Polhodes & inertia ellipsoid (in angular velocity space)
vs.
angular momentum sphere & kinetic energy ellipsoid (in angular momentum space)

我在研究該怎麼講探險家一號 Explorer 1 最後沿著 principle axis of the maximum moment of inertia 打轉。我查到一個演講，是從數值模擬的觀點講起。我也查到一個演講，是從理論講起。我覺得後者講得好多了。但是前者對歷史的敘述，比較完整。

1. 歷史回顧＆數值模擬

David A. Levinson （服務於Lockheed Martin Space Systems Company）於 15 November 2012受邀在 UC Davis Mechanical and Aerospace Engineering 的演講 Seminar
https://www.youtube.com/watch?v=RdDJtUxLwqQ
David Levinson on The Explorer I Anomaly.
注意看15:00 35:00 這些影片段落

2. 從理論講起，介紹

angular momentum sphere & kinetic energy ellipsoid (in angular momentum space）這段課本沒講，但是只要能舉一反三，就很容易懂了。

https://www.youtube.com/watch?v=luzs0pVqFvQ Kinematics - Motions of Spacecraft - 6.1.1 - Torque Free Motion Polhode Plots

以上連結已經消失！請改用以下連結

https://www.coursera.org/lecture/spacecraft-dynamics-kinetics/1-1-example-special-polhode-plots-9AOOO

Example: Special Polhode Plots

本補充教材就是要讓各位同學瞭解

1. 什麼是 Polhodes & inertia ellipsoid
https://en.wikipedia.org/wiki/Polhode
非常簡潔的文字，把所有重點都講過一遍了！
2. 為何沿著 principle axes of the maximum moment of inertia and the minimum moment of inertia 打轉，都相對很穩定，但是沿著 principle axis of the intermedium moment of inertia 打轉就不穩定了?
3. 介紹 Euler's equations for the motion of a rigid body 。用它來證明 2. 的陳述。證明過程教導學生如何進行平衡態的「穩定性分析」。

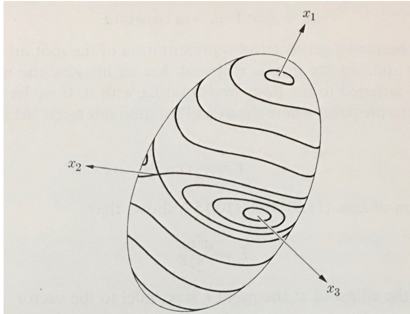


Fig. 11.3 Polhodes on a nondegenerate inertia ellipsoid.

Fig.1. 當 $I_3 > I_2 > I_1$ 時的 inertia ellipsoid 以及 可能的角速度變化軌跡Polhodes。此圖下載自課本 (Symon, 1960)。

4. 也可直接積分 Euler's equations 求出 角速度的軌跡線 Polhodes。

考慮一個在太空中自由翻滾的物體：

對於一個慣性座標中的觀察者而言，這個物體的轉動慣量張量的每一個分量大小，就可能會一直隨時間改變。但是因為外力與外力矩都為零，因此這個自由翻滾的物體，它的角動量會是一個守恆的向量。

若以該物體轉動慣量的主軸座標系（principle axes）來描述這個物體的運動，它的轉動慣量張量是一個對角線化的張量，且對角線上每一個轉動慣量的分量值，都是一個不變量。不過，用這個座標來描述物體的角動量，這個角動量向量的每一個分量，就可能會一直隨時間改變。不過因為外力與外力矩都為零，此物體的角動量的向量大小，仍會是一個不變量。

以下讓我們以自由翻滾物體的轉動慣量的主軸座標系（principle axes）$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, 來描述這個物體的運動。

若此物體的轉動慣量(moment of inertia) 為

$$\vec{\vec{I}} = I_1 \hat{e}_1 \hat{e}_1 + I_2 \hat{e}_2 \hat{e}_2 + I_3 \hat{e}_3 \hat{e}_3 \tag{1}$$

此物體的角速度(angular velocity)為

$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3 \tag{2}$$

此物體的角動量 angular momentum 為

$$\vec{L} = L_1 \hat{e}_1 + L_2 \hat{e}_2 + L_3 \hat{e}_3 \tag{3}$$

且

$$\vec{L} = \vec{\vec{I}} \cdot \vec{\omega} = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3 \tag{4}$$

Equations (3) and (4) yields

$$\begin{aligned} L_1 &= I_1 \omega_1 \\ L_2 &= I_2 \omega_2 \end{aligned} \tag{4a}$$

$$L_3 = I_3\omega_3$$

此自由翻滾的物體，它的動能 kinetic energy 為

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{\vec{I}}\cdot\vec{\omega} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 \tag{5}$$

我們也可以用物體的角動量 (angular momentum) 來描述它的動能 (kinetic energy). i.e.,

$$T = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} \tag{6}$$

Equation (5) and Equation (6) 都可以改寫成一個等動能的橢圓體方程式，但是這兩個橢圓體的外型，完全相反！首先讓我們看一下課本上所介紹的 inertia ellipsoid.

