太空科學研究所

Linear Algebra
Lecture 7c (Chap. 6)
Introduction to Positive Definite Matrices, Similar Matrices, Jorden Form, &
Singular Value Decomposition (SVD)

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Definition: Positive Definite Matrices

- The eigenvalues of positive definite matrices are all positive real numbers.
- The Inertial tensor of rotation and the thermal pressure tensor are positive definite matrices.

Definition: Similar Matrices

- Matrices have the same eigenvalues are similar matrices.

Let Λ and S be the eigenvalue matrix and the eigenvector matrix of A.

$\Rightarrow A = S\Lambda S^{-1}$

Let M be any invertable matrix and let $B = M^{-1}AM$.

$\Rightarrow B$ and A are similar matrices. They have the same eigenvalues.

Proof :

$B = M^{-1}AM = M^{-1}(S\Lambda S^{-1})M = (M^{-1}S)\Lambda(M^{-1}S)^{-1}$

$\Rightarrow$ The Λ and $(M^{-1}S)$ are the eigenvalue matrix and the eigenvector matrix of B.

The eigenvalues and eigenvectors of the similar matrices

- 經過 orthogonal matrix 的座標轉換, 所產生的 inertial tensor 其 eigenvalues 不變, 因此 這些形式不同的 inertial tensor，彼此為 similar matrices.
- 因為經過座標轉換, 所以每個形式不同的 inertial tensor, 其 eigenvectors 也都不相同！（也須經過適當的座標轉換！）
- The eigenvectors change with Q, where $B=Q^TAQ=Q^{-1}AQ$

更多similar matrices的相關特性,
請參考課本page 344 最下方的Table

Jordan Form

- 如果一個 nxn matrix A 有repeated eigenvalues, 則 A 的 eigenvectors 的個數,“可能”少於 n. 這時 A 無法被對角線化,最多只能化簡為　　對角線的blocks形態,如右。
  這樣的 matrix 就叫做 Jordan form.

$$\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

- 如果 $B=M^{-1}AM$
  則 A, B 會有相同的 Jordan form,且都無法對角線化!

Singular Value Decomposition (緣起)

- 物理問題中，所遇到的矩陣，多為 nxn 的 symmetric matrix or Hermitian matrix.
- 隨著科技進步，觀測資料越來越多。分析觀測資料時，常常遇到“長方矩陣”的相關問題（請參考 Least Squares Fit 那份講義的範例說明）。
- Singular value decomposition 就是數學家針對“長方矩陣”，或是非對稱square matrix 所研發出來的一種新的decomposition method.

Singular Value Decomposition (SVD)

- 如果矩陣A 是一個“長方矩陣”，或是非對稱的 square matrix, 則不再考慮 AS=SL, 而是考慮 AV=US, where matrices U and V are orthogonal matrices, and matrix S is a “semidiagonal” matrix.

  Example of semidiagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Matrix A can be decomposed into $A=USV^{-1}$.
- Objectives of this study is to find a way to determine the matrices U and V and the “singular matrix” S.

Singular Value Decomposition (SVD)

- 在介紹 projection matrix 以及 least square fit 時，我們都遇到 “長方矩陣” 的問題。那時就介紹過一種將“長方矩陣” 的問題，化為正方矩陣的問題的方法：
- If A is an mxn matrix then A^TA is an nxn matrix and AA^T is an mxm matrix.
- If a matrix A can be decomposed into $A=USV^{-1}$, then it can be shown that the matrices S^TS and SS^T are the eigenvalue matrices of matrices A^TA and AA^T, respectively. Whereas, the matrices V_{nxn} and U_{mxm} are the corresponding eigenvector matrices of the matrices A^TA_{nxn} and AA^T_{mxm}, respectively.

Eigenvalue matrix and eigenvector matrix of $A^T A$

Let the singular value decomposition of matrix $A = U\Sigma V^{-1}$

where matrices U and V are invertable orthogonal matrices

$\Rightarrow U^{-1} = U^T$ and $V^{-1} = V^T$

$$A^T A = (U\Sigma V^{-1})^T U\Sigma V^{-1} = (V^{-1})^T \Sigma^T U^T U\Sigma V^{-1} = V(\Sigma^T \Sigma)V^{-1}$$

$\Rightarrow$

V is the eigenvector matrix of the symmetric matrix $A^T A$

$\Sigma^T \Sigma$ is the eigenvalue matrix of the symmetric matrix $A^T A$

Eigenvalue matrix and eigenvector matrix of AA^T

Let the singular value decomposition of matrix $A = U\Sigma V^{-1}$

where matrices U and V are invertable orthogonal matrices

$\Rightarrow U^{-1} = U^T$ and $V^{-1} = V^T$

$$AA^T = U\Sigma V^{-1}(U\Sigma V^{-1})^T = U\Sigma V^{-1}(V^{-1})^T\Sigma^T U^T = U(\Sigma\Sigma^T)U^{-1}$$

