v}_\perp - \frac{\vec{E}\times\vec{B}}{B^2}\right]$$

where $\vec{v}_\perp = \vec{v} - \vec{v}_\parallel = \vec{v} - \vec{v}\cdot\hat{B}\hat{B} = \vec{v} - \vec{v}\cdot\frac{\vec{B}\vec{B}}{B^2}$. Let $\vec{V}_\perp = \vec{v}_\perp - \frac{\vec{E}\times\vec{B}}{B^2}$, it yields

$$\frac{d^2\vec{V}_\perp}{dt^2} = -\frac{q^2B^2}{m^2}\vec{V}_\perp$$

$$\frac{d^2\vec{v}_\parallel}{dt^2} = 0$$

Thus, we have $\vec{v}_\perp = \frac{\vec{E}\times\vec{B}}{B^2} + \vec{V}_\perp = \vec{v}_{E\times B} + \vec{v}_{gyro}$

$$\left(\vec{A}\times\vec{B}\right)\times\vec{C} = \left(\vec{A}\cdot\vec{C}\right)\vec{B} - \left(\vec{B}\cdot\vec{C}\right)\vec{A}$$

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“二元”或 “三元”「聯立的」 微分方程式 (System ODEs)

Part 2:

3-4 Kinematics in a plane

3-5 Kinematics in three dimensions

3-6 Elements of vector analysis

Kinematic Equations in Different Coordinate Systems

	Cartesian	Cylindrical	Spherical
Coord.	$[x(t), y(t), z(t)]$	$[r(t), \theta(t), z(t)]$	$[r(t), \theta(t), \phi(t)]$
position	$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$	$\vec{r}(t) = r(t)\hat{r}(\theta(t)) + z(t)\hat{z}$	$\vec{r}(t) = r(t)\hat{r}(\theta(t), \phi(t))$
$\vec{v} = \dot{\vec{r}}$ $\vec{a} = \ddot{\vec{r}}$	$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$	$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z}$ $\vec{a} = a_r\hat{r} + a_\theta\hat{\theta} + a_z\hat{z}$	$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}$ $\vec{a} = a_r\hat{r} + a_\theta\hat{\theta} + a_\phi\hat{\phi}$
	$v_x(t) = \dot{x}(t)$	$v_r = \dot{r}$	$v_r = \dot{r}$
	$v_y(t) = \dot{y}(t)$	$v_\theta = r\dot{\theta}$	$v_\theta = r\dot{\theta}$
	$v_z(t) = \dot{z}(t)$	$v_z = \dot{z}$	$v_\phi = r\sin\theta\,\dot{\phi}$
	$a_x(t) = \ddot{x}(t)$	$a_r = \ddot{r} - r\dot{\theta}^2$	$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2$
	$a_y(t) = \ddot{y}(t)$	$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$	$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2$
	$a_z(t) = \ddot{z}(t)$	$a_z = \ddot{z}$	$a_\phi = 2\dot{r}\sin\theta\,\dot{\phi} + 2r\cos\theta\,\dot{\theta}\dot{\phi} +$ $r\sin\theta\,\ddot{\phi}$

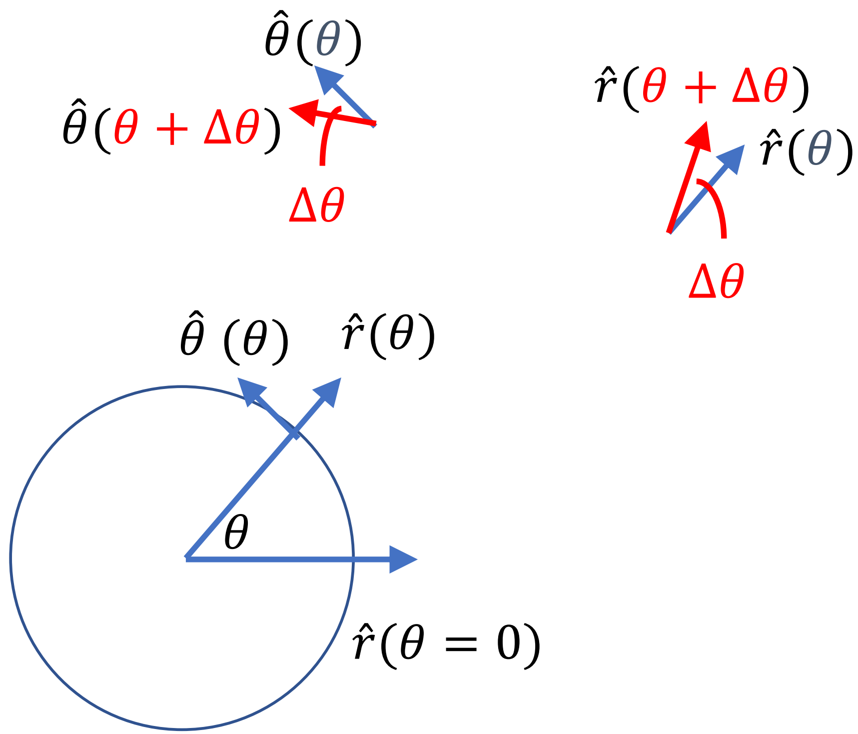
Definitions of Basis Vectors in Different Coordinate Systems

