Introduction to Mechanics

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Symon (1960): Chapter 1: Elements of Newtonian Mechanics

- Review 1-1, 1-2, 1-3, 1-6 & Quiz

  1-1 Mechanics, an exact science
  1-2 Kinematics, the description of motion
  1-3 Dynamics. Mass and force
  1-4 Newton's laws of motion
  1-5 Gravitation
  1-6 Units and dimensions
  1-7 Some elementary problems in mechanics

Outlines

- **The Research Fields of Mechanics**
- **Units and "Dimensional Analysis"**
- **Kinematic Equations** $(d\vec{x}/dt = \vec{v}, d\vec{v}/dt = \vec{a})$ **in Different Coordinate Systems**

The Research Fields of Mechanics

- **Kinematics** $(d\vec{x}/dt = \vec{v}, d\vec{v}/dt = \vec{a})$
  - 運動學: 可以是研究粒子的運動，也可以是研究光影或波動的移動情形。
  - 牛頓以前的科學家，完全不了解天體為何會運行，但是還是可以記錄星星在天空中的位置，這樣的研究，就是一種運動學的研究方式。
- **Statics** (a study of equilibrium states or solitary waves with $\partial/\partial t = 0$)
  - 靜力學 (研究「平衡態」或「固型波」解)
  - 平衡態範例：星球的大氣模型 hydrodynamic equilibrium
其中，「熱壓梯度力」與「重力」達到平衡
  - 要研究簡諧運動前，先要找出一個穩定的平衡態。
  - 動能＋位能 只有在不隨時間變化的位能場中，才會守恆

Q1: 請舉例說明：動能＋位能 只有在不隨時間變化的位能場中，才會守恆

- **Dynamics** (a self-consistent study of macroscopic fluid motion)
  - 動力學: 研究一群粒子的宏觀運動 (in $(\vec{x}, t)$ space) ，以及他們的運動如何造成周圍場的改變。這些場又怎麼影響這些粒子的運動情形。必須用到 Kinematics 來描述粒子或流體運動軌跡。
  - 基本上符合 $d\vec{p}/dt = \vec{F}$ 的因果關係。無法定義 Force 的系統，不能掛名 Dynamics（要掛名Mechanics ，見後）。
  - Dynamics 範例：Hydrodynamics ， Atmospheric Dynamics 大氣動力學, Fluid Dynamics 流體力學（不考慮流體中的波動） ， Magnetohydrodynamics 磁流體力學， Relativistic Dynamics 相對論力學（不涵蓋廣義相對論）
- **Kinetics** (a self-consistent study of microscopic collective motion & changes on heat and entropy)
  - 動力學:研究一群粒子的微觀運動 (in $(\vec{x}, \vec{v}, t)$ space) 與Dynamics 類似。但是談到熱 & 熵的改變，Dynamics用參數方式處理之，Kinetics 可用微觀的方式，探究其原因。
  - Kinetics 範例：Kinetic plasma physics 電漿動力學

Q2: 請說明什麼是「場」(field)?

- **Mechanics** (a true physics)
  - （廣義的）力學：包含可以定義 Force 的系統，以及無法定義Force的系統之物理過程與因果機制的探究。
  - 包含：Dynamics , Kinetics , Statics ，再加上波動、彎曲空間（無Force的概念）、量子波函數（無Force的概念）、電磁波、以及最簡單的粒子運動、等等。接近完整的物理（Physics)
  - Mechanics 範例 ：（無法完全用Force來描述的力學系統）
    Quantum Mechanics 量子力學，
    Wave Mechanics 波動力學，
    Statistical Mechanics 統計力學，
    Relativistic Mechanics 相對論力學（包含廣義相對論），
    Fluid Mechanics 流體力學（包含流體中的波動）。

Units and “Dimensional Analysis”

- 請注意區分 dimensions (因次) 與 units (單位) 的差異。
  - A ***dimension*** is a measure of a physical variable (without numerical values), while a ***unit*** is a way to assign a number or measurement to that dimension.
    For example, length is a *dimension*, but it is measured in *units* of feet (ft) or meters (m).
    https://www.me.psu.edu/cimbala/Learning/General/units.htm
- 只有相同「因次」 (dimensions) 的物理量才可以相加減。
  - 因此要檢查一道複雜的方程式，是否正確合理，最快的方式就是檢查方程式中每一項的「因次」是否相同。(作業與考試作答結果，檢查 1.因次 2.純量、向量、張量)
- 科學家為了確保推導方程式的過程中不犯錯，常常會把具有物理量的方程式改寫為「無因次」(dimensionless) 的形式。這樣的過程叫做「無因次化」 ，或 normalization 「歸一化」 (因為1是常數，沒有因次)。
  - 一個dimensionless 無因次的方程式，有助於我們比較方程式中，各項大小差異。針對不同的時空尺度現象，科學家會將數量相對很小的「項」剔除，以簡化方程式，將有助分析預測物理過程的變化趨勢。
  - 一個dimensionless 無因次的方程式，也有助於我們選擇正確的時空解析度，來進行觀測或數值模擬。

Base Dimensions in SI & in Gauss Units

- For **SI unit**, there are four base dimensions
  L : Length
  T : Time
  M : Mass
  Q : Electric charge
- For **Gaussian unit**, there are only three base dimensions
  L : Length
  T : Time
  M : Mass
  The charge is included in the dimensions of the electric field and the magnetic field in the Gaussian unit.

