Review:

Inertia (a scalar) = mass = m (慣量 即 質量 是一種純量)

Inertia tensor = Moment of inertia = $\vec{\vec{I}}$ (轉動慣量 是一種二階張量)

Velocity $\vec{v} = d\vec{x}/dt$ (速度 是一種向量，方向沿著前進方向，速度單位：長度/時間)

Angular velocity $\vec{\omega}$ (角速度 是一種向量，方向沿著轉軸方向，因為多數人是右撇子，所以用右手定則定義角速度的方向。角速度與磁場等，都是 pseudo vectors。他們的方向，若用左手定則來定義，則全部反向。 角速度單位：1/時間，$|\vec{\omega}| = d\theta/dt$)

$\vec{v} = \vec{\omega} \times \vec{r}$ (旋轉時，速度 與 角速度的關係。因為旋轉時 位移為弧長 $\Delta s = r\,\Delta\theta$ 因此旋轉速度大小 $v_\theta = \lim_{\Delta t \to 0} \Delta s/\Delta t = \lim_{\Delta t \to 0} r\,\Delta\theta/\Delta t \;= r\omega$ 。方向，就請自己比畫一下了)

Momentum $\vec{p} = m\vec{v}$ (動量 或 線動量 是一種向量，方向沿著前進方向，單位：____)

Angular momentum $\vec{L} = \vec{r} \times \vec{p}$ (角動量 是一種向量，方向沿著轉軸方向，右手定則，單位：__________)

現在考慮一個剛體的旋轉。把剛體切成N小塊。若此剛體沿著一個轉軸以角速度 $\vec{\omega}$ 旋轉。若第k小塊距離轉軸的距離向量為 $\vec{r}_k$ 質量為 m_k，則此剛體的角動量為

$$\vec{L} = \sum_{k=1}^{N} \vec{r}_k \times \vec{p}_k = \sum_{k=1}^{N} \vec{r}_k \times [m_k \vec{v}_k] = \sum_{k=1}^{N} \vec{r}_k \times [m_k(\vec{\omega} \times \vec{r}_k)] \tag{1}$$

由質心的定義可得，若剛體質心距離轉軸的距離向量為$\vec{R}$ 且剛體總質量為M 則

$$\vec{R} = \frac{\sum_{k=1}^{N} m_k \vec{r}_k}{\sum_{k=1}^{N} m_k} = \frac{1}{M}\sum_{k=1}^{N} m_k \vec{r}_k \tag{2}$$

我們總是希望了解「剛體質心運動」在剛體的角動量中所扮演的角色，所以我們將「第k小塊距離轉軸的距離向量$\vec{r}_k$」拆解為

$$\vec{r}_k = (\vec{r}_k - \vec{R}) + \vec{R} \tag{3}$$

Substituting Equation (3) into Equation (1) to replace all the $\vec{r}_k$ in Equation (1), it yields

$$\begin{aligned}\vec{L} &= \sum_{k=1}^{N} \vec{r}_k \times [m_k(\vec{\omega} \times \vec{r}_k)] = \sum_{k=1}^{N} [(\vec{r}_k - \vec{R}) + \vec{R}] \times \{m_k \vec{\omega} \times [(\vec{r}_k - \vec{R}) + \vec{R}]\} \\ &= \sum_{k=1}^{N} [(\vec{r}_k - \vec{R})] \times \{m_k \vec{\omega} \times [(\vec{r}_k - \vec{R})]\} + M\vec{R} \times (\vec{\omega} \times \vec{R})\end{aligned} \tag{4}$$

where

$$\sum_{k=1}^{N}[(\vec{r}_k-\vec{R})+\vec{R}]\times\{m_k\vec{\omega}\times[(\vec{r}_k-\vec{R})+\vec{R}]\}$$

$$=\sum_{k=1}^{N}[(\vec{r}_k-\vec{R})]\times\{m_k\vec{\omega}\times[(\vec{r}_k-\vec{R})]\}+\sum_{k=1}^{N}[(\vec{r}_k-\vec{R})]\times\{m_k\vec{\omega}\times[\vec{R}]\}$$

$$+\sum_{k=1}^{N}[\vec{R}]\times\{m_k\vec{\omega}\times[(\vec{r}_k-\vec{R})]\}+\sum_{k=1}^{N}[\vec{R}]\times\{m_k\vec{\omega}\times[\vec{R}]\}$$