$$\hat{e}_1 = \frac{\nabla u_1}{|\nabla u_1|} = h_1 \nabla u_1$$

Cartesian coordinate system	$\hat{x} = \frac{\nabla x}{\lvert\nabla x\rvert} = \nabla x$	$\hat{y} = \frac{\nabla y}{\lvert\nabla y\rvert} = \nabla y$	$\hat{z} = \frac{\nabla z}{\lvert\nabla z\rvert} = \nabla z$
Cylindrical coordinate system	$\hat{r} = \frac{\nabla r}{\lvert\nabla r\rvert} = \nabla r$	$\hat{\theta} = \frac{\nabla \theta}{\lvert\nabla \theta\rvert} = r\nabla \theta$	$\hat{z} = \frac{\nabla z}{\lvert\nabla z\rvert} = \nabla z$
Spherical coordinate system	$\hat{r} = \frac{\nabla r}{\lvert\nabla r\rvert} = \nabla r$	$\hat{\theta} = \frac{\nabla \theta}{\lvert\nabla \theta\rvert} = r\nabla \theta$	$\hat{\phi} = \frac{\nabla \phi}{\lvert\nabla \phi\rvert} = r \sin\theta \, \nabla \phi$

用“幾何”的方式求 單位向量的微分結果

cylindrical coordinate system
$$\frac{d\hat{r}}{d\theta} = \hat{\theta}$$
$$\frac{d\hat{\theta}}{d\theta} = -\hat{r}$$



spherical coordinate system
$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$$
$$\frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$$
$$\frac{\partial\hat{r}}{\partial\phi} = \sin\theta\ \hat{\phi}$$
$$\frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\ \hat{\phi}$$
$$\frac{d\hat{\phi}}{d\phi} = -\sin\theta\ \hat{r} - \cos\theta\ \hat{\theta}$$

用“代數”的方式求 單位向量的微分結果

cylindrical coordinate system	spherical coordinate system
$\hat{r} = \cos\theta\ \hat{x} + \sin\theta\ \hat{y}$ $\hat{\theta} = -\sin\theta\ \hat{x} + \cos\theta\ \hat{y}$	$\hat{r} = \cos\theta\ \hat{z} + \sin\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$ $\hat{\theta} = -\sin\theta\ \hat{z} + \cos\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$ $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$
$\frac{d\hat{r}}{d\theta} = -\sin\theta\ \hat{x} + \cos\theta\ \hat{y} = \hat{\theta}$ $\frac{d\hat{\theta}}{d\theta} = -\cos\theta\ \hat{x} - \sin\theta\ \hat{y} = -\hat{r}$	$\frac{\partial\hat{r}}{\partial\theta} = -\sin\theta\ \hat{z} + \cos\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) = \hat{\theta}$ $\frac{\partial\hat{r}}{\partial\phi} = \sin\theta\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = \sin\theta\,\hat{\phi}$
	$\frac{\partial\hat{\theta}}{\partial\theta} = -\cos\theta\ \hat{z} - \sin\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) = -\hat{r}$ $\frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = \cos\theta\,\hat{\phi}$
	$\frac{d\hat{\phi}}{d\phi} = -\cos\phi\,\hat{x} - \sin\phi\,\hat{y}$
	$-\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$ $= -\sin\theta\,[\cos\theta\ \hat{z} + \sin\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})]$ $- \cos\theta\,[-\sin\theta\ \hat{z} + \cos\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})]$ $= -\cos\phi\,\hat{x} - \sin\phi\,\hat{y}$
	$\frac{d\hat{\phi}}{d\phi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$

Vector Analysis 直角坐標

$$d\boldsymbol{r} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

Volumn integration of a scalar field, such as the density field $\rho(\boldsymbol{r}) = \rho(x,y,z)$

$$\iiint [\rho(\boldsymbol{r})]d\boldsymbol{r} = \iiint [\rho(x,y,z)]\, d^3x = \iiint [\rho(x,y,z)]\, dx\, dy\, dz \quad (1)$$