HW1. Find the dimensions of the following quantities based on their definitions

Quantities	Definitions	Dimensions
Velocity		
Force		
Momentum		
Energy		
Pressure		
Temperature		
Work		
Power		
Gravitational field		
Angular momentum		
Angular velocity		
Torque		
Rotational inertia		
Electric field		

Kinematic Equations in Different Coordinate Systems

	Cartesian	Cylindrical	Spherical
Coord.	$[x(t), y(t), z(t)]$	$[r(t), \theta(t), z(t)]$	$[r(t), \theta(t), \phi(t)]$
position	$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$	$\vec{r}(t) = r(t)\hat{r}(\theta(t)) + z(t)\hat{z}$	$\vec{r}(t) = r(t)\hat{r}(\theta(t), \phi(t))$
$\vec{v} = \dot{\vec{r}}$ $\vec{a} = \ddot{\vec{r}}$	$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$	$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z}$ $\vec{a} = a_r\hat{r} + a_\theta\hat{\theta} + a_z\hat{z}$	$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}$ $\vec{a} = a_r\hat{r} + a_\theta\hat{\theta} + a_\phi\hat{\phi}$
	$v_x(t) = \dot{x}(t)$	$v_r = \dot{r}$	$v_r = \dot{r}$
	$v_y(t) = \dot{y}(t)$	$v_\theta = r\dot{\theta}$	$v_\theta = r\dot{\theta}$
	$v_z(t) = \dot{z}(t)$	$v_z = \dot{z}$	$v_\phi = r\sin\theta\,\dot{\phi}$
	$a_x(t) = \ddot{x}(t)$	$a_r = \ddot{r} - r\dot{\theta}^2$	$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2$
	$a_y(t) = \ddot{y}(t)$	$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$	$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2$
	$a_z(t) = \ddot{z}(t)$	$a_z = \ddot{z}$	$a_\phi = 2\dot{r}\sin\theta\,\dot{\phi} + 2r\cos\theta\,\dot{\theta}\dot{\phi} +$ $r\sin\theta\,\ddot{\phi}$

Definitions of Basis Vectors in Different Coordinate Systems

$$\hat{e}_1 = \frac{\nabla u_1}{|\nabla u_1|} = h_1 \nabla u_1$$

Cartesian coordinate system	$\hat{x} = \frac{\nabla x}{|\nabla x|} = \nabla x$	$\hat{y} = \frac{\nabla y}{|\nabla y|} = \nabla y$	$\hat{z} = \frac{\nabla z}{|\nabla z|} = \nabla z$
Cylindrical coordinate system	$\hat{r} = \frac{\nabla r}{|\nabla r|} = \nabla r$	$\hat{\theta} = \frac{\nabla \theta}{|\nabla \theta|} = r\nabla \theta$	$\hat{z} = \frac{\nabla z}{|\nabla z|} = \nabla z$
Spherical coordinate system	$\hat{r} = \frac{\nabla r}{|\nabla r|} = \nabla r$	$\hat{\theta} = \frac{\nabla \theta}{|\nabla \theta|} = r\nabla \theta$	$\hat{\phi} = \frac{\nabla \phi}{|\nabla \phi|} = r \sin\theta \, \nabla \phi$

Kinematic Equations in the Cylindrical Coordinate System

Let us consider the motion of a particle in a cylindrical coordinate system, where the basis vectors are $\{\hat{r}, \hat{\theta}, \hat{z}\}$, where the unit vector $\hat{r}$ and $\hat{\theta}$ will point to different direction at different θ. Thus, we have $\hat{r} = \hat{r}(\theta)$, and $\hat{\theta} = \hat{\theta}(\theta)$.

Let the coordinate of a particle at time t be $[r(t), \theta(t), z(t)]$.

The position vector of the particle is $\vec{r} = r\hat{r} + z\hat{z}$, which must be a function of the particle's coordinate $[r(t), \theta(t), z(t)]$. The position vector can be rewritten as

$$\vec{r}[r(t), \theta(t), z(t)] = r(t)\hat{r}[\theta(t)] + z(t)\hat{z}$$

Let $\dot{\vec{r}} = d\vec{r}/dt$ and $\ddot{\vec{r}} = d^2\vec{r}/dt^2$

The velocity of the particle is $\vec{v} = \dot{\vec{r}} = v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z}$

The acceleration of the particle is $\vec{a} = \ddot{\vec{r}} = a_r\hat{r} + a_\theta\hat{\theta} + a_z\hat{z}$.

