Motion of a Particle in a One-Dimensional System

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Ling-Hsiao Lyu

Department of Space Science and Engineering

Symon (1960): Chapter 2:
Motion of a Particle in One Dimension

“一元”微分(積分)方程式

Review 2-1, 2-2 & Quiz

等應數課先教，再講 2-3~2-11，有些題目適合數值模擬求解

Homeworks (常考問題) 2-4, 2-7, 2-9

2-1 Momentum and energy theorems (高中)

2-2 Discussion of the general problem of one-dimensional motion (高中)

2-3 Applied force depending on the time (大學)

2-4 Damping force depending on the velocity (大學)

2-5 Conservative force depending on position. Potential energy (高中)

2-6 Falling bodies (高中)

2-7 The simple harmonic oscillator (高中 ＆ 大學)

2-8 Linear differential equations with constant coefficients (大學)

2-9 The damped harmonic oscillator (大學)

2-10 The forced harmonic oscillator (大學)

2-11 The principle of superposition. Harmonic oscillator with arbitrary applied force

Outlines

- Momentum and Energy theorems
- One-dimensional motions

Kinematic Equations

- **Kinematic Equations**

$$\frac{d\vec{x}}{dt} = \_\_$$

$$\frac{d\vec{v}}{dt} = \_\_$$

- **Newton's Second Law:**

$$\vec{F} = m\_\_ = m\_\_\_$$

where $\vec{F}$ is the force

Kinematic Equations

- **Kinematic Equations**

$$\frac{d\vec{x}}{dt} = \vec{v}$$

$$\frac{d\vec{v}}{dt} = \vec{a}$$

- **Newton's Second Law:**

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

where $\vec{F}$ is the force

Linear Momentum & Momentum Theorem

- **Definition of linear momentum $\vec{p}$ (non-relativistic)**

$$\vec{p} = \_\_\_\_$$

- Since force $\vec{F} = m\vec{a} = m(d\vec{v}/dt)$, it yields

  The (differential) momentum theorem:

  - **For one-dimensional problem:**

$$\frac{dp(t)}{dt} = \dot{p}(t) = \_\_\_\_$$

  - **For two- or three-dimensional problem:**

$$\frac{d\vec{p}}{dt} = \_\_ \Rightarrow \begin{cases} \dot{p}_x(t) = m\dot{v}_x(t) = \_\_ \\ \dot{p}_y(t) = m\dot{v}_y(t) = \_\_ \\ \dot{p}_z(t) = m\dot{v}_z(t) = \_\_ \end{cases}$$

Linear Momentum & Momentum Theorem

- **Definition of linear momentum (non-relativistic)**

$$\vec{p} = m\vec{v}$$

Relativistic monentum

$$\vec{p} = \gamma m\vec{v}$$

where γ is the Lorentz factor

$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

- Since force $\vec{F} = m\vec{a} = m(d\vec{v}/dt)$, it yields **The (differential) momentum theorem:**
  - **For one-dimensional problem:**

$$\frac{dp(t)}{dt} = \dot{p}(t) = F(t)$$

$$\frac{dp(t)}{dt} = \dot{p}(t) = m\dot{v}(t) = F[x(t), v(t), t]$$

  - **For two- or three-dimensional problem:**

$$\frac{d\vec{p}}{dt} = \vec{F} \Rightarrow \begin{cases} \dot{p}_x(t) = m\dot{v}_x(t) = F_x\,[x(t), y(t), z(t), v_x(t), v_y(t), v_z(t), t] \\ \dot{p}_y(t) = m\dot{v}_y(t) = F_y[x(t), y(t), z(t), v_x(t), v_y(t), v_z(t), t] \\ \dot{p}_z(t) = m\dot{v}_z(t) = F_z[x(t), y(t), z(t), v_x(t), v_y(t), v_z(t), t] \end{cases}$$

Momentum Theorem (cont.)