$$=\sum_{k=1}^{N}[(\vec{r}_k-\vec{R})]\times\{m_k\vec{\omega}\times[(\vec{r}_k-\vec{R})]\}+\left\{\sum_{k=1}^{N}m_k(\vec{r}_k-\vec{R})\right\}\times(\vec{\omega}\times\vec{R})$$

$$+\vec{R}\times\left\{\vec{\omega}\times\sum_{k=1}^{N}m_k(\vec{r}_k-\vec{R})\right\}+\vec{R}\times(\vec{\omega}\times\vec{R})\left[\sum_{k=1}^{N}m_k\right]$$

$$=\sum_{k=1}^{N}[(\vec{r}_k-\vec{R})]\times\{m_k\vec{\omega}\times[(\vec{r}_k-\vec{R})]\}+(M\vec{R}-M\vec{R})\times(\vec{\omega}\times\vec{R})$$

$$+\vec{R}\times\{\vec{\omega}\times(M\vec{R}-M\vec{R})\}+\vec{R}\times(\vec{\omega}\times\vec{R})M$$

$$=\sum_{k=1}^{N}[(\vec{r}_k-\vec{R})]\times\{m_k\vec{\omega}\times[(\vec{r}_k-\vec{R})]\}+0+0+M\vec{R}\times(\vec{\omega}\times\vec{R})$$

Since

$$\vec{R}\times(\vec{\omega}\times\vec{R})=(\vec{R}\cdot\vec{R})\vec{\omega}-\vec{R}\vec{R}\cdot\vec{\omega}$$

Equation (4) can be rewritten as

$$\vec{L}=\left\{\vec{\vec{I}}_G+M\left[(\vec{R}\cdot\vec{R})\vec{\vec{1}}-\vec{R}\vec{R}\right]\right\}\cdot\vec{\omega} \tag{5}$$

where

$$\vec{\vec{I}}_G=\sum_{k=1}^{N}m_k\left[(\vec{r}_k-\vec{R})\cdot(\vec{r}_k-\vec{R})\vec{\vec{1}}-(\vec{r}_k-\vec{R})(\vec{r}_k-\vec{R})\right] \tag{6}$$

is the inertia tensor with respect to a rotating axis passing through the center of mass of the rigid body.

舉例說明：

Let $\vec{R} = R_x\hat{x} + R_y\hat{y} + R_z\hat{z}$. It yields

$$M\left[\left(\vec{R}\cdot\vec{R}\right)\vec{\vec{1}} - \vec{R}\vec{R}\right]$$

$$= M\left\{\left(R_x^2 + R_y^2 + R_z^2\right)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}R_x^2 & R_xR_y & R_xR_z\\ R_yR_x & R_y^2 & R_yR_z\\ R_zR_x & R_zR_y & R_z^2\end{bmatrix}\right\} \quad (7)$$

$$= M\begin{bmatrix}R_y^2 + R_z^2 & -R_xR_y & -R_xR_z\\ -R_yR_x & R_x^2 + R_z^2 & -R_yR_z\\ -R_zR_x & -R_zR_y & R_x^2 + R_y^2\end{bmatrix}$$

Likewise

$$\vec{\vec{I}}_G = \sum_{k=1}^{N} m_k\left[\left(\vec{r}_k - \vec{R}\right)\cdot\left(\vec{r}_k - \vec{R}\right)\vec{\vec{1}} - \left(\vec{r}_k - \vec{R}\right)\left(\vec{r}_k - \vec{R}\right)\right] = \begin{bmatrix}I_{Gxx} & I_{Gxy} & I_{Gxz}\\ I_{Gyx} & I_{Gyy} & I_{Gyz}\\ I_{Gzx} & I_{Gzy} & I_{Gzz}\end{bmatrix} \quad (8)$$

where

$$I_{Gxx} = \sum_{k=1}^{N} m_k\left\{\left[\left(\vec{r}_k - \vec{R}\right)_y\right]^2 + \left[\left(\vec{r}_k - \vec{R}\right)_z\right]^2\right\}$$

$$I_{Gyy} = \sum_{k=1}^{N} m_k\left\{\left[\left(\vec{r}_k - \vec{R}\right)_x\right]^2 + \left[\left(\vec{r}_k - \vec{R}\right)_z\right]^2\right\}$$