HW2. Please find $d\hat{r}/dt$, $d\hat{\theta}/dt$, (v_r, v_θ, v_z), and (a_r, a_θ, a_z).

Hint: 先用幾何 and/or 代數的方法求 $d\hat{r}/d\theta$ & $d\hat{\theta}/d\theta$ ，再用 chain rule 方法求 $d\hat{r}/dt$ & $d\hat{\theta}/dt$ 。

Kinematic Equations in the Spherical Coordinate System

Let us consider the motion of a particle in a cylindrical coordinate system, where the basis vectors are $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$, where the unit vectors $\hat{r}$ and $\hat{\theta}$ will point to different direction at different θ or different ϕ. Thus, we have $\hat{r} = \hat{r}(\theta, \phi)$, and $\hat{\theta} = \hat{\theta}(\theta, \phi)$. The unit vector $\hat{\phi}$ will point to different direction at different ϕ. Thus, we have $\hat{\phi} = \hat{\phi}(\phi)$.

Let the coordinate of a particle at time t be $[r(t), \theta(t), \phi(t)]$.

The position vector of the particle is $\vec{r} = r\hat{r}$, which must be a function of the particle's coordinate $[r(t), \theta(t), \phi(t)]$. The position vector can be rewritten as

$$\vec{r}[r(t), \theta(t), \phi(t)] = r(t)\hat{r}[\theta(t), \phi\ (t)]$$

Let $\dot{\vec{r}} = d\vec{r}/dt$ and $\ddot{\vec{r}} = d^2\vec{r}/dt^2$

The velocity of the particle is $\vec{v} = \dot{\vec{r}} = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{z}$

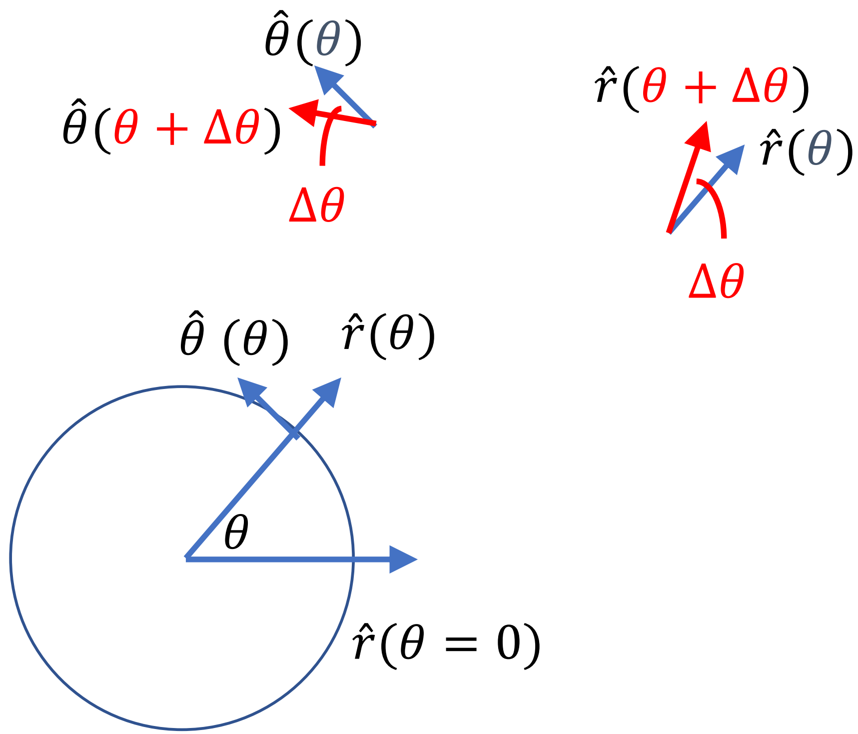
The acceleration of the particle is $\vec{a} = \ddot{\vec{r}} = a_r\hat{r} + a_\theta\hat{\theta} + a_\phi\hat{z}$.

HW3. Please find $d\hat{r}/dt$, $d\hat{\theta}/dt$, $d\hat{\phi}/dt$, (v_r, v_θ, v_ϕ), and (a_r, a_θ, a_ϕ).