- **An integrated form of the momentum theorem:**
  - **For one-dimensional problem:**

$$p = \int F dt$$

or

$$p(t_2) - p(t_1) = \int_{t_1}^{t_2} F(t) dt$$

$$p(t_2) - p(t_1) = \int_{t_1}^{t_2} F[x(t), v(t), t] dt$$

Kinetic Energy & Energy Theorem

- **Definition of Kinetic Energy T (non-relativistic)**

$$T = \_\_\_\_$$

- **The (differential) Energy theorem:**
  - **For one-dimensional problem:**

$$\frac{dT(t)}{dt} = \_\_\_\_$$

  - **For two- or three-dimensional problem:**

$$\frac{dT}{dt} = \_\_\_\_$$

Kinetic Energy & Energy Theorem

- **Definition of Kinetic Energy T (non-relativistic)**

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

- **The (differential) Energy theorem:**
  - **For one-dimensional problem:**

$$\frac{dT(t)}{dt} = mv \frac{dv}{dt} = vF$$

  - **For two- or three-dimensional problem:**

$$\frac{dT}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \vec{F}$$

where $\vec{v} \cdot \vec{F}$ is the power

Energy Theorem (cont.)

- **An integrated form of the Energy theorem:**
  - **For one-dimensional problem:**

$$T(t_2) - T(t_1) = \Delta T = \int_{t_1}^{t_2} F(t)v(t)dt$$

If the force F is known as a function of x, then

$$\Delta T = \int_{t_1}^{t_2} F[x(t)]v(t)dt = \int_{x_1}^{x_2} F(x)dx$$

  - **For two- or three-dimensional problem:**

$$\Delta T = \int_{t_1}^{t_2} \vec{F}[\vec{x}(t)] \cdot \vec{v}(t)dt = \int_{x_1}^{x_2} \vec{F}(x) \cdot d\vec{x}$$

where $\int_{x_1}^{x_2} \vec{F}(\vec{x}) \cdot d\vec{x}$ is the **work** done by the force $\vec{F}$

One-dimensional motion of a particle

2-3 Applied force depending on the time

$$\frac{dv(t)}{dt} = \frac{F(t)}{m} \Rightarrow v(t_1) = v(t_0) + \_\_\_\_\_\_\_\_$$

Δv 可由 $F(t)/m$ 曲線下的面積求得。

2-4 Damping force depending on the velocity

$$\frac{dv}{dt} = \frac{F(v)}{m} = \frac{1}{m}(-bv)$$

將 v 放到等號同一側，將 t 放到等號另一側，積分即得。

2-5 Conservative force depending on position. Potential energy

$$\frac{dv}{dt} = \frac{F(x)}{m} = \frac{1}{m}[-\frac{dV(x)}{dx}]$$

將等號兩側同乘以 v ，在對時間積分即得 動能＋位能守恆。

再由動能換算 v

2-6 Falling bodies

$$\frac{d^2z}{dt^2} = \frac{dv_z}{dt} = -\frac{mg}{m} = -g$$

請自己算！

One-dimensional motion of a particle

2-7 The simple harmonic oscillator

$$m\ddot{x} = -\boldsymbol{kx} \Rightarrow \ddot{x} = -\left(\frac{k}{m}\right)x$$

考慮一下，什麼樣的函數，微分兩次，結果跟原來的函數，
只差一個負號，再多乘以一些常數？就用這個函數帶回去，就有解了！

2-8 Linear differential equations with constant coefficients

$$a_2\ddot{x} + a_1\dot{x} + a_0x = f(t)$$

2-9 The damped harmonic oscillator

$$m\ddot{x} = -kx - bv$$

or

$$m\ddot{x} + b\dot{x} + kx = 0$$

2-10 The forced harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = F_0\cos(\omega_0 t + \theta_0)$$

