Review Vectors' Expression

- Expressions of vector
  - Hand writing: $\vec{v}$ or $\underline{v}$ or |v> **or** $\vec{v}^T$ or $\underline{v}^T$ or <v|
  - Typeface: boldface for vector, i.e., $\mathbf{v}$ **or** $\mathbf{v}^T$
  - Matrix expression of vectors: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ **or** $[\; v_1 \quad v_2 \quad v_3 \;]$

  where the vector is defined by

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 = \sum_i v_i\hat{e}_i = v_i\hat{e}_i$$

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = \sum_i v_i\mathbf{e}_i = v_i\mathbf{e}_i$$

Matrix Expression of Vector Products

- inner product: $\mathbf{v}\cdot\mathbf{w}=[\,v_1 \quad v_2 \quad v_3\,]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

  The multiplying of the projection of one vector to the other vector

- cross product: $\mathbf{v}\times\mathbf{w}=\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

  Size and normal direction of the area determined by the two vectors

- dyad product: $\mathbf{v}\mathbf{w}=\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}[\,w_1 \quad w_2 \quad w_3\,]=\begin{bmatrix} v_1w_1 & v_1w_2 & v_1w_3 \\ v_2w_1 & v_2w_2 & v_2w_3 \\ v_3w_1 & v_3w_2 & v_3w_3 \end{bmatrix}$

Review of Matrix Multiplication: Ux, $x^T U^T$

$$U\mathbf{x} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} \uparrow \\ \mathbf{u}_1 \\ \downarrow \end{bmatrix} + x_2 \begin{bmatrix} \uparrow \\ \mathbf{u}_2 \\ \downarrow \end{bmatrix} + x_3 \begin{bmatrix} \uparrow \\ \mathbf{u}_3 \\ \downarrow \end{bmatrix}$$

$$\mathbf{x}^T U^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \\ \leftarrow & \mathbf{u}_2^T & \rightarrow \\ \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix}$$

$$= x_1 \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \end{bmatrix} + x_2 \begin{bmatrix} \leftarrow & \mathbf{u}_2^T & \rightarrow \end{bmatrix} + x_3 \begin{bmatrix} \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix}$$

Review of Matrix Multiplication: UDU^T

$$UDU^T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \\ \leftarrow & \mathbf{u}_2^T & \rightarrow \\ \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ d_1\mathbf{u}_1 & d_2\mathbf{u}_2 & d_3\mathbf{u}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \\ \leftarrow & \mathbf{u}_2^T & \rightarrow \\ \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix}$$

$$= d_1 \begin{bmatrix} \uparrow \\ \mathbf{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \end{bmatrix} + d_2 \begin{bmatrix} \uparrow \\ \mathbf{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_2^T & \rightarrow \end{bmatrix} + d_3 \begin{bmatrix} \uparrow \\ \mathbf{u}_3 \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix}$$

Changing Basis
(Changing Coordinate System)

- **v** in the original coordinate system:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = \sum_j v_j \mathbf{e}_j$$

- **v** in the new coordinate system:

$$\mathbf{v} = v_1^*\mathbf{e}_1^* + v_2^*\mathbf{e}_2^* + v_3^*\mathbf{e}_3^* = \sum_j v_j^* \mathbf{e}_j^*$$

- How to determine v_j^* ?

Changing Basis
How to determine v_j^* ?

- 用計算式的方式（直式）證明

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\mathbf{v} = v_1^*\mathbf{e}_1^* + v_2^*\mathbf{e}_2^* + v_3^*\mathbf{e}_3^* = v_1^*\begin{bmatrix} \uparrow \\ \mathbf{e}_1^* \\ \downarrow \end{bmatrix} + v_2^*\begin{bmatrix} \uparrow \\ \mathbf{e}_2^* \\ \downarrow \end{bmatrix} + v_3^*\begin{bmatrix} \uparrow \\ \mathbf{e}_3^* \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}\begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix}$$

Changing Basis
How to determine v_j^* ?

- 用計算式的方式（直式）證明（續）

$$\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix}$$

$$\begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \overset{\quad}{\underset{\text{if } \mathbf{e}_i^{*T}\mathbf{e}_j^* = \delta_{ij}}{=\!=\!=}} \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

where $\delta_{ij} = 1$, if $i = j$, but $\delta_{ij} = 0$, if $i \neq j$

Changing Basis
How to determine v_j^* ?