$$I_{Gzz} = \sum_{k=1}^{N} m_k\left\{\left[\left(\vec{r}_k - \vec{R}\right)_x\right]^2 + \left[\left(\vec{r}_k - \vec{R}\right)_y\right]^2\right\}$$

$$I_{Gxy} = I_{Gyx} = -\sum_{k=1}^{N} m_k\left(\vec{r}_k - \vec{R}\right)_x\cdot\left(\vec{r}_k - \vec{R}\right)_y$$

$$I_{Gxz} = I_{Gzx} = -\sum_{k=1}^{N} m_k\left(\vec{r}_k - \vec{R}\right)_x\cdot\left(\vec{r}_k - \vec{R}\right)_z$$

$$I_{Gyz} = I_{Gzy} = -\sum_{k=1}^{N} m_k\left(\vec{r}_k - \vec{R}\right)_y\cdot\left(\vec{r}_k - \vec{R}\right)_z$$

若取旋轉軸為 z 軸，則 $\vec{\omega} = \omega_z\hat{z}$ and $\vec{R} = R_x\hat{x} + R_y\hat{y}$。

因此

$$M\left[\left(\vec{R}\cdot\vec{R}\right)\vec{\vec{1}} - \vec{R}\vec{R}\right]\cdot\vec{\omega} = M\begin{bmatrix}R_y^2 & -R_xR_y & 0\\ -R_yR_x & R_x^2 & 0\\ 0 & 0 & R_x^2 + R_y^2\end{bmatrix}\cdot\begin{bmatrix}0\\ 0\\ \omega_z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ M\left(R_x^2 + R_y^2\right)\omega_z\end{bmatrix}$$

Likewise

$$\vec{\vec{I}}_G \cdot \vec{\omega} = \left\{ \sum_{k=1}^{N} m_k \left[(\vec{r}_k - \vec{R}) \cdot (\vec{r}_k - \vec{R}) \vec{\vec{1}} - (\vec{r}_k - \vec{R})(\vec{r}_k - \vec{R}) \right] \right\} \cdot \vec{\omega}$$
$$= \begin{bmatrix} I_{Gxx} & I_{Gxy} & 0 \\ I_{Gyx} & I_{Gyy} & 0 \\ 0 & 0 & I_{Gzz} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_{Gzz}\omega_z \end{bmatrix} = I_{Gzz}\omega_z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where

$$I_{Gxx} = \sum_{k=1}^{N} m_k \left\{ \left[(\vec{r}_k - \vec{R})_y \right]^2 \right\}$$
$$I_{Gyy} = \sum_{k=1}^{N} m_k \left\{ \left[(\vec{r}_k - \vec{R})_x \right]^2 \right\}$$
$$I_{Gzz} = \sum_{k=1}^{N} m_k \left\{ \left[(\vec{r}_k - \vec{R})_x \right]^2 + \left[(\vec{r}_k - \vec{R})_y \right]^2 \right\}$$
$$I_{Gxy} = I_{Gyx} = -\sum_{k=1}^{N} m_k (\vec{r}_k - \vec{R})_x \cdot (\vec{r}_k - \vec{R})_y$$

and the matrix representation of the unit vector $\hat{z}$ is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

因此我們得到平行軸定理：

$$\vec{L} = \left\{ \vec{\vec{I}}_G + M \left[(\vec{R} \cdot \vec{R}) \vec{\vec{1}} - \vec{R}\vec{R} \right] \right\} \cdot \vec{\omega} = \left[I_{Gzz} + M \left(R_x^2 + R_y^2 \right) \right] \omega_z \hat{z} = (I_{Gzz} + MR^2) \omega_z \hat{z}$$

現在讓我們考慮另一個簡單的範例，讓我們考慮一個類似手機的長方體，長度為 L，寬度為 W，厚度為 D，且 $D < W < L$，若我們取沿著長度方向為x 軸，寬度方向為y 軸，厚度方向為z 軸，並取質心為原點。則根據方程式(8)可知

$$\vec{\vec{I}} = \vec{\vec{I}}_G = \begin{bmatrix} I_{Gxx} & I_{Gxy} & I_{Gxz} \\ I_{Gyx} & I_{Gyy} & I_{Gyz} \\ I_{Gzx} & I_{Gzy} & I_{Gzz} \end{bmatrix} = \begin{bmatrix} I_{Gxx} & 0 & 0 \\ 0 & I_{Gyy} & 0 \\ 0 & 0 & I_{Gzz} \end{bmatrix}$$

