太空科學研究所

Linear Algebra
Lecture 7b (Chap. 6)
Eigenvalue and Eigenvector
-- A General Discussion

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Objectives

- Examine the the eigenvalues and eigenvectors of the following matrices
    1. Matrices with repeated eigenvalues
        a) Projection matrix
        b) Upper triangle matrix with repeated diagonal terms
    2. Antisymmetric (skew-symmetric) real matrices
    3. Symmetric real matrices
    4. Hermitian matrices
    5. Markov matrices
- Application to differential equations

Matrices with repeated eigenvalues (projection matrix)

- Example 1: Projection matrix
  - Let P be a projection matrix that project a vector to a space that was spanned by the three vectors $\mathbf{a}_1, \quad \mathbf{a}_2, \quad \mathbf{a}_3$

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

  - What are the eigenvalues and the corresponding eigenvectors of P?

Matrices with repeated eigenvalues (projection matrix)

- 由 projection matrix 的特性可得:
- P should have the following eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = \lambda_4 = 1$$

- 由 $\mathbf{a}_1, \quad \mathbf{a}_2, \quad \mathbf{a}_3$ 的特性可得:

The corresponding orthonormal eigenvectors are

$$\mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Matrices with repeated eigenvalues (projection matrix)

- The corresponding eigenvectors of $\lambda_2 = \lambda_3 = \lambda_4 = 1$ are not unique. Here are other choices

$$\mathbf{e}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

Matrices with repeated eigenvalues (nondiagonalizable matrix)

- Example 2: Upper triangle matrix with repeated diagonal terms $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

- Matrix A has repeated eigenvalues 3, 3
- Matrix A has only one eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Antisymmetric Real Matrices

- If a matrix A satisfies $A_{ij} = -A_{ji}$ and $A_{ij} \in R$ then A is an antisymmetric real matrix

Case 1

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = +i, \text{ or } -i$$

Case 2

$$A = \begin{bmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{bmatrix} \quad \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & -B_z & B_y \\ B_z & -\lambda & -B_x \\ -B_y & B_x & -\lambda \end{bmatrix} = -\lambda^3 - \lambda(B_x^2 + B_y^2 + B_z^2) = 0$$

$$\Rightarrow \lambda = 0, \text{ or } \pm iB, \text{ where } B = \sqrt{B_x^2 + B_y^2 + B_z^2}. \text{ The corresponding eigen vector of } \lambda = 0 \text{ is } \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Antisymmetric Real Matrices

- The eigenvalues of antisymmetric real matrices are real numbers or complex conjugate numbers.
- In Case 1, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda = +i, \text{ or } -i \Rightarrow \|\lambda\| = 1$
- The length of the eigenvalue of the orthogonal matrix $A^T A = I$ is always equal to 1.

Proof: $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow \overline{\mathbf{x}}^T A^T = \overline{\lambda}\overline{\mathbf{x}}^T \Rightarrow \overline{\mathbf{x}}^T A^T A\mathbf{x} = \overline{\lambda}\overline{\mathbf{x}}^T \lambda\mathbf{x}$

$A^T A = I \Rightarrow \overline{\mathbf{x}}^T \mathbf{x} = \overline{\lambda}\lambda\overline{\mathbf{x}}^T \mathbf{x}$ Since $\overline{\mathbf{x}}^T \mathbf{x} = \|\mathbf{x}\|^2 > 0$, it yields $\overline{\lambda}\lambda = \|\lambda\|^2 = 1$

同理可證: The length of the eigenvalue of the Hermitian matrix $\overline{A}^T A = I$ is always equal to 1.

Real Matrices

- For all real matrices, the eigenvectors of the complex conjugate eigenvalues are complex conjugate to each other.