Equation (5) can be rewritten as

$$1 = \frac{\omega_1^2}{2T/I_1} + \frac{\omega_2^2}{2T/I_2} + \frac{\omega_3^2}{2T/I_3} = \frac{\omega_1^2}{a_1^2} + \frac{\omega_2^2}{a_2^2} + \frac{\omega_3^2}{a_3^2} \tag{7}$$

where $a_1 = \sqrt{2T/I_1}, a_2 = \sqrt{2T/I_2}$, $a_3 = \sqrt{2T/I_3}$. 也就是說在主軸座標系（principle axes）$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$中，物體角速度的解，將會落在一個 ω-space 中「等動能橢圓體結

構的表面」上。此ω-space 中「等動能橢圓面」整體大小正比於$\sqrt{T}$。沿 $\omega_1, \omega_2, \omega_3$ 各方向的軸長分別正比於$1/\sqrt{I_1}$, $1/\sqrt{I_2}$, $1/\sqrt{I_3}$。角速度的解會根據初始條件，沿著此橢圓面上某一特定軌跡路徑（Polhodes）隨時間而改變。然而其改變速度快慢，通常需要用數值模擬的方式，解一組 尤拉方程式 （Euler's equations for the motion of a rigid body 後面會介紹）來決定，並無法直接由軌跡圖（Polhodes）來判斷。

舉例說明 Equation (5) & Equation (7) 所對應的等動能 inertia ellipsoid：

如果我們考慮一個自由翻滾的手機形狀或書本形狀的長方體，它的$I_1 > I_2 > I_3$。則我們知道$\hat{e}_1, \hat{e}_2, \hat{e}_3$ 的方向分別如圖 2 所示。這個手機自由翻轉時所對應的 inertia ellipsoid 將類似於圖 3 所示。

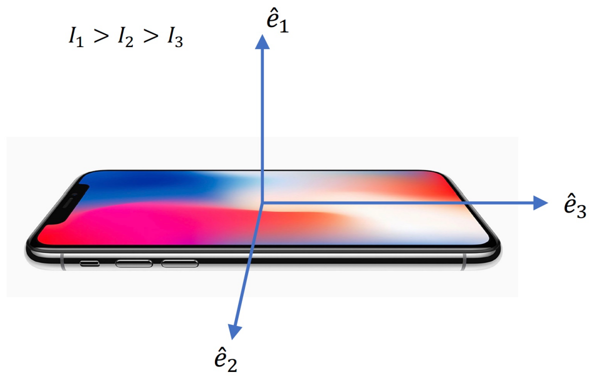


Fig. 2. 手機所對應的轉種慣量$I_1 > I_2 > I_3$ 時，主軸$\hat{e}_1, \hat{e}_2, \hat{e}_3$的分佈情形。

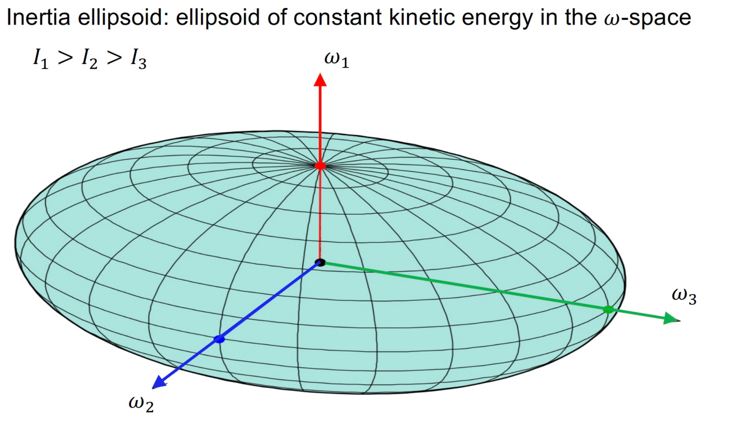


Fig. 3.當 $I_1 > I_2 > I_3$ 時的等動能 inertia ellipsoid （ω-space 橢圓體）。其整體大小正比於$\sqrt{T}$。$\omega_1, \omega_2, \omega_3$各方向的軸長分別正比於$1/\sqrt{I_1}$, $1/\sqrt{I_2}$, $1/\sqrt{I_3}$。

接著讓我們看一下以下教學影片 1 中上所介紹的等動能 ellipsoid.