其中因為此長方體以x 軸、y 軸、或 z 軸旋轉時，都對稱於質心，因此 $I_{Gxy} = I_{Gxz} = I_{Gyz} = 0$。所以轉動動量$\vec{L} = \vec{\vec{I}} \cdot \vec{\omega}$ 的矩陣表示式可寫為

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{Gxx} & 0 & 0 \\ 0 & I_{Gyy} & 0 \\ 0 & 0 & I_{Gzz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} I_{Gxx}\omega_x \\ I_{Gyy}\omega_y \\ I_{Gzz}\omega_z \end{bmatrix}$$

向量表示式為

$$L_x\hat{x} + L_y\hat{y} + L_z\hat{z} = I_{Gxx}\omega_x\hat{x} + I_{Gyy}\omega_y\hat{y} + I_{Gzz}\omega_z\hat{z}$$

如何將以上「矩陣表示式」與「向量表示式」做一個連結呢？

為了方便討論，我們令 基底 $\mathcal{B} = \{\hat{x}, \hat{y}, \hat{z}\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

The matrix representation of $\vec{e}_1$ on the basis $\mathcal{B}$ is $(\vec{e}_1)_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

The matrix representation of $\vec{e}_2$ on the basis $\mathcal{B}$ is $(\vec{e}_2)_{\mathcal{B}} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

The matrix representation of $\vec{e}_3$ on the basis $\mathcal{B}$ is $(\vec{e}_3)_{\mathcal{B}} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

Thus,

$$\left(\vec{L}\right)_{\mathcal{B}} = L_x(\vec{e}_1)_{\mathcal{B}} + L_y(\vec{e}_2)_{\mathcal{B}} + L_z(\vec{e}_3)_{\mathcal{B}} = L_x\begin{bmatrix}1\\0\\0\end{bmatrix} + L_y\begin{bmatrix}0\\1\\0\end{bmatrix} + L_z\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}L_x\\L_y\\L_z\end{bmatrix}$$

$$(\vec{\omega})_{\mathcal{B}} = \omega_x(\vec{e}_1)_{\mathcal{B}} + \omega_y(\vec{e}_2)_{\mathcal{B}} + \omega_z(\vec{e}_3)_{\mathcal{B}} = \omega_x\begin{bmatrix}1\\0\\0\end{bmatrix} + \omega_y\begin{bmatrix}0\\1\\0\end{bmatrix} + \omega_z\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\omega_x\\\omega_y\\\omega_z\end{bmatrix}$$

Likewise,

$$\begin{bmatrix}I_{Gxx}\omega_x\\I_{Gyy}\omega_y\\I_{Gzz}\omega_z\end{bmatrix} = I_{Gxx}\omega_x\begin{bmatrix}1\\0\\0\end{bmatrix} + I_{Gyy}\omega_y\begin{bmatrix}0\\1\\0\end{bmatrix} + I_{Gzz}\omega_z\begin{bmatrix}0\\0\\1\end{bmatrix} = I_{Gxx}\omega_x\hat{x} + I_{Gyy}\omega_y\hat{y} + I_{Gzz}\omega_z\hat{z}$$

以下要說明 二階張量 $\vec{\vec{I}}_G$的「物理張量表示式」 $\vec{\vec{I}}_G = I_{Gxx}\hat{x}\hat{x} + I_{Gyy}\hat{y}\hat{y} + I_{Gzz}\hat{z}\hat{z}$

與 $\vec{\vec{I}}_G$在基底 $\mathcal{B}$ 中的「矩陣表示式」 兩者之間的關係

$$\begin{bmatrix}I_{Gxx} & 0 & 0\\0 & I_{Gyy} & 0\\0 & 0 & I_{Gzz}\end{bmatrix} = I_{Gxx}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + I_{Gyy}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + I_{Gzz}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$= I_{Gxx}\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} + I_{Gyy}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} + I_{Gzz}\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

$$= I_{Gxx}|e_1><e_1| + I_{Gyy}|e_2><e_2| + I_{Gzz}|e_3><e_3|$$

$$= I_{Gxx}(\vec{e}_1)_{\mathcal{B}}(\vec{e}_1)_{\mathcal{B}}^T + I_{Gyy}(\vec{e}_2)_{\mathcal{B}}(\vec{e}_2)_{\mathcal{B}}^T + I_{Gzz}(\vec{e}_3)_{\mathcal{B}}(\vec{e}_3)_{\mathcal{B}}^T$$