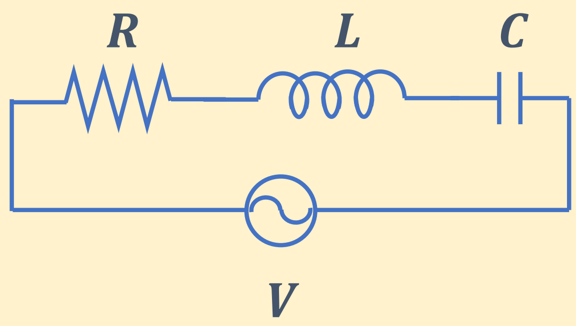
One-dimensional motion of a particle

課本範例 **RLC circuit**：

考慮一組串聯電路，包含有一個電阻R、一個電感 L 、一個電容 C 、以及一個外加電源 $V(t)$ 。 此電路應滿足以下方程式：

$$V(t) = IR + L\frac{dI}{dt} + \frac{Q}{C}$$

Since



$$I = \frac{dQ}{dt}$$

It yields

$$L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C} = V(t)$$

這個式子是不是跟 **2-8, 2-9, & 2-10** 的式子很像？所以它們的解，也很像。

若外加電壓滿足 $V(t) = V_0\cos(\omega_0 t + \theta_0)$，就形成一個 forced harmonic oscillator ，因此實驗室裡，常用電路來進行力學實驗。

2-3 & 2-4 解答

2-3 Applied force depending on the time

$$\frac{dv(t)}{dt} = \frac{F(t)}{m}$$

$$\Rightarrow v(t_1) = v(t_0) + \frac{1}{m}\int_{t_0}^{t_1} F(t)dt$$

$\Delta v = v(t_1) - v(t_0)$ 可由 $F(t)/m$ 曲線下的面積求得。or

$$v(t) = v(t_0) + \frac{1}{m}\int_{t_0}^{t} F(t')dt'$$

注意：解答不可套用等加速度的公式，因為 $F(t)/m$ 是一個變加速度。所以

$$v(t) \neq v(t_0) + \frac{F(t)}{m}t$$

2-4 Damping force depending on the velocity

$$\frac{dv}{dt} = \frac{F(v)}{m} = \frac{1}{m}(-bv)$$

將 v 放到等號同一側，將 t 放到等號另一側，積分即得。

$$\frac{dv}{v} = d\ln(v) = \frac{-b}{m}dt$$

$$\int_{\ln v(t_0)}^{\ln v(t_1)} d\ln(v) = \frac{-b}{m}\int_{t_0}^{t_1} dt$$

$$\ln\left(\frac{v(t_1)}{v(t_0)}\right) = \frac{-b}{m}(t_1 - t_0)$$

$$v(t_1) = v(t_0)\exp\left[\frac{-b}{m}(t_1 - t_0)\right]$$

Or, for $v_0 = v(t = 0)$, we have

$$v(t) = v_0 e^{-bt/m}$$

2-5 解答

2-5 Conservative force depending on position. Potential energy

$$\frac{dv}{dt}=\frac{F(x)}{m}=\frac{1}{m}[-\frac{dV(x)}{dx}]$$

將等號兩側同乘以 v ，在對時間積分即得 動能＋位能守恆。再由動能換算 v 。

其中 $dV(x)=-F(x)dx$ 或 $V(x)-V(x_0)=V-V_0=-\int_{x_0}^{x}F(x)dx$

依照以上解題提示得

$$mv\frac{dv}{dt}=mv\frac{1}{m}\left[-\frac{dV(x)}{dx}\right]=v\left[-\frac{dV(x)}{dx}\right]$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right)=\left[-\frac{dV}{dx}\right]\frac{dx}{dt}=-\frac{dV}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right)+\frac{dV}{dt}=\frac{d}{dt}\left(\frac{1}{2}mv^2+V\right)=0$$