Proof: $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$

Example 1: Rotation matrices

The eigenvalues and eigenvectors of the rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are

$$\lambda_1 = \cos\theta + i\sin\theta,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix},\ \text{and}\ \lambda_2 = \bar{\lambda}_1 = \cos\theta - i\sin\theta,\quad \mathbf{x}_2 = \bar{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Antisymmetric Real Matrices

- Example 2: Antisymmetric Real Matrices

$$A = \begin{bmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{bmatrix} \Rightarrow \lambda_1 = 0,\ \mathbf{x}_1 = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}, \text{ where } B = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

$$\lambda_2 = iB: \text{ If } (B_x, B_y) \neq (0,0) \text{ then } \mathbf{x}_2 = \begin{bmatrix} -B_x B_z - iB_y B \\ -B_y B_z + iB_x B \\ B^2 - B_z^2 \end{bmatrix}. \text{ If } B_x = B_y = 0, \text{ then } \mathbf{x}_2 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

$$\text{and } \lambda_3 = \bar{\lambda}_2 = -iB,\ \mathbf{x}_3 = \bar{\mathbf{x}}_2$$

$A=S\Lambda S^{-1}$

- 如果矩陣A有n個eigenvalue and n 個 independent eigenvectors 則

$$A\begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & & \uparrow \\ \lambda_1\mathbf{x}_1 & \dots & \lambda_n\mathbf{x}_n \\ \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

$$\text{Let } S = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ \downarrow & & \downarrow \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \Rightarrow AS = S\Lambda \quad \Rightarrow A = S\Lambda S^{-1}$$

Symmetric Real Matrices

- If a matrix A satisfies $A_{ij} \in R$ and $A_{ij} = A_{ji}$ or $A = A^T$ then A is a symmetric real matrix
  - The eigenvalues of a symmetric real matrix are all real numbers.
  - The eigenvectors of a symmetric real matrix that corresponding to different eigenvlaues are perpendicular to each other.
  - One can always find a set of orthonormal eigenvectors to diagonalized the symmetric matrix.

Hermitian Matrices

- If a matrix A satisfies $A_{ij} \in C$ and $A_{ij} = \overline{A}_{ji}$ or $A = \overline{A}^T$ then A is a Hermitian matrix
  - The eigenvalues of a Hermitian matrix are all real numbers.
  - The eigenvectors of a Hermitian matrix that corresponding to different eigenvlaues are perpendicular to each other.
  - One can always find a set of orthonormal eigenvectors to diagonalized the Hermitian matrix.

Proof:
The eigenvalues of a Hermitian matrix are all real numbers

(1) $A = \overline{A}^T$

(2) $A\mathbf{x} = \lambda \mathbf{x}$

Taking complex conjugate and transpose of equation (2), it yields

(3) $\overline{\mathbf{x}}^T \overline{A}^T = \overline{\lambda} \overline{\mathbf{x}}^T$

Substituting equation (1) into equation (3), it yields

(4) $\overline{\mathbf{x}}^T A = \overline{\lambda} \overline{\mathbf{x}}^T$

$\overline{\mathbf{x}}^T (2) \Rightarrow$

(5) $\overline{\mathbf{x}}^T A\mathbf{x} = \lambda \overline{\mathbf{x}}^T \mathbf{x}$

Substituting equation (4) into equation (5), it yields

(6) $\overline{\lambda} \overline{\mathbf{x}}^T \mathbf{x} = \lambda \overline{\mathbf{x}}^T \mathbf{x}$

Since $\overline{\mathbf{x}}^T \mathbf{x} = \|\mathbf{x}\|^2 > 0$, equation (6) yields $\overline{\lambda} = \lambda$. That is $\lambda \in R$.

Proof:
The eigenvectors of a Hermitian matrix that corresponding to different eigenvlaues are perpendicular to each other

(1) $A = \overline{A}^T$

(2) $\lambda_1 \neq \lambda_2$, and both of them are real numbers

(3) $A\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$

(4) $A\mathbf{x}_2 = \lambda_2 \mathbf{x}_2$

Taking complex conjugate and transpose of equation (4), it yields

(5) $\overline{\mathbf{x}}_2^T \overline{A}^T = \lambda_2 \overline{\mathbf{x}}_2^T$

Substituting equation (1) into equation (5), it yields

(6) $\overline{\mathbf{x}}_2^T A = \lambda_2 \overline{\mathbf{x}}_2^T$

$\overline{\mathbf{x}}_2^T (3) \Rightarrow$

(7) $\overline{\mathbf{x}}_2^T A\mathbf{x}_1 = \lambda_1 \overline{\mathbf{x}}_2^T \mathbf{x}_1$