$$\vec{\vec{I}}\cdot\vec{\omega} = \vec{\vec{I}}_G\cdot\vec{\omega} = \left(I_{Gxx}\hat{x}\hat{x} + I_{Gyy}\hat{y}\hat{y} + I_{Gzz}\hat{z}\hat{z}\right)\cdot\left(\omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}\right)$$

$$= I_{Gxx}\omega_x\hat{x} + I_{Gyy}\omega_y\hat{y} + I_{Gzz}\omega_z\hat{z}$$

Or

$$\left[I_{Gxx}(\vec{e}_1)_{\mathcal{B}}(\vec{e}_1)_{\mathcal{B}}^T + I_{Gyy}(\vec{e}_2)_{\mathcal{B}}(\vec{e}_2)_{\mathcal{B}}^T + I_{Gzz}(\vec{e}_3)_{\mathcal{B}}(\vec{e}_3)_{\mathcal{B}}^T\right]\left[\omega_x(\vec{e}_1)_{\mathcal{B}} + \omega_y(\vec{e}_2)_{\mathcal{B}} + \omega_z(\vec{e}_3)_{\mathcal{B}}\right]$$

$$= \left[I_{Gxx}\omega_x(\vec{e}_1)_{\mathcal{B}} + I_{Gyy}\omega_y(\vec{e}_2)_{\mathcal{B}} + I_{Gzz}\omega_z(\vec{e}_3)_{\mathcal{B}}\right]$$

Note that $(\vec{e}_i)_{\mathcal{B}}^T\left(\vec{e}_j\right)_{\mathcal{B}} = \delta_{ij} = \begin{cases}1 & \text{if } i = j\\0 & \text{if } i \neq j\end{cases}$

現在若重新選一組 基底 $\mathcal{B}' = \{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ 。若 $\mathcal{B}'$ basis is an orthonormal basis（正交保長的基底）則

$$\vec{e}_i' \cdot \vec{e}_j' = \delta_{ij} = \begin{Bmatrix} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{Bmatrix}$$

Let

$$\vec{e}_1' = A_{11}\vec{e}_1 + A_{21}\vec{e}_2 + A_{31}\vec{e}_3$$
$$\vec{e}_2' = A_{12}\vec{e}_1 + A_{22}\vec{e}_2 + A_{32}\vec{e}_3$$
$$\vec{e}_3' = A_{13}\vec{e}_1 + A_{23}\vec{e}_2 + A_{33}\vec{e}_3$$

Name, $\vec{e}_j' = \sum_{i=1}^{3} A_{ij}\vec{e}_i$. Namely

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1')_{\mathcal{B}} & (\vec{e}_2')_{\mathcal{B}} & (\vec{e}_3')_{\mathcal{B}} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Then, a vector

$$\vec{L} = L_1\vec{e}_1 + L_2\vec{e}_2 + L_3\vec{e}_3 = L_1'\vec{e}_1' + L_2'\vec{e}_2' + L_3'\vec{e}_2'$$

i.e.,

$$\vec{L} = \sum_{i=1}^{3} L_i\vec{e}_i = \sum_{j=1}^{3} L_j'\vec{e}_j' = \sum_{j=1}^{3} L_j' \sum_{i=1}^{3} A_{ij}\vec{e}_i = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} A_{ij}L_j' \right) \vec{e}_i$$

It yields

$$L_i = \sum_{j=1}^{3} A_{ij}L_j'$$

or

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1')_{\mathcal{B}} & (\vec{e}_2')_{\mathcal{B}} & (\vec{e}_3')_{\mathcal{B}} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix}$$

It yields

$$\begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1')_{\mathcal{B}} & (\vec{e}_2')_{\mathcal{B}} & (\vec{e}_3')_{\mathcal{B}} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

For the orthonormal basis, it can be shown that

$$\begin{bmatrix} \leftarrow & (\vec{e}_1')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_2')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_3')_{\mathcal{B}}^T & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1')_{\mathcal{B}} & (\vec{e}_2')_{\mathcal{B}} & (\vec{e}_3')_{\mathcal{B}} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Namely