教學影片1: https://www.coursera.org/lecture/spacecraft-dynamics-kinetics/1-1-example-special-polhode-plots-9AOOO

Example: Special Polhode Plots

Equation (6) can be rewritten as

$$1 = \frac{L_1^2}{2TI_1} + \frac{L_2^2}{2TI_2} + \frac{L_3^2}{2TI_3} = \frac{L_1^2}{b_1^2} + \frac{L_2^2}{b_2^2} + \frac{L_3^2}{b_3^2} \quad (8)$$

where $b_1 = \sqrt{2TI_1}, b_2 = \sqrt{2TI_2}, b_3 = \sqrt{2TI_3}$. 也就是說在主軸座標系（principle axes）$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$中，物體角動量的解，將會落在一個 L-space 的「等動能橢圓體結構的表面」上。此L-space 中「等動能橢圓面」整體大小正比於$\sqrt{T}$。沿 L_1, L_2, L_3 各方向的軸長分別正比於$\sqrt{I_1}, \sqrt{I_2}, \sqrt{I_3}$。如圖 4 所示。

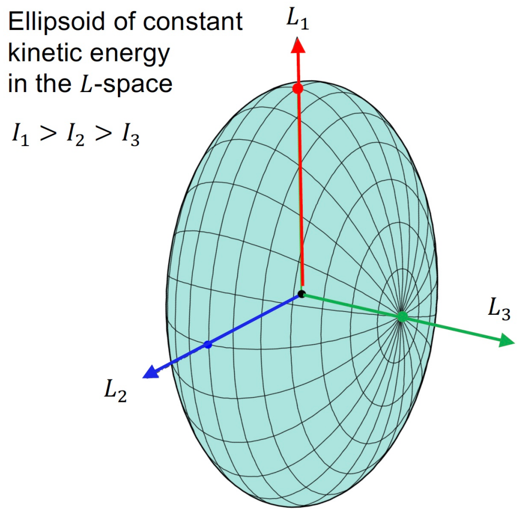


Fig. 4.當 $I_1 > I_2 > I_3$ 時的 ellipsoid of constant kinetic energy in the L-space。

至於角動量的解，因為自由翻滾的物體，外力矩為零，所以不論$\hat{e}_1, \hat{e}_2, \hat{e}_3$怎麼轉，角動量的大小不會因為座標轉換而有所變動。因此 Equation (3) yields

$$L_1^2 + L_2^2 + L_3^2 = L^2 \tag{9}$$

所以物體角動量的解，也必需要位在此L-space 中的等角動量球面上。因此物體角動量的解就是 Equation (8)所描繪的「等動能橢圓面」與 Equation (9)所描繪的「等角動量球面」兩者的交集所構成的曲線。這些曲線，可以畫在「等動能橢圓面」上，也可以畫在「等角動量球面」上。教學影片 1 中就是選擇畫在等角動量球面上。因為如果翻滾的物體不是一個堅實的剛體，而是包含了液體的容器，或是結構不夠堅實的固體，則物體雖然不受外力，但是會有摩擦等內力耗損其動能，使得它的等動能橢圓面逐漸縮小。因此等動能橢圓面與等角動量球面的交集所構成的曲線，也會隨之發生變動。最後穩定態應為

$$\hat{e}_1(t \to \infty) = \frac{\vec{L}}{L} \tag{10}$$

$$\vec{\omega}(t \to \infty) = \omega_1 \hat{e}_1 = (L/I_1)\,\hat{e}_1 \tag{11}$$

$$T(t \to \infty) = T_{min} = \frac{L^2}{2I_1} \tag{12}$$

因為三度空間的圖比較難畫，所以我們用二度空間的剖面圖來取代。

A cross-section plot of the kinetic energy ellipsoid and angular momentum sphere at $L_2 = 0$ in the L-space