Hint: 先用幾何 and/or 代數的方法求 $\partial\hat{r}/\partial\theta$ & $\partial\hat{r}/\partial\phi$ ，再用 chain rule 方法求 $d\hat{r}/dt$ 。

用“幾何”的方式求 單位向量的微分結果

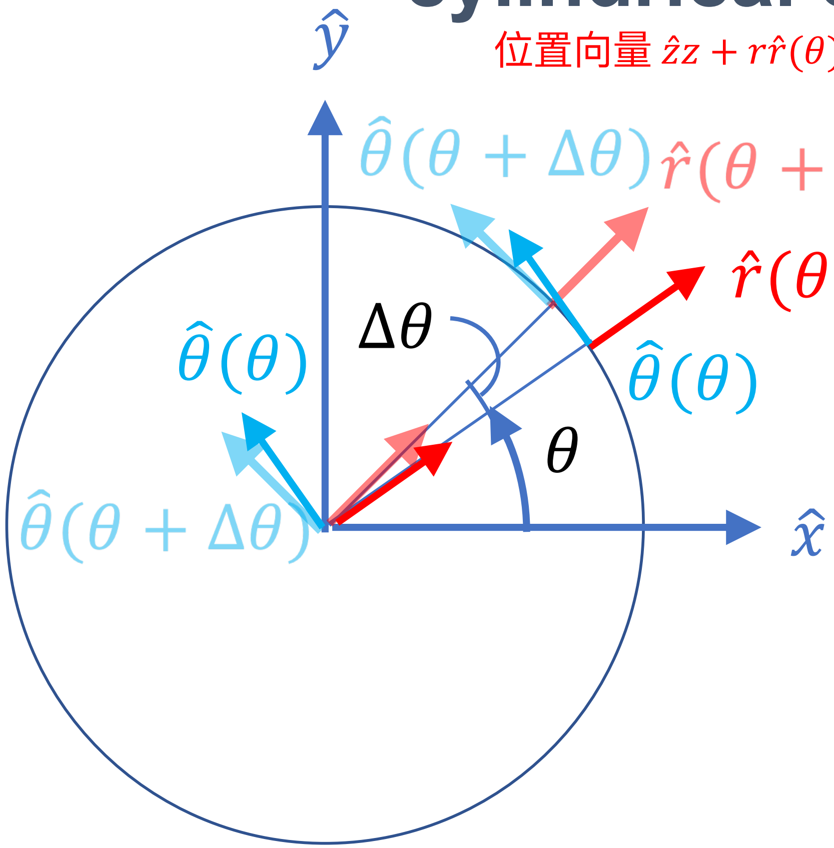
cylindrical coordinate system	spherical coordinate system
$\frac{d\hat{r}}{d\theta} = \hat{\theta}$	$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$
$\frac{d\hat{\theta}}{d\theta} = -\hat{r}$	$\frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$
	$\frac{\partial\hat{r}}{\partial\phi} = \sin\theta\,\hat{\phi}$
	$\frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\,\hat{\phi}$
	$\frac{d\hat{\phi}}{d\phi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$

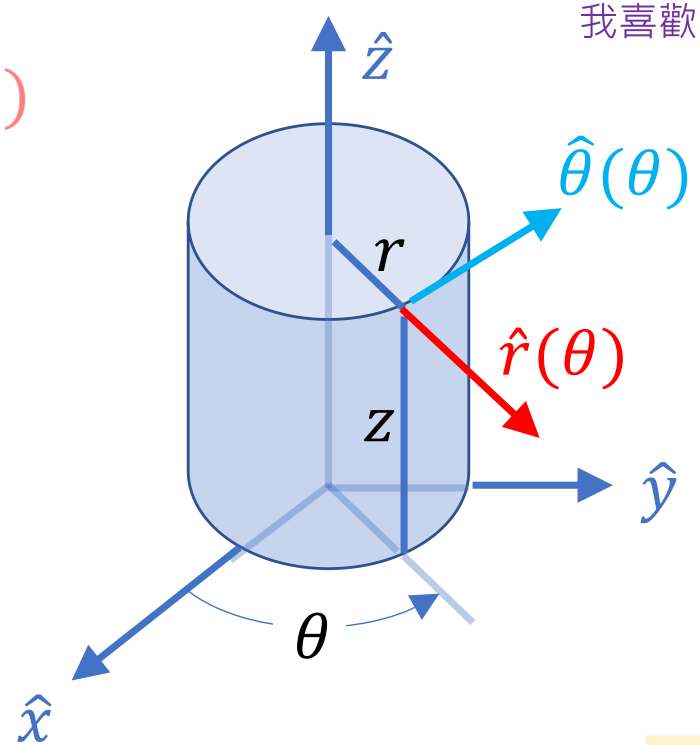


用“幾何”的方式求 單位向量的微分結果

cylindrical coordinate system

位置向量 $\hat{z}z + r\hat{r}(\theta)$ ($= \vec{S} = \vec{r} = \vec{x} = \vec{\xi}$) 喜歡用什麼符號都可以，只要說清楚即可。
我喜歡用 $\vec{x}$ 表示位置向量





$$\frac{d\hat{r}}{d\theta} = \lim_{\Delta\theta\to 0} \frac{\hat{r}(\theta+\Delta\theta) - \hat{r}(\theta)}{\Delta\theta} = \hat{\theta}\left(\frac{|\hat{r}|\cdot\Delta\theta}{\Delta\theta}\right) = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = \lim_{\Delta\theta\to 0} \frac{\hat{\theta}(\theta+\Delta\theta) - \hat{\theta}(\theta)}{\Delta\theta} = -\hat{r}\left(\frac{|\hat{\theta}|\cdot\Delta\theta}{\Delta\theta}\right) = -\hat{r}$$