$$\frac{1}{2}mv^2+V=\frac{1}{2}mv_0^2+V_0$$

where $v_0=v(t=0)$, $x_0=x(t=0)$ and $V_0=V(x=x_0)$. It yields

$$v^2=v_0^2+\frac{2}{m}\left(V_0-V(x)\right)\geq 0$$

但是真正要解 $v(t)$ 如何隨時間變化，還是需要「聯立」以下方程式，用數值方法求解

$\frac{dx(t)}{dt}=v(\mathrm{t})$	$\frac{dv(t)}{dt}=\frac{F[x(t)]}{m}$

2-6 & 2-7 解答

2-6 Falling bodies

$$\frac{d^2z}{dt^2} = \frac{dv_z}{dt} = -\frac{mg}{m} = -g$$

其中，我們取 z 向上 為正。重力加速度方向向下，所以取負號。

這是一個「等加速度」的問題，高中學過了，這裡不再贅述。

2-7 The simple harmonic oscillator

$$m\ddot{x} = -\boldsymbol{kx} \Rightarrow \ddot{x} = -\left(\frac{k}{m}\right)x$$

考慮一下，什麼樣的函數，微分兩次，結果跟原來的函數，
只差一個負號，再多乘以一些常數？就用這個函數帶回去，就有解了！

（猜）Let $x(t) = A_0\cos(\omega t + \phi)$ it yields $\dot{x}(t) = -\omega A_0\sin(\omega t + \phi)$ and

$$\ddot{x}(t) = -\omega^2 A_0\cos(\omega t + \phi) = -\omega^2 x(t)$$

所以可知，此彈簧系統的震盪角頻率 $\omega = \sqrt{k/m}$

至於常數 $A_0 > 0$ 以及初始相位 ϕ 就要由初始條件（initial condition）來決定。

例如：

若 $x(t=0) = x_0$	$\dot{x}(t=0) = 0$	則 $A_0 =$	$\phi =$
若 $x(t=0) = 0$	$\dot{x}(t=0) = v_0$	則 $A_0 =$	$\phi =$

補充 The simple harmonic oscillator

- 當一個質點達到力平衡時（所受的淨力為零時），可能是處在一個stable equilibrium, unstable equilibrium, or neutral equilibrium.
- 由力的觀點看： 當我們考慮將此質點由原來的平衡態做一個小的虛擬位移 （a virtual displacement），
  - 如果位移後所受的淨力，與位移方向相反，我們就說它受到一個恢復力 restoring force ，而且稱原來的平衡態為 **穩定平衡態 stable equilibrium.** 穩定平衡態附近會有 簡諧震盪 harmonic oscillation 出現
  - 如果位移後所受的淨力，與位移方向相同，我們就說它受到一個使它位移放大的力，這是一種正回饋效應 positive feedback ，因此稱原來的平衡態為 **不穩定平衡態 unstable equilibrium.**
  - 如果位移後所受的淨力仍為零。我們就稱原來的平衡態為 **隨遇平衡態 neutral equilibrium.**
- 由能量的觀點看： 如果一個平衡態系統中任何兩區域物質互換位置後，
  - 整個系統位能會增加，這個系統就是一個 stable equilibrium.
  - 整個系統位能會減少，這個系統就是一個 unstable equilibrium.
  - 整個系統位能不變，這個系統就是一個 neutral equilibrium.
- **以上描述，你能舉例說明嗎？你會由力的觀點說明單擺與彈簧的簡諧震盪嗎？你會由力矩的觀點，說明複擺的簡諧震盪嗎？**

2-9 解答

2-9 The damped harmonic oscillator

$$m\ddot{x} = -kx - bv$$

For $v = \dot{x},$ it yields the governing equation

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

Note that the dimension of b/m is (Time)$^{-1}$.
The dimension of k/m is (Time)$^{-2}$.