Vector differential operator

$$\nabla = \frac{\partial}{\partial \boldsymbol{r}} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Given a scalar field $f(\boldsymbol{r}) = f(\vec{r}) = f(x,y,z)$, then the gradient of $f(\boldsymbol{r})$ is

$$\text{grad } f(\boldsymbol{r}) = \nabla f(\boldsymbol{r}) = \hat{x}\frac{\partial f(x,y,z)}{\partial x} + \hat{y}\frac{\partial f(x,y,z)}{\partial y} + \hat{z}\frac{\partial f(x,y,z)}{\partial z} \quad (2)$$

Given a vector field $\boldsymbol{A}(\boldsymbol{r}) = \vec{A}(\vec{r}) = \hat{x}A_x(x,y,z) + \hat{y}A_y(x,y,z) + \hat{z}A_z(x,y,z)$, then the divergence of $\boldsymbol{A}(\boldsymbol{r})$ is

$$\text{div } \boldsymbol{A}(\boldsymbol{r}) = \nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = \frac{\partial A_x(x,y,z)}{\partial x} + \frac{\partial A_y(x,y,z)}{\partial y} + \frac{\partial A_z(x,y,z)}{\partial z} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (3)$$

and the curl of $\boldsymbol{A}(\boldsymbol{r})$ is

$$\text{curl } \boldsymbol{A}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad (4)$$

Vector Analysis 柱面坐標

$$d\boldsymbol{r} = \hat{r}dr + \hat{\theta}rd\theta + \hat{z}dz$$

Volumn integration of a scalar field, such as the density field $\rho(\boldsymbol{r}) = \rho(r,\theta,z)$

$$\iiint [\rho(\boldsymbol{r})]d\boldsymbol{r} = \iiint [\rho(r,\theta,z)]\, d^3x = \iiint [\rho(r,\theta,z)]\, dr\, rd\theta\, dz \tag{5}$$

Vector differential operator

$$\nabla = \frac{\partial}{\partial \boldsymbol{r}} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{z}\frac{\partial}{\partial z}$$

Given a scalar field $f(\boldsymbol{r}) = f(\vec{r}) = f(r,\theta,z)$, then the gradient of $f(\boldsymbol{r})$ is

$$\text{grad } f(\boldsymbol{r}) = \nabla f(\boldsymbol{r}) = \hat{r}\frac{\partial f(r,\theta,z)}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f(r,\theta,z)}{\partial \theta} + \hat{z}\frac{\partial f(r,\theta,z)}{\partial z} \tag{6}$$

Given a vector field $\boldsymbol{A}(\boldsymbol{r}) = \vec{A}(\vec{r}) = \hat{r}(\theta)\, A_r(r,\theta,z) + \hat{\theta}(\theta)\, A_\theta(r,\theta,z) + \hat{z}\, A_z(r,\theta,z)$, then the divergence of $\boldsymbol{A}(\boldsymbol{r})$ is

$$\text{div } \boldsymbol{A}(\boldsymbol{r}) = \nabla\cdot\boldsymbol{A}(\boldsymbol{r}) = \frac{\partial A_r}{\partial r} + \frac{A_r}{r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \tag{7}$$

and the curl of $\boldsymbol{A}(\boldsymbol{r})$ is

$$\text{curl } \boldsymbol{A}(\boldsymbol{r}) = \nabla\times\boldsymbol{A}(\boldsymbol{r}) = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\theta}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} + \frac{A_\theta}{r}\right) \tag{8}$$

Vector Analysis 球面坐標

$$d\boldsymbol{r} = \hat{r}\,dr + \hat{\theta}\, r\, d\theta + \hat{\phi}\, r\sin\theta\, d\phi$$

Volumn integration of a scalar field, such as the density field $\rho(\boldsymbol{r}) = \rho(r,\theta,\phi)$

$$\iiint [\rho(\boldsymbol{r})]d\boldsymbol{r} = \iiint [\rho(r,\theta,\phi)]\, d^3x = \iiint [\rho(r,\theta,\phi)]\, dr\, rd\theta\, r\sin\theta\, d\phi = \iiint [\rho(r,\theta,\phi)]\, r^2\sin\theta\;\, dr\, d\theta\, d\phi \qquad (9)$$

Vector differential operator

$$\nabla = \frac{\partial}{\partial \boldsymbol{r}} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Given a scalar field $f(\boldsymbol{r}) = f(\vec{r}) = f(r,\theta,\phi)$, then the gradient of $f(\boldsymbol{r})$ is

$$\text{grad}\, f(\boldsymbol{r}) = \nabla f(\boldsymbol{r}) = \hat{r}\frac{\partial f(r,\theta,\phi)}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f(r,\theta,\phi)}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f(r,\theta,\phi)}{\partial \phi} \qquad (10)$$