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}$$

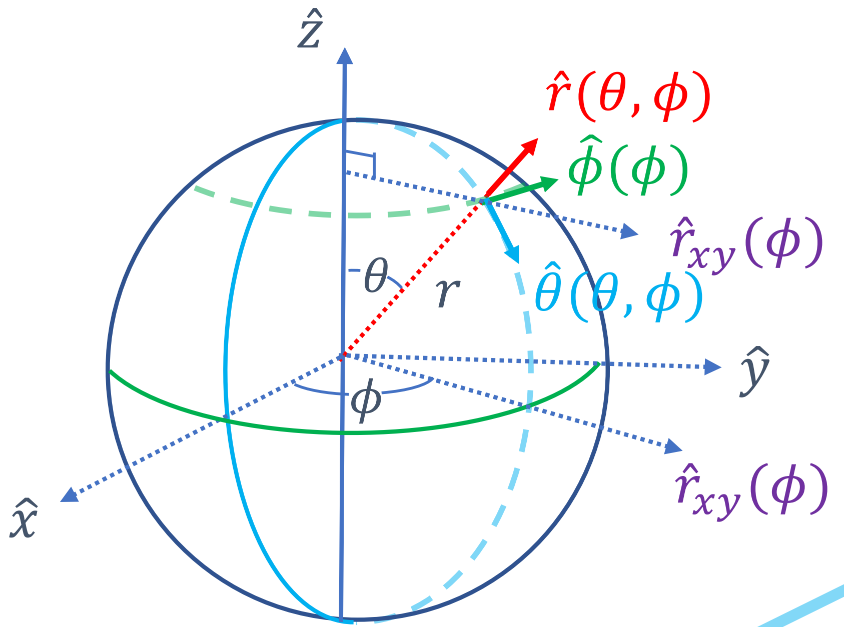
$$\frac{d\hat{\theta}}{d\theta} = -\hat{r}$$

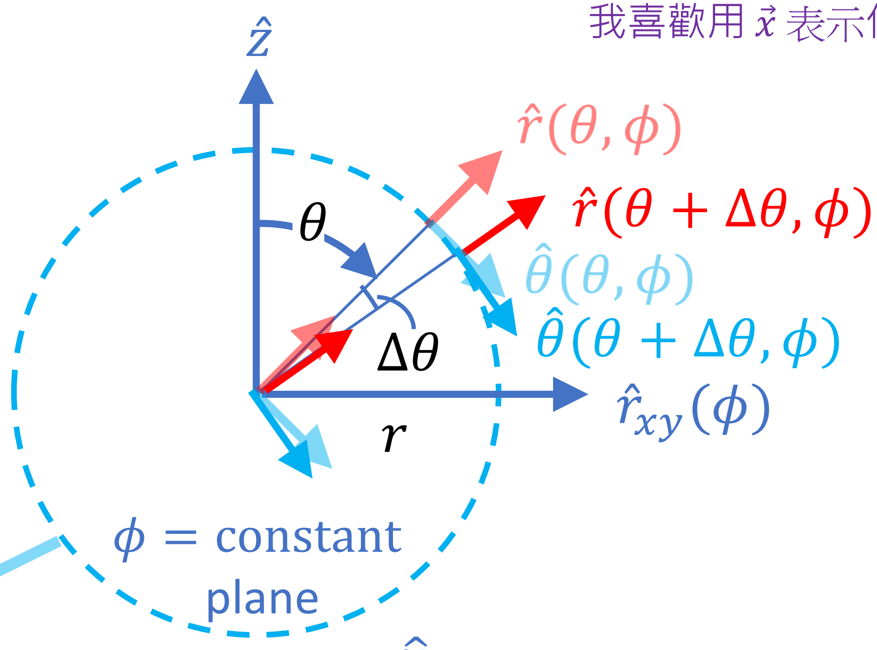
用“幾何”的方式求 單位向量的微分結果

spherical coordinate system

位置向量 $r(t)\hat{r}\big(\theta(t),\phi(t)\big)(=\vec{S}=\vec{r}=\vec{x}=\vec{\xi}\,)$ 喜歡用什麼符號都可以，只要說清楚即可。