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1')_{\mathcal{B}} & (\vec{e}_2')_{\mathcal{B}} & (\vec{e}_3')_{\mathcal{B}} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1} = \begin{bmatrix} \leftarrow & (\vec{e}_1')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_2')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_3')_{\mathcal{B}}^T & \rightarrow \end{bmatrix}$$

Thus, we have

$$\begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = \begin{bmatrix} \leftarrow & (\vec{e}_1')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_2')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_3')_{\mathcal{B}}^T & \rightarrow \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \tag{9}$$

Likewise, for

$$\vec{\omega} = \omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3 = \omega'_1\vec{e}'_1 + \omega'_2\vec{e}'_2 + \omega'_3\vec{e}'_2$$

we have

$$\begin{bmatrix}\omega'_1\\ \omega'_2\\ \omega'_3\end{bmatrix} = \begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \tag{10}$$

Then, how about

$$\vec{\vec{I}} = I_{11}\vec{e}_1\vec{e}_1 + I_{22}\vec{e}_2\vec{e}_2 + I_{33}\vec{e}_3\vec{e}_3 = \sum_{i=1}^{3}\sum_{j=1}^{3} I'_{ij}\,\vec{e}'_i\vec{e}'_j$$

Find I'_{ij} =?

Since $\vec{L} = \vec{\vec{I}}\cdot\vec{\omega}$, it yields $\left(\vec{L}\right)_{\mathcal{B}'} = \left(\vec{\vec{I}}\cdot\vec{\omega}\right)_{\mathcal{B}'} = \left(\vec{\vec{I}}\right)_{\mathcal{B}'}(\vec{\omega})_{\mathcal{B}'}$. Namely

$$\begin{bmatrix}L'_1\\ L'_2\\ L'_3\end{bmatrix} = \begin{bmatrix}I'_{11} & I'_{12} & I'_{13}\\ I'_{21} & I'_{22} & I'_{23}\\ I'_{31} & I'_{32} & I'_{33}\end{bmatrix}\begin{bmatrix}\omega'_1\\ \omega'_2\\ \omega'_3\end{bmatrix} \tag{11}$$

Substituting Equations (9) and (10) into Equation (11) to eliminate $\left(\vec{L}\right)_{\mathcal{B}'}$ and $(\vec{\omega})_{\mathcal{B}'}$ respectively, it yields

$$\begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix}\begin{bmatrix}L_1\\ L_2\\ L_3\end{bmatrix} = \begin{bmatrix}I'_{11} & I'_{12} & I'_{13}\\ I'_{21} & I'_{22} & I'_{23}\\ I'_{31} & I'_{32} & I'_{33}\end{bmatrix}\begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \tag{12}$$

Equation (12) yields

$$\begin{bmatrix}L_1\\ L_2\\ L_3\end{bmatrix} = \begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix}^{-1}\begin{bmatrix}I'_{11} & I'_{12} & I'_{13}\\ I'_{21} & I'_{22} & I'_{23}\\ I'_{31} & I'_{32} & I'_{33}\end{bmatrix}\begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \tag{13}$$

We recall that

$$\begin{bmatrix}L_1\\ L_2\\ L_3\end{bmatrix} = \begin{bmatrix}I_{11} & 0 & 0\\ 0 & I_{22} & 0\\ 0 & 0 & I_{33}\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \tag{14}$$

Equations (13) and (14) yields

$$\begin{bmatrix}I_{11} & 0 & 0\\ 0 & I_{22} & 0\\ 0 & 0 & I_{33}\end{bmatrix} = \begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix}^{-1}\begin{bmatrix}I'_{11} & I'_{12} & I'_{13}\\ I'_{21} & I'_{22} & I'_{23}\\ I'_{31} & I'_{32} & I'_{33}\end{bmatrix}\begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix} \tag{15}$$

or

$$\begin{aligned}\begin{bmatrix}I'_{11} & I'_{12} & I'_{13}\\ I'_{21} & I'_{22} & I'_{23}\\ I'_{31} & I'_{32} & I'_{33}\end{bmatrix} &= \begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix}\begin{bmatrix}I_{11} & 0 & 0\\ 0 & I_{22} & 0\\ 0 & 0 & I_{33}\end{bmatrix}\begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix}^{-1}\\ &= \begin{bmatrix}\leftarrow & (\vec{e}'_1)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_2)^T_{\mathcal{B}} & \rightarrow\\ \leftarrow & (\vec{e}'_3)^T_{\mathcal{B}} & \rightarrow\end{bmatrix}\begin{bmatrix}I_{11} & 0 & 0\\ 0 & I_{22} & 0\\ 0 & 0 & I_{33}\end{bmatrix}\begin{bmatrix}\uparrow & \uparrow & \uparrow\\ (\vec{e}'_1)_{\mathcal{B}} & (\vec{e}'_2)_{\mathcal{B}} & (\vec{e}'_3)_{\mathcal{B}}\\ \downarrow & \downarrow & \downarrow\end{bmatrix}\end{aligned} \tag{16}$$