$\Rightarrow$

U is the eigenvector matrix of the symmetric matrix AA^T

$\Sigma\Sigma^T$ is the eigenvalue matrix of the symmetric matrix AA^T

SVD example 1

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \text{ and } AA^T = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$$

$$\det(A^T A - \lambda I) = (5-\lambda)^2 - 25 = 0 \Rightarrow \lambda = 0 \text{ or } \lambda = 10$$

$$\text{For } \lambda_1 = 0, \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \text{ For } \lambda_2 = 10, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \Rightarrow V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\det(AA^T - \lambda I) = (8-\lambda)(2-\lambda) - 16 = 0 \Rightarrow \lambda = 0 \text{ or } \lambda = 10$$

$$\text{For } \lambda_1 = 0, \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \text{ For } \lambda_2 = 10, \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \Rightarrow U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\text{Let } \Sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{10} \end{bmatrix} \Rightarrow U\Sigma V^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{10} \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = A$$

SVD example 2a

$$A = \begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow A^T A = \begin{bmatrix} 1 & \sqrt{5} & -2 \\ -1 & \sqrt{5} & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \text{ and } AA^T = \begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{5} & -2 \\ -1 & \sqrt{5} & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 10 & 0 \\ -4 & 0 & 8 \end{bmatrix}$$

$$\det(A^T A - \lambda I) = (10-\lambda)^2 = 0 \Rightarrow \lambda = 10 \Rightarrow V = ? \quad \text{Possible choices: } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \text{ etc.}$$

$$\det(AA^T - \lambda I) = (2-\lambda)(10-\lambda)(8-\lambda) - 16(10-\lambda) = 0 \Rightarrow -\lambda(\lambda^2 - 20\lambda + 100) = 0 \Rightarrow \lambda = 10, 10, 0$$

$$\text{For } \lambda_1 = \lambda_2 = 10, \mathbf{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \quad \text{For } \lambda_3 = 0, \mathbf{x}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}. \Rightarrow U = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 0 & 2 \\ 0 & \sqrt{5} & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\text{Let } \Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \Rightarrow U\Sigma V^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 0 & 2 \\ 0 & \sqrt{5} & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 2 \\ 0 & \sqrt{5} & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \sqrt{5} \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix} = A$$

SVD example 2b

$$A = \begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow A^T A = \begin{bmatrix} 1 & \sqrt{5} & -2 \\ -1 & \sqrt{5} & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \text{ and } AA^T = \begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{5} & -2 \\ -1 & \sqrt{5} & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 10 & 0 \\ -4 & 0 & 8 \end{bmatrix}$$

$$\det(A^T A - \lambda I) = (10-\lambda)^2 = 0 \Rightarrow \lambda = 10 \Rightarrow V = ? \quad \text{Possible choices: } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ etc.}$$

$$\det(AA^T - \lambda I) = (2-\lambda)(10-\lambda)(8-\lambda) - 16(10-\lambda) = 0 \Rightarrow -\lambda(\lambda^2 - 20\lambda + 100) = 0 \Rightarrow \lambda = 10, 10, 0$$

$$\text{For } \lambda_1 = \lambda_2 = 10, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \quad \text{For } \lambda_3 = 0, \mathbf{x}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}. \Rightarrow U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & \sqrt{5} & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\text{Let } \Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \Rightarrow U\Sigma V^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & \sqrt{5} & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & \sqrt{5} & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix} = A$$

SVD : Summary of examples

$$A=\begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix}=U\Sigma V^{-1}=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 0 & 2 \\ 0 & \sqrt{5} & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-1} \text{ or}$$

$$A=\begin{bmatrix} 1 & -1 \\ \sqrt{5} & \sqrt{5} \\ -2 & 2 \end{bmatrix}=U\Sigma V^{-1}=\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 0 & 2 \\ 0 & \sqrt{5} & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right)^{-1} \text{ or ...}$$

likewise

$$A=\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}=U\Sigma V^{-1}=\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{10} \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right)^{-1} \text{ or}$$

$$A=\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}=U\Sigma V^{-1}=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{10} \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-1} \text{ or ....}$$

$\Rightarrow$ Matrix A has nonunique singular value decomposition form, but has unique "singular values," which are the elements in the semi-diagonal matrix Σ.

Summary of SVD ($A=USV^{-1}$) p. 355

- Meaning of A**x**
  1. Rotate the vector **x** based on the row vectors of V (or change to the new basis which consists of the column vectors of V)
  2. Stretch corresponding components by S
  3. Rotate the resulting vector based on the column vectors of U (or change to the new basis, which consists of the row vectors of U)

Summary of SVD (AV=US) p. 356

- Matrix A maps the vectors in one orthogonal matrix V to the vectors in another orthogonal matrix U (with non-zero stretches).
- The function of A is very similar to the function of conformal mapping!
- Check for advanced example : (http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=1196456&fromcon)

Matrices for the Derivatives and Integral pp. 372-373

- Matrix form of derivatives

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

- Matrix form of integration

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

- One-sided inverse: $AA^{-1}=?$; $A^{-1}A=?$