我喜歡用 $\vec{x}$ 表示位置向量





$\hat{r}=\hat{r}(\theta,\phi)$

$\hat{\theta}=\hat{\theta}(\theta,\phi)$

$\hat{\phi}=\hat{\phi}(\phi)$

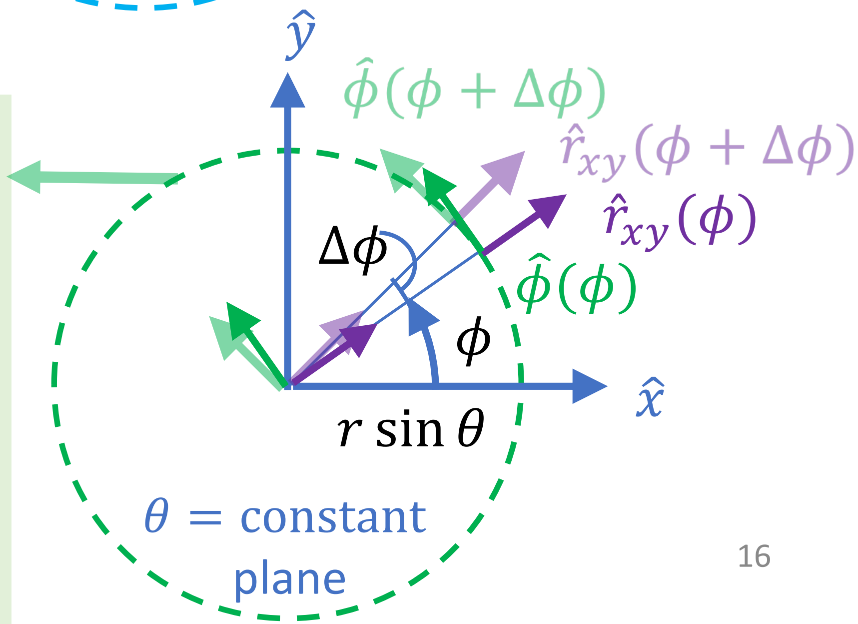
$\hat{r}_{xy}=\hat{r}_{xy}(\phi)$

$$\frac{\partial\hat{r}}{\partial\theta}=\hat{\theta}$$

$$\frac{\partial\hat{\theta}}{\partial\theta}=-\hat{r}$$

$$\frac{d\hat{r}_{xy}}{d\phi}=\hat{\phi}$$

$$\frac{d\hat{\phi}}{d\phi}=-\hat{r}_{xy}$$



幾何方式 1：利用 $\hat{r}_{xy}$，求 各單位向量對 ϕ 的微分

spherical coordinate system

$$\left[\frac{d}{d\phi}\hat{r}_{xy}(\phi)\right] = \hat{\phi}$$

$$\hat{r}(\theta,\phi) = \hat{z}\cos\theta + \hat{r}_{xy}(\phi)\sin\theta$$

$$\frac{\partial\hat{r}}{\partial\phi} = \left[\frac{d}{d\phi}\hat{r}_{xy}(\phi)\right]\sin\theta = \hat{\phi}\sin\theta$$

$$\hat{\theta}(\theta,\phi) = -\hat{z}\sin\theta + \hat{r}_{xy}(\phi)\cos\theta$$

$$\frac{\partial\hat{\theta}}{\partial\phi} = \left[\frac{d}{d\phi}\hat{r}_{xy}(\phi)\right]\cos\theta = \hat{\phi}\cos\theta$$

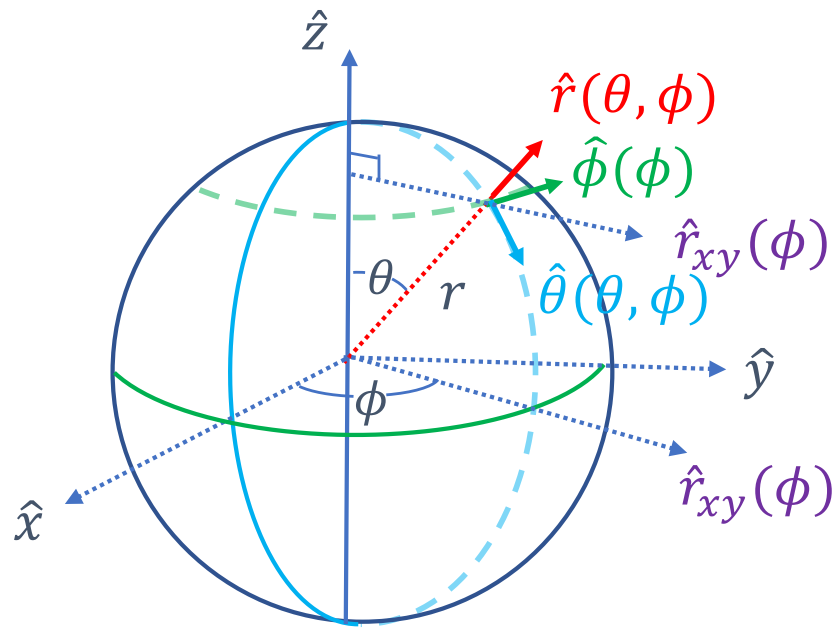
$$\frac{d\hat{\phi}}{d\phi} = -\hat{r}_{xy}(\phi)$$