Given a vector field $\boldsymbol{A}(\boldsymbol{r}) = \vec{A}(\vec{r}) = \hat{r}(\theta,\phi)\, A_r(r,\theta,\phi) + \hat{\theta}(\theta,\phi)\, A_\theta(r,\theta,\phi) + \hat{\phi}(\phi)\, A_\phi(r,\theta,\phi)$, then the divergence of $\boldsymbol{A}(\boldsymbol{r})$ is

$$\text{div}\, \boldsymbol{A}(\boldsymbol{r}) = \nabla\cdot\boldsymbol{A}(\boldsymbol{r}) = \frac{\partial A_r}{\partial r} + \frac{2A_r}{r} + \frac{\cos\theta\, A_\theta}{r\sin\theta} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \qquad (11)$$

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and the curl of $\boldsymbol{A}(\boldsymbol{r})$ is

$$\text{curl}\, \boldsymbol{A}(\boldsymbol{r}) = \nabla\times\boldsymbol{A}(\boldsymbol{r})$$

$$= \hat{r}\left(\frac{1}{r}\frac{\partial A_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi} + \frac{\cos\theta\, A_\phi}{r\sin\theta}\right) + \hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial A_\phi}{\partial r} - \frac{A_\phi}{r}\right) + \hat{\phi}\left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} + \frac{A_\theta}{r}\right) \qquad (12)$$

Gauss' theorem & Stokes' theorem

- the divergence theorem, or Gauss' theorem:

微觀的物理量　　宏觀的物理量

$$\iiint_{Vol.(S)} \nabla \cdot \vec{E}\, d^3x = \oiint_{S(Vol.)} \vec{E} \cdot d\vec{a}$$

- Stokes' theorem:

$$\iint_{S(C)} \left(\nabla \times \vec{E}\right) \cdot d\vec{a} = \oint_{C(S)} \vec{E} \cdot d\vec{l}$$

$$\iint_{S(C)} \left(\nabla \times \vec{V}\right) \cdot d\vec{a} = \oint_{C(S)} \vec{V} \cdot d\vec{l}$$

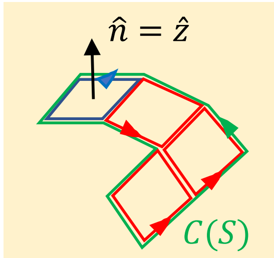
If $\vec{V}$ is the flow field(流場), then $\nabla \times \vec{V}$ is called the vorticity (渦度), and $\oint_{C(S)} \vec{V} \cdot d\vec{l}$ is called the circulation along the closed loop $C(S)$.

以 Stokes' theorem 為範例證明之

$$\iint_{S(C)} (\nabla\times\vec{E})\cdot d\vec{a} = \oint_{C(S)} \vec{E}\cdot d\vec{l}$$

將 $S(C)$ 切成很多小塊的方型切面。每塊面的法線方向不一定相同。但是只要每一塊面都滿足上式，則加起來，相鄰兩面右側積分互相抵消，最後只剩最外圍的積分 $C(S)$ 。現在取 local coordinate 。考慮其中一個小塊方型切面 S_0 ，法線方向為 $\hat{z}$ 。整塊面位在 $z=0$ 處。方型切面兩邊分別沿 $\hat{x}$ and $\hat{y}$ 方向。若方型切面左下角位在 (x,y) ，右上角位在 $(x+\Delta x\,,y+\Delta y)$ 。再令 $(x_0,y_0)=(x+\Delta x/2\,,y+\Delta y/2)$ 。因為

$$\nabla\times\vec{E} = \hat{x}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+\hat{y}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\hat{z}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)$$



在 $(x_0,y_0,z=0)$ 處

$$\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)_{\substack{x=x_0\\y=y_0\\z=0}} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[\frac{E_y(x+\Delta x,y_0)-E_y(x,y_0)}{\Delta x}-\frac{E_x(x_0,y+\Delta y)-E_x(x_0,y)}{\Delta y}\right]_{z=0}$$

$$\iint_{S_0}(\nabla\times\vec{E})\cdot d\vec{a} = \left[(\nabla\times\vec{E})\cdot\hat{z}(\Delta x\Delta y)\right]_{\substack{x=x_0\\y=y_0\\z=0}} = \left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)_{\substack{x=x_0\\y=y_0\\z=0}}(\Delta x\Delta y)$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[\frac{E_y(x+\Delta x,y_0)-E_y(x,y_0)}{\Delta x}-\frac{E_x(x_0,y+\Delta y)-E_x(x_0,y)}{\Delta y}\right]_{z=0}(\Delta x\Delta y)$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ z=0}}\left\{\left[E_y(x+\Delta x,y_0)-E_y(x,y_0)\right]\Delta y-\left[E_x(x_0,y+\Delta y)-E_x(x_0,y)\right]\Delta x\right\}$$

$$= \oint_{C(S_0)}\vec{E}\cdot d\vec{l}$$