（猜）Let $x = C\exp(\gamma t)$ it yields

$\dot{x} = \gamma C\exp(\gamma t)$ and $\ddot{x} = \gamma^2 C\exp(\gamma t)$

Substituting the expressions x, $\dot{x}$, and $\ddot{x}$ into the governing equation, it yields

$$\left[\gamma^2 + \frac{b}{m}\gamma + \frac{k}{m}\right] C\exp(\gamma t) = 0$$

For $C\exp(\gamma t) \neq 0$, it yields

$$\gamma^2 + \frac{b}{m}\gamma + \frac{k}{m} = 0$$

The solution of γ are

$$\gamma = \gamma_{\substack{1\\2}} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

Thus,

$$x = C_1\exp(\gamma_1 t) + C_2\exp(\gamma_2 t)$$

For

$$\frac{k}{m} > \left(\frac{b}{2m}\right)^2 > 0$$

it yields

$\gamma_1 = -(b/2m) + i\omega$ and $\gamma_2 = -(b/2m) - i\omega$

where $\omega = \sqrt{(k/m) - (b/2m)^2}$

Thus,

$$x = \exp\left(\frac{-b}{2m}t\right)[D_1\sin(\omega t) + D_2\cos(\omega t)]$$

Or

$$x = \exp\left(\frac{-b}{2m}t\right)[A_0\cos(\omega t + \phi)]$$

The constants C_1 & C_2 , or D_1 & D_2 , or A_0 & ϕ can be determined based on the intial conditions.

比較 2-4, 2-7, & 2-9

(2–9 對大二學生有點難，僅供參考)

	$b>0\ \&\ k>0$	$x(t=0)=x_0\ \&\ v(t=0)=v_0$
2-4	$\frac{dv}{dt}=\frac{1}{m}(-bv)$	$x(t)=x_0+(v_0/\gamma)(1-e^{-\gamma t})$ $v(t)=v_0e^{-\gamma t}$ where $\gamma=b/m$
2-7	$m\ddot{x}=-kx$	$x(t)=(x_0/\sin\phi_0)\sin(\omega t+\phi_0)$ $v(t)=(v_0/\cos\phi_0)\cos(\omega t+\phi_0)$ where $\omega=\sqrt{k/m}\ \&\ \phi_0=\tan^{-1}[x_0/(v_0/\omega)]$
2-9	$m\ddot{x}=-kx-bv$	For $\sqrt{k/m}>b/2m$ $x(t)=\frac{x_0}{\sin\phi_0}e^{-\gamma_0 t}\sin(\omega_d t+\phi_0)$ $v(t)=\frac{v_0}{\cos(\phi_0+\phi_1)}e^{-\gamma_0 t}\cos(\omega_d t+\phi_0+\phi_1)$ where $\gamma_0=b/2m,\ \omega_d=\sqrt{(k/m)-(b/2m)^2}$ $\phi_1=\tan^{-1}(\gamma_0/\omega_d),\phi_0=\tan^{-1}\left\{\frac{x_0}{\left[\left(\frac{v_0}{\omega}\right)-\frac{x_0\gamma_0}{\omega_d}\right]}\right\}$

比較 2-4, 2-7, & 2-9

(2–9 對大二學生有點難，僅供參考)

	$b > 0\ \&\ k > 0$	$x(t=0) = x_0\ \&\ v(t=0) = v_0$
2-9	$m\ddot{x} = -kx - bv$	For $\sqrt{k/m} < b/2m$ $$x(t) = \frac{\gamma_2 x_0 + v_0}{\gamma_2 - \gamma_1} e^{-\gamma_1 t} - \frac{\gamma_1 x_0 + v_0}{\gamma_2 - \gamma_1} e^{-\gamma_2 t}$$ $$v(t) = -\gamma_1 \frac{\gamma_2 x_0 + v_0}{\gamma_2 - \gamma_1} e^{-\gamma_1 t} + \gamma_2 \frac{\gamma_1 x_0 + v_0}{\gamma_2 - \gamma_1} e^{-\gamma_2 t}$$ where $$\gamma_1 = \gamma_0 - \sqrt{(b/2m)^2 - (k/m)}$$ $$\gamma_2 = \gamma_0 + \sqrt{(b/2m)^2 - (k/m)}$$
2-9	$m\ddot{x} = -kx - bv$	For $\sqrt{k/m} = b/2m$ $$x(t) = x_0 e^{-\gamma_0 t} + (\gamma_0 x_0 + v_0) t e^{-\gamma_0 t}$$ $$v(t) = v_0 e^{-\gamma_0 t} - \gamma_0(\gamma_0 x_0 + v_0) t e^{-\gamma_0 t}$$ where $\gamma_0 = b/2m$