- 用文字敘述的方式（橫式）證明，比較抽象，但為傳統證法

$$\text{Let } \mathbf{e}_j^* = \sum_i A_{ij}\mathbf{e}_i \Rightarrow \quad \mathbf{v} = \sum_i v_i \mathbf{e}_i = \sum_j v_j^* \mathbf{e}_j^* = \sum_j v_j^* \sum_i A_{ij}\mathbf{e}_i = \sum_i \sum_j A_{ij} v_j^* \mathbf{e}_i$$

$$\Downarrow$$

$$\Rightarrow \sum_i v_i \mathbf{e}_i = \sum_i \Big(\sum_j A_{ij} v_j^*\Big)\mathbf{e}_i \quad \Rightarrow \quad v_i = \sum_j A_{ij} v_j^*$$

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = A \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} \quad \Rightarrow \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} = A^{-1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Changing Basis
(Changing Coordinate System)

- **P** in the original coordinate system:

  dyad product (in Physics) $\mathbf{P} = \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j$

  matrix multiplication (in Math) $P = \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j^T$

- **P** in the new coordinate system:

$$P = \sum_j \sum_i P_{ij}^* \mathbf{e}_i^* (\mathbf{e}_j^*)^T$$

- How to determine P_{ij}^* ?

Changing Basis
How to determine P_{ij}^* ?

用文字敘述的方式（橫式）證明

$$P = \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j^T = \sum_k \sum_l P_{lk}^* \mathbf{e}_l^* (\mathbf{e}_k^*)^T$$

$$= \sum_k \sum_l P_{lk}^* (\sum_i A_{il} \mathbf{e}_i)(\sum_j A_{jk} \mathbf{e}_j)^T$$

$$= \sum_j \sum_i [\sum_k \sum_l A_{il} P_{lk}^* (A^T)_{kj}] \mathbf{e}_i \mathbf{e}_j^T$$

$$\Rightarrow \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j^T = \sum_j \sum_i [\sum_k \sum_l A_{il} P_{lk}^* (A^T)_{kj}] \mathbf{e}_i \mathbf{e}_j^T$$

$$\Rightarrow P_{ij} = \sum_k \sum_l A_{il} P_{lk}^* (A^T)_{kj}$$

Let $\mathbf{e}_j^* = \sum_i A_{ij} \mathbf{e}_i \Rightarrow$

$$\Downarrow$$

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Changing Basis
How to determine P_{ij}* ?

計算式的方式（直式）證明（續）

Namely,

$$P=\sum_j\sum_i P_{ij}\mathbf{e}_i\mathbf{e}_j^T=\begin{bmatrix}\uparrow & \uparrow & \uparrow\\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \downarrow & \downarrow & \downarrow\end{bmatrix}\begin{bmatrix}P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ P_{31} & P_{32} & P_{33}\end{bmatrix}\begin{bmatrix}\leftarrow & \mathbf{e}_1^T & \rightarrow\\ \leftarrow & \mathbf{e}_2^T & \rightarrow\\ \leftarrow & \mathbf{e}_3^T & \rightarrow\end{bmatrix}$$

Likewise

$$P=\sum_k\sum_l P_{lk}^*\mathbf{e}_l^*(\mathbf{e}_k^*)^T=\begin{bmatrix}\uparrow & \uparrow & \uparrow\\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^*\\ \downarrow & \downarrow & \downarrow\end{bmatrix}\begin{bmatrix}P_{11}^* & P_{12}^* & P_{13}^*\\ P_{21}^* & P_{22}^* & P_{23}^*\\ P_{31}^* & P_{32}^* & P_{33}^*\end{bmatrix}\begin{bmatrix}\leftarrow & \mathbf{e}_1^{*T} & \rightarrow\\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow\\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow\end{bmatrix}$$

Changing Basis
How to determine P_{ij}* ?

計算式的方式（直式）證明（續）

$$P=\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}=\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}\begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix}\begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix}=\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1}\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}\begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix}^{-1}$$

$$\underset{\text{if } \mathbf{e}_i^{*T}\mathbf{e}_j^* = \delta_{ij}}{\Rightarrow} \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix}=\begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$