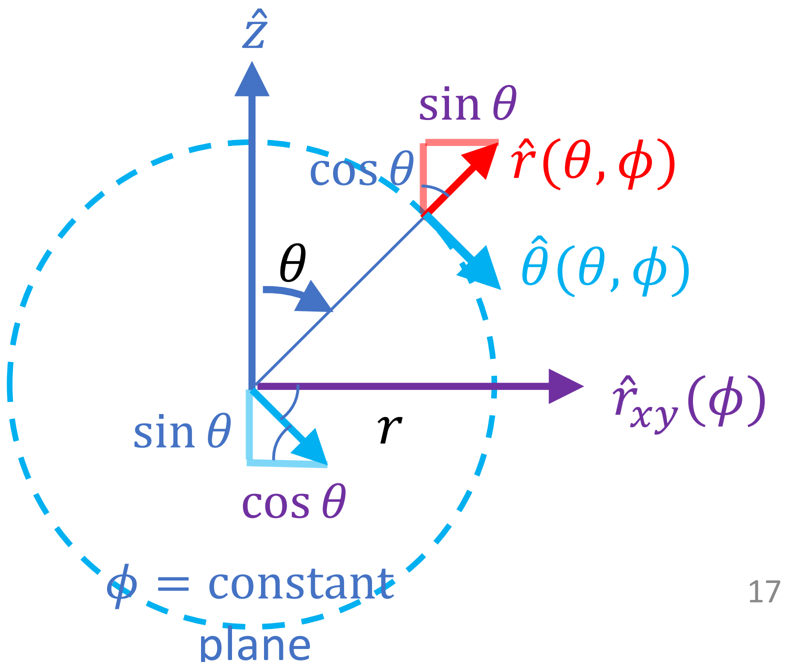
Since

$$\left[\hat{r}(\theta,\phi)\sin\theta + \hat{\theta}(\theta,\phi)\cos\theta\right] = \hat{r}_{xy}(\phi)$$

It yields

$$\frac{d\hat{\phi}}{d\phi} = -\left[\hat{r}\sin\theta + \hat{\theta}\cos\theta\right]$$