$I_1 > I_2 > I_3$

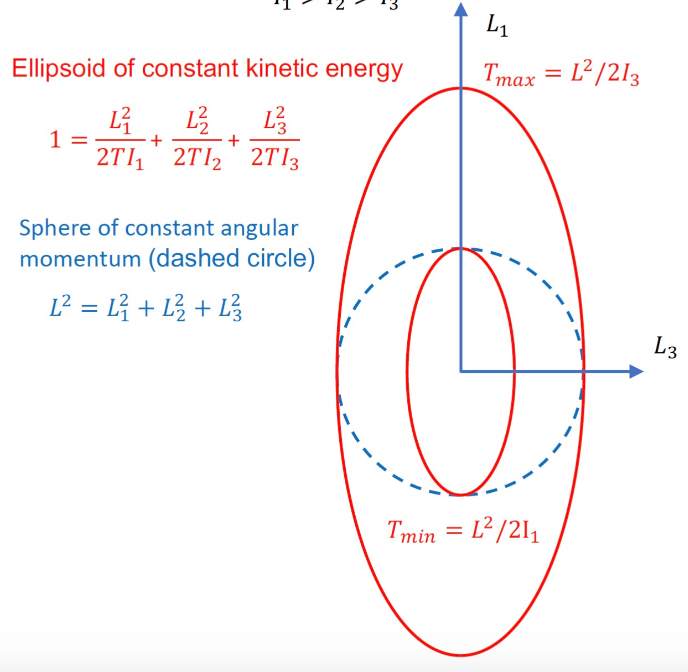


Fig. 5. L-space 的等動能橢圓面（實線）與等角動量球面（虛線）在$L_2 = 0$ 的剖面圖，其中轉動慣量滿足$I_1 > I_2 > I_3$。圖中顯示最初的旋轉若是沿著最小轉動慣量的軸，最後因為阻尼生熱，轉動動能由$T_{max} = L^2/2I_3$ 降低至 $T_{min} = L^2/2I_1$ ，整個物體將沿著最大轉動慣量的軸旋轉。

圖 5 繪出L-space 中，等動能橢圓面與等角動量球面在$L_2 = 0$ 的剖面圖。圖中顯示最初的旋轉若是沿著最小轉動慣量的軸，最後因為阻尼生熱，轉動動能由$T_{max} = L^2/2I_3$ 降低至 $T_{min} = L^2/2I_1$ ，整個物體將沿著最大轉動慣量的軸旋轉。

事實上我們也可以模仿這個例子，用課本中所介紹的 inertia ellipsoid 來解釋如果有能量耗散發生時，整個物體將沿著最大轉動慣量的軸旋轉。因為外力矩為零，角動量守恆。將 Equation (4a) 代入 Equation (9)，可得

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = L^2$$

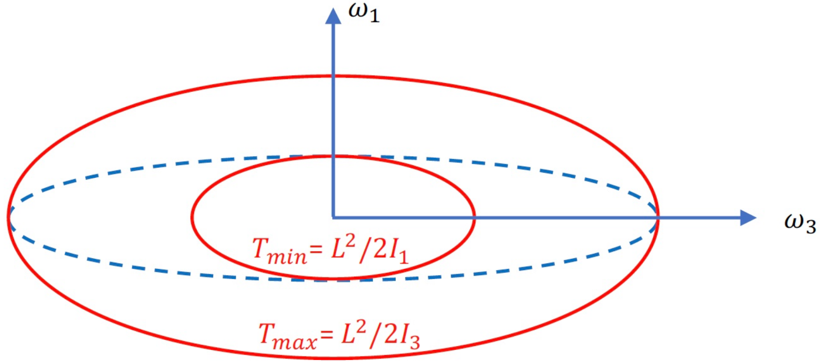
所以在ω-space 中，以上這個角動量守恆方程式可改寫為一個橢圓方程式

$$1 = \frac{\omega_1^2}{L^2/I_1^2} + \frac{\omega_2^2}{L^2/I_2^2} + \frac{\omega_3^2}{L^2/I_3^2} = \frac{\omega_1^2}{c_1^2} + \frac{\omega_2^2}{c_2^2} + \frac{\omega_3^2}{c_2^2} \tag{13}$$

where $c_1 = L/I_1, c_2 = L/I_2$, and $c_3 = L/I_3$. 此ω-space 中「等角動量橢圓面」整體大小正比於L。沿 $\omega_1, \omega_2, \omega_3$ 各方向的軸長分別正比於$1/I_1$, $1/I_2$, $1/I_3$。因此與 Equation (7) 相比，在ω-space 中「等角動量橢圓面」是一個比「等動能橢圓面」更橢圓的橢圓面。

A cross-section plot of two types of inertia ellipsoids at $\omega_2 = 0$ in the ω-space

$I_1 > I_2 > I_3$



Ellipsoid of constant kinetic energy (inertia ellipsoid)

$$1 = \frac{\omega_1^2}{2T/I_1} + \frac{\omega_2^2}{2T/I_2} + \frac{\omega_3^2}{2T/I_3}$$