因此只要設計出一組基底 $\mathcal{B}' = \{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ 並寫出 $(\vec{e}_1')_{\mathcal{B}}$ $(\vec{e}_2')_{\mathcal{B}}$ $(\vec{e}_3')_{\mathcal{B}}$ 就可以得到 $(\vec{L})_{\mathcal{B}'}$, $(\vec{\omega})_{\mathcal{B}'}$ 以及一組很複雜的 $(\vec{\vec{I}})_{\mathcal{B}'}$

反過來，如何由此 很複雜的 $(\vec{\vec{I}})_{\mathcal{B}'}$ 回推 $(\vec{e}_1')_{\mathcal{B}}$ $(\vec{e}_2')_{\mathcal{B}}$ $(\vec{e}_3')_{\mathcal{B}}$ 與很簡單的 $(\vec{\vec{I}})_{\mathcal{B}}$ 呢？ 原來 $(\vec{\vec{I}})_{\mathcal{B}'}$ 的 eigen values 就是 I_{11} , I_{22} , I_{33} , 對應的 eigen vectors 就是 $(\vec{e}_1)_{\mathcal{B}'}$, $(\vec{e}_2)_{\mathcal{B}'}$, $(\vec{e}_3)_{\mathcal{B}'}$

Note that, by definition

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1')_{\mathcal{B}} & (\vec{e}_2')_{\mathcal{B}} & (\vec{e}_3')_{\mathcal{B}} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1)_{\mathcal{B}'} & (\vec{e}_2)_{\mathcal{B}'} & (\vec{e}_3)_{\mathcal{B}'} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Thus, we have

$$\begin{bmatrix} \leftarrow & (\vec{e}_1')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_2')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_3')_{\mathcal{B}}^T & \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1)_{\mathcal{B}'} & (\vec{e}_2)_{\mathcal{B}'} & (\vec{e}_3)_{\mathcal{B}'} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$\left(\vec{\vec{I}}\right)_{\mathcal{B}'} = \begin{bmatrix} I_{11}' & I_{12}' & I_{13}' \\ I_{21}' & I_{22}' & I_{23}' \\ I_{31}' & I_{32}' & I_{33}' \end{bmatrix} = \begin{bmatrix} \leftarrow & (\vec{e}_1')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_2')_{\mathcal{B}}^T & \rightarrow \\ \leftarrow & (\vec{e}_3')_{\mathcal{B}}^T & \rightarrow \end{bmatrix} \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1')_{\mathcal{B}} & (\vec{e}_2')_{\mathcal{B}} & (\vec{e}_3')_{\mathcal{B}} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ (\vec{e}_1)_{\mathcal{B}'} & (\vec{e}_2)_{\mathcal{B}'} & (\vec{e}_3)_{\mathcal{B}'} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \leftarrow & (\vec{e}_1)_{\mathcal{B}'}^T & \rightarrow \\ \leftarrow & (\vec{e}_2)_{\mathcal{B}'}^T & \rightarrow \\ \leftarrow & (\vec{e}_3)_{\mathcal{B}'}^T & \rightarrow \end{bmatrix}$$

$$= I_{11} \begin{bmatrix} \uparrow \\ (\vec{e}_1)_{\mathcal{B}'} \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & (\vec{e}_1)_{\mathcal{B}'}^T & \rightarrow \end{bmatrix} + I_{22} \begin{bmatrix} \uparrow \\ (\vec{e}_2)_{\mathcal{B}'} \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & (\vec{e}_2)_{\mathcal{B}'}^T & \rightarrow \end{bmatrix}$$

$$+ I_{33} \begin{bmatrix} \uparrow \\ (\vec{e}_3)_{\mathcal{B}'} \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & (\vec{e}_3)_{\mathcal{B}'}^T & \rightarrow \end{bmatrix}$$