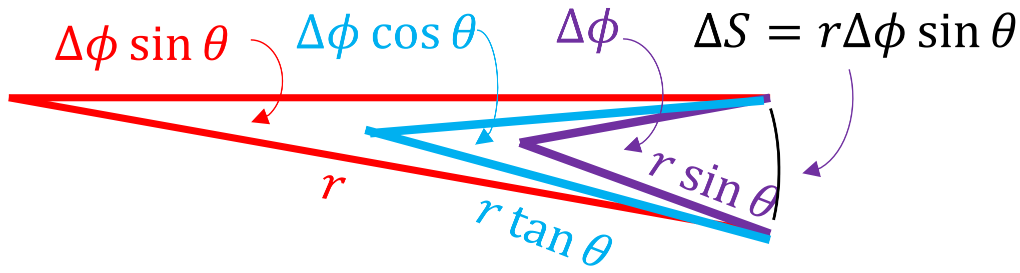


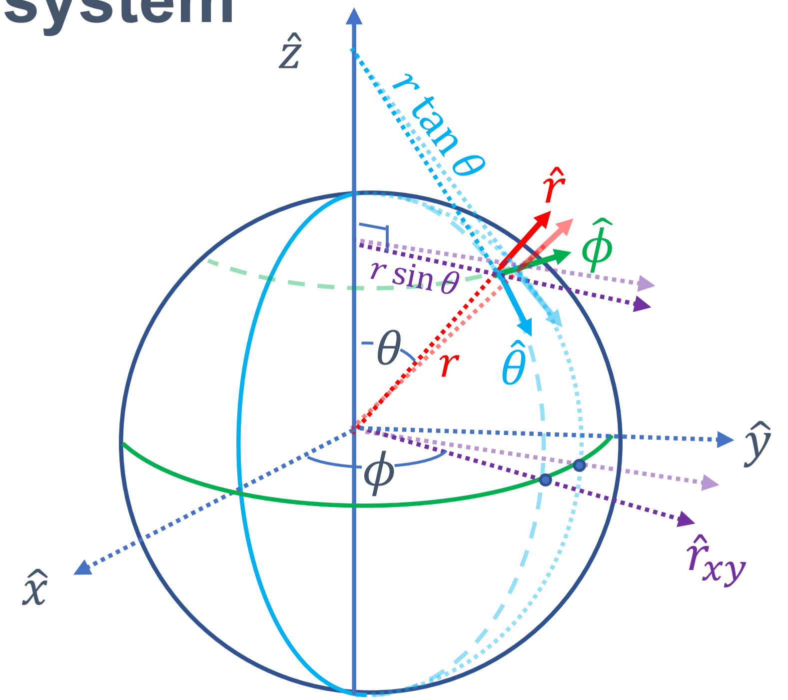


幾何方式 2：不用$\hat{r}_{xy}$，直接求 各單位向量對 ϕ 的微分

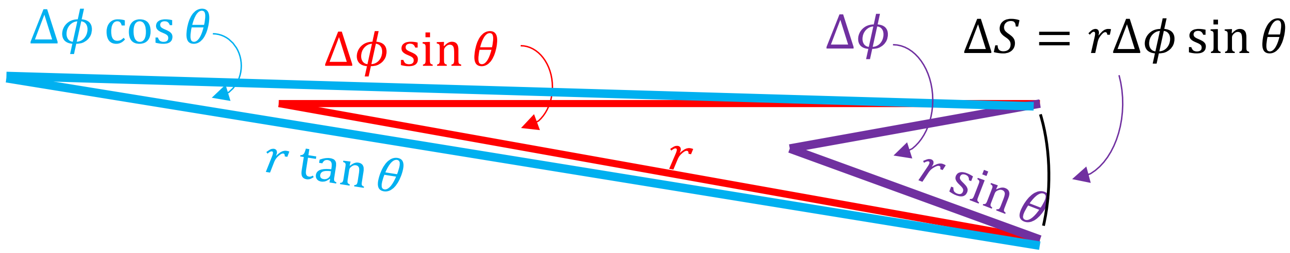
spherical coordinate system

$\theta < 45°$





$\theta > 45°$



$$\frac{\partial \hat{r}}{\partial \phi} = \hat{\phi} \sin\theta$$

$$\frac{\partial \hat{\theta}}{\partial \phi} = \hat{\phi} \cos\theta$$

Since $\hat{\phi} = \hat{r} \times \hat{\theta}$, it yields,

$$\frac{d\hat{\phi}}{d\phi} = \frac{\partial \hat{r}}{\partial \phi} \times \hat{\theta} + \hat{r} \times \frac{\partial \hat{\theta}}{\partial \phi} = \hat{\phi} \times \hat{\theta} \sin\theta + \hat{r} \times \hat{\phi} \cos\theta = -\hat{r} \sin\theta - \hat{\theta} \cos\theta$$

用“幾何”的方式求 單位向量的微分結果

cylindrical coordinate system	spherical coordinate system
$\frac{d\hat{r}}{d\theta} = \hat{\theta}$	$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$
$\frac{d\hat{\theta}}{d\theta} = -\hat{r}$	$\frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$
	$\frac{\partial\hat{r}}{\partial\phi} = \sin\theta\,\hat{\phi}$
	$\frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\,\hat{\phi}$
	$\frac{d\hat{\phi}}{d\phi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$

用“代數”的方式求 單位向量的微分結果

cylindrical coordinate system
$\hat{r} = \cos\theta\ \hat{x} + \sin\theta\ \hat{y}$ $\hat{\theta} = -\sin\theta\ \hat{x} + \cos\theta\ \hat{y}$
$\frac{d\hat{r}}{d\theta} = -\sin\theta\ \hat{x} + \cos\theta\ \hat{y} = \hat{\theta}$ $\frac{d\hat{\theta}}{d\theta} = -\cos\theta\ \hat{x} - \sin\theta\ \hat{y} = -\hat{r}$

spherical coordinate system
$\hat{r} = \cos\theta\ \hat{z} + \sin\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$ $\hat{\theta} = -\sin\theta\ \hat{z} + \cos\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$ $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$
$\frac{\partial\hat{r}}{\partial\theta} = -\sin\theta\ \hat{z} + \cos\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) = \hat{\theta}$ $\frac{\partial\hat{r}}{\partial\phi} = \sin\theta\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = \sin\theta\,\hat{\phi}$
$\frac{\partial\hat{\theta}}{\partial\theta} = -\cos\theta\ \hat{z} - \sin\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) = -\hat{r}$ $\frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = \cos\theta\,\hat{\phi}$
$\frac{d\hat{\phi}}{d\phi} = -\cos\phi\,\hat{x} - \sin\phi\,\hat{y}$
$-\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$ $= -\sin\theta\,[\cos\theta\ \hat{z} + \sin\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})]$ $- \cos\theta\,[-\sin\theta\ \hat{z} + \cos\theta\ (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})]$ $= -\cos\phi\,\hat{x} - \sin\phi\,\hat{y}$
$\frac{d\hat{\phi}}{d\phi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$

提醒：（請謹記）

- 回家的路，至少要兩條以上，才安全。
- 解問題，至少要用兩種以上不同的方式，都得到相同的解答，結果才「可能是」正確的答案。
- 若用一種方法求解，發生錯誤的機率是1/100 。
  則用兩種不同的方法，發生相同的錯誤，機率會降到1/10000 。餘此類推！
- 從現在起，請養成好習慣，要用兩種以上的方式解題、或驗證、或說明你的研究結果。
  - 用數學方式得到的結果，要了解它背後的「物理意義」。
  - 從直觀物理上推論出來的結果，要設法用數學歸納證明之。
  - To understand is to know how to calculate. --Dirac

Questions? or Comments? 有問題或建議嗎？