Ellipsoid of constant angular momentum (dashed curve)

$$1 = \frac{\omega_1^2}{L^2/I_1^2} + \frac{\omega_2^2}{L^2/I_2^2} + \frac{\omega_3^2}{L^2/I_3^2}$$

Fig. 6. ω-space 的等動能橢圓面（實線）與等角動量橢圓面（虛線）在$\omega_2 = 0$ 的剖面圖，其中轉動慣量滿足$I_1 > I_2 > I_3$。圖中顯示最初的旋轉若是沿著最小轉動慣量的軸，最後因為阻尼生熱，轉動動能由$T_{max} = L^2/2I_3$ 降低至 $T_{min} = L^2/2I_1$ ，整個物體將沿著最大轉動慣量的軸旋轉。

圖 6 繪出ω-space 中，兩種 inertia ellipsoids 在$\omega_2 = 0$處的剖面圖，其中轉動慣量滿足$I_1 > I_2 > I_3$。圖中顯示最初的旋轉若是沿著最小轉動慣量的軸，最後因為阻尼生熱，轉動動能由$T_{max} = L^2/2I_3$ 降低至 $T_{min} = L^2/2I_1$ ，整個物體將沿著最大轉動慣量的軸旋轉。因此不論是圖 5 或圖 6 都可以得到相同的結論。

以下教學影片 2 就是描述 Equation (13)的「等角動量橢圓面」與 Equation (7)的「等動能橢圓面」相交處就是所謂的 Polhodes 角速度軸的軌跡圖。

教學影片2: https://www.youtube.com/watch?v=ei79Y_aqrm0
Dynamics of a rigid body in torque-free motion.

現在回到課本，看看什麼是 Euler's equations for the motion of a rigid body。

因為自由翻滾的物體M所受到的外力矩$\vec{N}$為零，所以

$$\frac{d}{dt}\vec{L} = \vec{N} = 0 \tag{14}$$

因為 I_1, I_2, I_3 為不變量，但是角速度大小 $\omega_1, \omega_2, \omega_3$, 以及 $\hat{e}_1, \hat{e}_2, \hat{e}_3$ 三軸的方向，都會隨時間改變，因此 Equation (14) can be rewritten as

$$\frac{d\vec{L}}{dt} = I_1\frac{d\omega_1}{dt}\hat{e}_1 + I_2\frac{d\omega_2}{dt}\hat{e}_2 + I_3\frac{d\omega_3}{dt}\hat{e}_3 + I_1\omega_1\frac{d\hat{e}_1}{dt} + I_2\omega_2\frac{d\hat{e}_2}{dt} + I_3\omega_3\frac{d\hat{e}_3}{dt} = 0 \tag{15}$$

用繪圖法可證明

$$\frac{d\hat{e}_1}{dt} = \omega_3(+\hat{e}_2) + \omega_2(-\hat{e}_3) \tag{16}$$

$$\frac{d\hat{e}_2}{dt} = \omega_3(-\hat{e}_1) + \omega_1(+\hat{e}_3) \tag{17}$$

$$\frac{d\hat{e}_3}{dt} = \omega_2(+\hat{e}_1) + \omega_1(-\hat{e}_2) \tag{18}$$

Substituting Equations (16), (17), and (18) into Equation (15) to eliminate$d\hat{e}_1/dt$, $d\hat{e}_2/dt$, and $d\hat{e}_3/dt$, respectively, it yields

$$\begin{aligned}
\frac{d\vec{L}}{dt} = I_1\frac{d\omega_1}{dt}\hat{e}_1 + I_2\frac{d\omega_2}{dt}\hat{e}_2 + I_3\frac{d\omega_3}{dt}\hat{e}_3 + I_1\omega_1[\omega_3(+\hat{e}_2) + \omega_2(-\hat{e}_3)] \\
+ I_2\omega_2[\omega_3(-\hat{e}_1) + \omega_1(+\hat{e}_3)] + I_3\omega_3[\omega_2(+\hat{e}_1) + \omega_1(-\hat{e}_2)] \\
= \hat{e}_1\left(I_1\frac{d\omega_1}{dt} - I_2\omega_2\omega_3 + I_3\omega_3\omega_2\right) \\
+ \hat{e}_2\left(I_2\frac{d\omega_2}{dt} + I_1\omega_1\omega_3 - I_3\omega_3\omega_1\right) \\
+ \hat{e}_3\left(I_3\frac{d\omega_3}{dt} - I_1\omega_1\omega_2 + I_2\omega_2\omega_1\right) = 0
\end{aligned} \tag{19}$$