Homework: Plot the above solutions with $x_0 > 0\ \&\ v_0 = 0$.

2-8 & 2-10 解答

2-8 Linear differential equations with constant coefficients

$$a_2\ddot{x} + a_1\dot{x} + a_0 x = f(t)$$

用數值解（Euler method, the 2nd-order Runge-Kutta method, or the 4th-order Runge-Kutta method）

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{a_1}{a_2}\dot{x} - \frac{a_0}{a_2}x + \frac{f(t)}{a_2} \end{bmatrix}$$

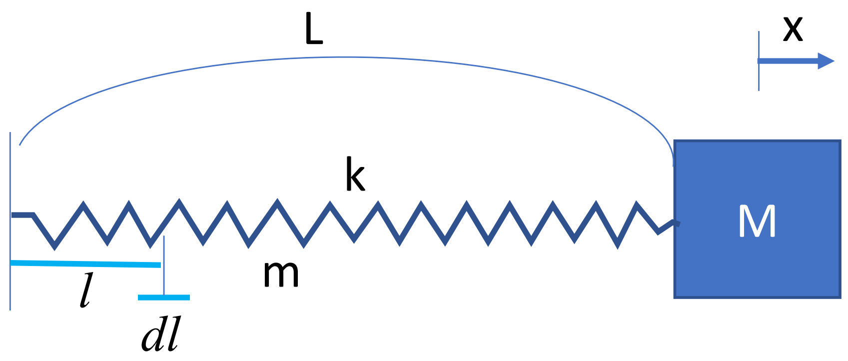
2-10 The forced harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = F_0\cos(\omega_0 t + \theta_0)$$

用數值解

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{F_0}{m}\cos(\omega_0 t + \theta_0) \end{bmatrix}$$

一般討論彈簧振盪時，都考慮輕彈簧。也就是彈簧的質量 m，遠小於所掛重物的質量M。現在若將彈簧的質量 m 納入考慮，則此彈簧系統的震盪週期將如何改變？



$$\frac{1}{2}kx^2+\frac{1}{2}M\dot{x}^2+\frac{1}{2}\frac{m}{L}\int_0^L\left(\frac{l}{L}\dot{x}\right)^2 dl=\text{constant}$$

$$\frac{1}{2}kx^2+\frac{1}{2}M\dot{x}^2+\frac{1}{2}\frac{m}{L^3}\dot{x}^2\int_0^L l^2 dl$$

$$=\frac{1}{2}kx^2+\frac{1}{2}M\dot{x}^2+\frac{1}{2}\frac{m}{L^3}\dot{x}^2\frac{L^3}{3}$$

$$=\frac{1}{2}kx^2+\frac{1}{2}(M+\frac{m}{3})\dot{x}^2 \quad =\text{constant}$$

The time derivative of the above conservation law yields

$$kx\dot{x}+\left(M+\frac{m}{3}\right)\dot{x}\ddot{x}=0$$

or,

$$\left(M+\frac{m}{3}\right)\ddot{x}=-kx$$

It yields

$$x(t)=\frac{x_0}{\sin\phi_0}\sin(\omega t+\phi_0)$$

where

$$\omega=\sqrt{\frac{k}{M+\frac{m}{3}}}$$