A=SLS^{-1}

- 如果矩陣A有n個eigenvalue and n 個 independent eigenvectors 則

$$A\begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{x}_1 & \ldots & \mathbf{x}_n \\ \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & & \uparrow \\ \lambda_1\mathbf{x}_1 & \ldots & \lambda_n\mathbf{x}_n \\ \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{x}_1 & \ldots & \mathbf{x}_n \\ \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

$$\text{Let } S = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{x}_1 & \ldots & \mathbf{x}_n \\ \downarrow & & \downarrow \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \Rightarrow AS = S\Lambda \quad \Rightarrow A = S\Lambda S^{-1}$$

Symmetric Real Matrices

- If a matrix A satisfies $A_{ij} \in R$ and $A_{ij} = A_{ji}$ or $A = A^T$ then A is a symmetric real matrix
  - The eigenvalues of a symmetric real matrix are all real numbers.
  - The eigenvectors of a symmetric real matrix that corresponding to different eigenvlaues are perpendicular to each other.
  - One can always find a set of orthonormal eigenvectors to diagonalized the symmetric matrix.

Hermitian Matrices

- If a matrix A satisfies $A_{ij} \in C$ and $A_{ij} = \overline{A}_{ji}$ or $A = \overline{A}^T$ then A is a Hermitian matrix
  - The eigenvalues of a Hermitian matrix are all real numbers.
  - The eigenvectors of a Hermitian matrix that corresponding to different eigenvlaues are perpendicular to each other.
  - One can always find a set of orthonormal eigenvectors to diagonalized the Hermitian matrix.

Proof:

The eigenvalues of a Hermitian matrix are all real numbers

(1) $A = \overline{A}^T$

(2) $A\mathbf{x} = \lambda \mathbf{x}$

Taking complex conjugate and transpose of equation (2), it yields

(3) $\overline{\mathbf{x}}^T \overline{A}^T = \overline{\lambda}\overline{\mathbf{x}}^T$

Substituting equation (1) into equation (3), it yields

(4) $\overline{\mathbf{x}}^T A = \overline{\lambda}\overline{\mathbf{x}}^T$

$\overline{\mathbf{x}}^T (2) \Rightarrow$

(5) $\overline{\mathbf{x}}^T A\mathbf{x} = \lambda \overline{\mathbf{x}}^T \mathbf{x}$

Substituting equation (4) into equation (5), it yields

(6) $\overline{\lambda}\overline{\mathbf{x}}^T \mathbf{x} = \lambda \overline{\mathbf{x}}^T \mathbf{x}$

Since $\overline{\mathbf{x}}^T \mathbf{x} = \|\mathbf{x}\|^2 > 0$, equation (6) yields $\overline{\lambda} = \lambda$. That is $\lambda \in R$.

Proof:
The eigenvectors of a Hermitian matrix that corresponding to different eigenvlaues are perpendicular to each other

(1) $A = \overline{A}^T$

(2) $\lambda_1 \neq \lambda_2$, and both of them are real numbers

(3) $A\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$

(4) $A\mathbf{x}_2 = \lambda_2 \mathbf{x}_2$

Taking complex conjugate and transpose of equation (4), it yields

(5) $\overline{\mathbf{x}}_2^T \overline{A}^T = \lambda_2 \overline{\mathbf{x}}_2^T$

Substituting equation (1) into equation (5), it yields

(6) $\overline{\mathbf{x}}_2^T A = \lambda_2 \overline{\mathbf{x}}_2^T$

$\overline{\mathbf{x}}_2^T (3) \Rightarrow$

(7) $\overline{\mathbf{x}}_2^T A\mathbf{x}_1 = \lambda_1 \overline{\mathbf{x}}_2^T \mathbf{x}_1$

Substituting equation (6) into equation (7), it yields

(8) $\lambda_2 \overline{\mathbf{x}}_2^T \mathbf{x}_1 = \lambda_1 \overline{\mathbf{x}}_2^T \mathbf{x}_1 \quad \Rightarrow (\lambda_2 - \lambda_1)\overline{\mathbf{x}}_2^T \mathbf{x}_1 = 0$

Since $\lambda_2 - \lambda_1 \neq 0$, equation (8) yields $\overline{\mathbf{x}}_2^T \mathbf{x}_1 = 0$. That is $\mathbf{x}_2 \perp \mathbf{x}_1$

True or False?

If λ is an eigenvalue of matrix A, and β is an eigenvalue of matrix B, then $\lambda\beta$ is an eigenvalue of matrix AB. True or False?
If it is True, Prove it. If it is False, explain why.

If λ is an eigenvalue of matrix A, and β is an eigenvalue of matrix B, then $\lambda + \beta$ is an eigenvalue of matrix $A + B$. True or False?
If it is True, Prove it. If it is False, explain why.