Equation (19) yields

$$\frac{d\omega_1}{dt} = \frac{I_2 - I_3}{I_1}\ \omega_2\omega_3 \tag{20}$$

$$\frac{d\omega_2}{dt} = \frac{I_3 - I_1}{I_2}\ \omega_3\omega_1 \tag{21}$$

$$\frac{d\omega_3}{dt} = \frac{I_1 - I_2}{I_3}\ \omega_1\omega_2 \tag{22}$$

Equations (20)~(22) are the Euler's equations for the motion of a rigid body.
給定一組初始條件 ω_1, ω_2, ω_3 我們可以用數值模擬方法直接積分以上這組 Euler's equations 得到一組$\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$ 在ω-space 的軌跡線。此即 Polhodes 角速度軸的軌跡線。

如果我們只想分析角速度軸在主軸附近的軌跡線，我們可以做以下的「穩定性分析」(stability analysis)。要做穩定性分析前，我們先要做確定角速度軸與主軸平行時，是一個平衡態(equilibrium state)。然後我們再加一個小振幅的擾動 (small amplitude perturbation) 看看這個小擾動的振幅會不會被放大(be amplified)。如果振幅會被放大，則此平衡態就是一個「不穩定的平衡態」（an unstable equilibrium state）。

檢驗是否為平衡態(equilibrium state)：

Case 1: $\vec{\omega}(t=0)=\omega_0\hat{e}_1$
i.e., without perturbations

$$\omega_1(t=0)=\omega_0 \tag{23}$$

$$\omega_2(t=0)=0 \tag{24}$$

$$\omega_3(t=0)=0 \tag{25}$$

Substituting Equations (23)~(25) into Equations (20)~(22), it yields,

$$\frac{d\omega_1}{dt}=\frac{d\omega_2}{dt}=\frac{d\omega_3}{dt}=0$$

Thus, $\vec{\omega}(t=0)=\omega_0\hat{e}_1$ is an equilibrium state.

Case 2: $\vec{\omega}(t=0)=\omega_0\hat{e}_2$
i.e., without perturbations

$$\omega_1(t=0)=0 \tag{26}$$

$$\omega_2(t=0)=\omega_0 \tag{27}$$

$$\omega_3(t=0)=0 \tag{28}$$

Substituting Equations (26)~(28) into Equations (20)~(22), it yields,

$$\frac{d\omega_1}{dt}=\frac{d\omega_2}{dt}=\frac{d\omega_3}{dt}=0$$

Thus, $\vec{\omega}(t=0)=\omega_0\hat{e}_2$ is an equilibrium state.

Case 3: $\vec{\omega}(t=0)=\omega_0\hat{e}_3$

i.e., without perturbations

$$\omega_1(t=0)=0 \tag{29}$$

$$\omega_2(t=0)=0 \tag{30}$$

$$\omega_3(t=0)=\omega_0 \tag{31}$$

Substituting Equations (29)~(31) into Equations (20)~(22), it yields,

$$\frac{d\omega_1}{dt}=\frac{d\omega_2}{dt}=\frac{d\omega_3}{dt}=0$$

Thus, $\vec{\omega}(t=0)=\omega_0\hat{e}_3$ is an equilibrium state.

所以 Cases 1~3 都是平衡態的解。以下就用穩定分析法，看看這些平衡態是「穩定的平衡態」（stable equilibrium）還是「不穩定的平衡態」（unstable equilibrium）。

在進行穩定分析之前，我們先分析一下$d^2\omega_1/dt^2$, $d^2\omega_2/dt^2$, $d^2\omega_3/dt^2$ 為何。

Taking the time derivative of Equations (20)~(22), and using Equations (20)~(22) to eliminate $d\omega_1/dt$, $d\omega_2/dt$, $d\omega_3/dt$ in the resulting equations, it yields

$$\begin{aligned}\frac{d^2\omega_1}{dt^2} &= \frac{I_2 - I_3}{I_1}\frac{d\omega_2}{dt}\omega_3 + \frac{I_2 - I_3}{I_1}\ \omega_2\frac{d\omega_3}{dt} \\ &= \frac{I_2 - I_3}{I_1}\left(\frac{I_3 - I_1}{I_2}\ \omega_3\omega_1\right)\omega_3 + \frac{I_2 - I_3}{I_1}\ \omega_2\left(\frac{I_1 - I_2}{I_3}\ \omega_1\omega_2\right) \\ &= \left[\frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}\ \omega_3^2 + \frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_3}\ \omega_2^2\right]\omega_1\end{aligned} \tag{32}$$

$$\frac{d^2\omega_2}{dt^2} = \frac{I_3 - I_1}{I_2}\frac{d\omega_3}{dt}\omega_1 + \frac{I_3 - I_1}{I_2}\ \omega_3\frac{d\omega_1}{dt}$$

$$= \frac{I_3 - I_1}{I_2}\left(\frac{I_1 - I_2}{I_3}\ \omega_1\omega_2\right)\omega_1 + \frac{I_3 - I_1}{I_2}\ \omega_3\left(\frac{I_2 - I_3}{I_1}\ \omega_2\omega_3\right) \tag{33}$$

$$= \left[\frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3}\omega_1^2 + \frac{(I_3 - I_1)(I_2 - I_3)}{I_2 I_1}\omega_3^2\right]\omega_2$$

$$\frac{d^2\omega_3}{dt^2} = \frac{I_1 - I_2}{I_3}\frac{d\omega_1}{dt}\omega_2 + \frac{I_1 - I_2}{I_3}\ \omega_1\frac{d\omega_2}{dt}$$

$$= \frac{I_1 - I_2}{I_3}\left(\frac{I_2 - I_3}{I_1}\ \omega_2\omega_3\right)\omega_2 + \frac{I_1 - I_2}{I_3}\ \omega_1\left(\frac{I_3 - I_1}{I_2}\ \omega_3\omega_1\right) \tag{34}$$

$$= \left[\frac{(I_1 - I_2)(I_2 - I_3)}{I_3 I_1}\omega_2^2 + \frac{(I_1 - I_2)(I_3 - I_1)}{I_3 I_2}\omega_1^2\right]\omega_3$$

Equations (32)~(34) can be rewritten as

$$\frac{d^2\omega_1}{dt^2} = (A_3\ \omega_3^2 + A_2\omega_2^2)\omega_1 \tag{35}$$

$$\frac{d^2\omega_2}{dt^2} = (A_1\omega_1^2 + A_3\omega_3^2)\omega_2 \tag{36}$$

$$\frac{d^2\omega_3}{dt^2} = (A_2\omega_2^2 + A_1\omega_1^2)\omega_3 \tag{37}$$

where

$$A_1 = \frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3} \tag{38}$$

$$A_2 = \frac{(I_1 - I_2)(I_2 - I_3)}{I_3 I_1} \tag{39}$$

$$A_3 = \frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2} \tag{40}$$

For $I_1 > I_2 > I_3$, it yields $A_1 < 0$, $A_2 > 0$, and $A_3 < 0$.

穩定分析 Stability Analysis:

Case 1: equilibrium state $\vec{\omega}_0 = \omega_0 \hat{e}_1$

Adding small amplitude perturbations to the equilibrium state, it yields

$$\omega_1 = \omega_0 + \delta\omega_1 \tag{41}$$

$$\omega_2 = \delta\omega_2 \tag{42}$$

$$\omega_3 = \delta\omega_3 \tag{43}$$

where

$$\left|\frac{\delta\omega_1}{\omega_0}\right| \approx \left|\frac{\delta\omega_2}{\omega_0}\right| \approx \left|\frac{\delta\omega_3}{\omega_0}\right| \approx \epsilon \ll 1$$

Substituting Equations (41)~(43) into Equations (35)~(37), it yields,

$$\frac{d^2\delta\omega_1}{dt^2} = (A_3\,\delta\omega_3^2 + A_2\delta\omega_2^2)(\delta\omega_1 + \omega_0) \approx (\epsilon^2\omega_0^2)\omega_0$$

$$\frac{d^2\delta\omega_2}{dt^2} = [A_1(\delta\omega_1 + \omega_0)^2 + A_3\delta\omega_3^2]\delta\omega_2 \approx (A_1 + A_3\epsilon^2)\omega_0^2\,\delta\omega_2$$

$$\frac{d^2\delta\omega_3}{dt^2} = [A_2\delta\omega_2^2 + A_1(\delta\omega_1 + \omega_0)^2]\delta\omega_3 \approx (A_2\epsilon^2 + A_1)\omega_0^2\,\delta\omega_3$$

Ignoring the ϵ^2 terms, it yields

$$\frac{d^2\delta\omega_1}{dt^2} \approx 0 \tag{44}$$

$$\frac{d^2\delta\omega_2}{dt^2} \approx A_1\omega_0^2\,\delta\omega_2 \tag{45}$$

$$\frac{d^2\delta\omega_3}{dt^2} = A_1\omega_0^2\,\delta\omega_3 \tag{46}$$

Since $A_1 < 0$, Equations (45) and (46) yield the small amplitude perturbations $\delta\omega_2$ and $\delta\omega_3$ will not be amplified. Thus, $\vec{\omega}_0 = \omega_0\hat{e}_1$ is a stable equilibrium state.

Case 2: equilibrium state $\vec{\omega}_0 = \omega_0\hat{e}_2$

Adding small amplitude perturbations to the equilibrium state, it yields

$$\omega_1 = \delta\omega_1 \tag{47}$$

$$\omega_2 = \omega_0 + \delta\omega_2 \tag{48}$$

$$\omega_3 = \delta\omega_3 \tag{49}$$

where

$$\left|\frac{\delta\omega_1}{\omega_0}\right| \approx \left|\frac{\delta\omega_2}{\omega_0}\right| \approx \left|\frac{\delta\omega_3}{\omega_0}\right| \approx \epsilon \ll 1$$

Substituting Equations (47)~(49) into Equations (35)~(37), it yields,

$$\frac{d^2\delta\omega_1}{dt^2} = [A_3\ \delta\omega_3^2 + A_2(\delta\omega_2 + \omega_0)^2]\delta\omega_1 \approx (A_3\epsilon^2 + A_2)\omega_0^2\ \delta\omega_1$$

$$\frac{d^2\delta\omega_2}{dt^2} = (A_1\delta\omega_1^2 + A_3\delta\omega_3^2)(\delta\omega_2 + \omega_0) \approx (\epsilon^2\omega_0^2)\omega_0$$

$$\frac{d^2\delta\omega_3}{dt^2} = [A_2(\delta\omega_2 + \omega_0)^2 + A_1\delta\omega_1^2]\delta\omega_3 \approx (A_2 + A_1\epsilon^2)\omega_0^2\ \delta\omega_3$$

Ignoring the ϵ^2 terms, it yields

$$\frac{d^2\delta\omega_1}{dt^2} \approx A_2\omega_0^2\ \delta\omega_1 \tag{50}$$

$$\frac{d^2\delta\omega_2}{dt^2} \approx 0 \tag{51}$$

$$\frac{d^2\delta\omega_3}{dt^2} = A_2\omega_0^2\,\delta\omega_3 \tag{52}$$

Since $A_2 > 0$, Equations (50) and (52) yield the small amplitude perturbations $\delta\omega_1$ and $\delta\omega_3$ will be amplified. Thus, $\vec{\omega}_0 = \omega_0\hat{e}_2$ is an unstable equilibrium state.

Case 3: equilibrium state $\vec{\omega}_0 = \omega_0\hat{e}_3$
Adding small amplitude perturbations to the equilibrium state, it yields

$$\omega_1 = \delta\omega_1 \tag{53}$$

$$\omega_2 = \delta\omega_2 \tag{54}$$

$$\omega_3 = \omega_0 + \delta\omega_3 \tag{55}$$

where

$$\left|\frac{\delta\omega_1}{\omega_0}\right| \approx \left|\frac{\delta\omega_2}{\omega_0}\right| \approx \left|\frac{\delta\omega_3}{\omega_0}\right| \approx \epsilon \ll 1$$

Substituting Equations (53)~(55) into Equations (35)~(37), it yields,

$$\frac{d^2\delta\omega_1}{dt^2} = [A_3\,(\delta\omega_3 + \omega_0)^2 + A_2\delta\omega_2^2]\delta\omega_1 \approx (A_3 + A_2\epsilon^2)\omega_0^2\,\delta\omega_1$$

$$\frac{d^2\delta\omega_2}{dt^2} = [A_1\delta\omega_1^2 + A_3(\delta\omega_3 + \omega_0)^2]\delta\omega_2 \approx (A_1\epsilon^2 + A_3)\omega_0^2\,\delta\omega_2$$

$$\frac{d^2\delta\omega_3}{dt^2} = (A_2\delta\omega_2^2 + A_1\delta\omega_1^2)(\delta\omega_3 + \omega_0) \approx (\epsilon^2\omega_0^2)\omega_0$$

Ignoring the ϵ^2 terms, it yields

$$\frac{d^2\delta\omega_1}{dt^2} \approx A_3\omega_0^2\,\delta\omega_1 \tag{56}$$

$$\frac{d^2\delta\omega_2}{dt^2} \approx A_3\omega_0^2\,\delta\omega_2 \tag{57}$$

$$\frac{d^2\delta\omega_3}{dt^2} = 0 \tag{58}$$

Since $A_3 < 0$, Equations (56) and (57) yield the small amplitude perturbations $\delta\omega_2$ and $\delta\omega_3$ will not be amplified. Thus, $\vec{\omega}_0 = \omega_0\hat{e}_3$ is a stable equilibrium state.