Substituting equation (6) into equation (7), it yields

(8) $\lambda_2 \overline{\mathbf{x}}_2^T \mathbf{x}_1 = \lambda_1 \overline{\mathbf{x}}_2^T \mathbf{x}_1 \quad \Rightarrow (\lambda_2 - \lambda_1)\overline{\mathbf{x}}_2^T \mathbf{x}_1 = 0$

Since $\lambda_2 - \lambda_1 \neq 0$, equation (8) yields $\overline{\mathbf{x}}_2^T \mathbf{x}_1 = 0$. That is $\mathbf{x}_2 \perp \mathbf{x}_1$

Example of Hermitian Matrix

$$A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \Rightarrow A = \bar{A}^T \Rightarrow A \text{ is a Hermitian matrix}$$

Find the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = 0 \Rightarrow \det\begin{bmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = 0 \Rightarrow \lambda = 0 \text{ or } \lambda{=}2$$

$$\text{Let } \lambda_1 = 0,\ \ (A - \lambda_1 I)\mathbf{x}_1 = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\mathbf{x}_1 = 0 \Rightarrow \mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ -1 \end{bmatrix}$$

$$\text{Let } \lambda_2 = 0,\ \ (A - \lambda_2 I)\mathbf{x}_2 = \begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix}\mathbf{x}_2 = 0 \Rightarrow \mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Please read the table: "Real versus Complex" on p. 491.

$$\Rightarrow S = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \Rightarrow S^{-1} = \bar{S}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} \quad (\Rightarrow S \text{ is an unitary matrix.})$$

$$\text{If } A = S\Lambda S^{-1} \Rightarrow \Lambda = S^{-1}AS = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\text{Since } \lambda_1 \neq \lambda_2, \text{ check if } \mathbf{x}_1 \perp \mathbf{x}_2: \quad \bar{\mathbf{x}}_1^T \mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix} = 0$$

Markov Matrix

- Markov Matrix:
  - All elements are greater than or equal to zero and less than or equal to 1.
  - The sum of each column is equal to 1.
  - 1 is one of the eigenvalues of the Markov Matrix
- Examples:

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \quad B = \begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.8 & 0.4 & 0.1 \\ 0.1 & 0. & 0.6 \end{bmatrix}$$

Markov Matrix

- Example:
- Mary and John play an unfair game. Mary will give John 20% of her money. John will give Marry 30% of his money. At beginning, John received 5000 dollars. Show that the final equilibrium state Mary will have 3000 dollars and John will have 2000 dollars.

$$\text{Let } A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} Mary's\ money \\ John's\ money \end{bmatrix} \Rightarrow \mathbf{u}_{n+1} = A\mathbf{u}_n$$

$$\text{For } \mathbf{u}_0 = \begin{bmatrix} 0 \\ 5000 \end{bmatrix} \Rightarrow \mathbf{u}_{n+1} = A^n\mathbf{u}_0 = S\Lambda^n S^{-1}, \quad \text{where } A \text{ has two eigen values: 1 and 0.5}$$

$$\text{Find } A^n = ?, \quad \mathbf{u}_{n+1} = ?, \quad \text{and } A^\infty = ?, \quad \mathbf{u}_\infty = ?$$

Eigen Mode Solutions of Differential Equations (Case 1)

Let $u_1 = c_1 e^{\gamma t}, \quad u_2 = c_2 e^{\gamma t}$

$$\frac{du_1}{dt} = au_1 + bu_2 \Rightarrow \gamma u_1 = au_1 + bu_2$$

$$\frac{du_2}{dt} = cu_1 + du_2 \Rightarrow \gamma u_2 = cu_1 + du_2$$

$$\Rightarrow \quad \gamma \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow A\mathbf{u} = \gamma\mathbf{u} \Rightarrow \det(A - \gamma I) = 0$

$\Rightarrow$ Solution $\gamma = \gamma_r + i\gamma_i$ is the eigenvalue of matrix A.

If $\gamma_r < 0$, then the magnitude of the corresponding eigenvector $\mathbf{u}$ decreases with time (damping).

If $\gamma_r > 0$, then the magnitude of the corresponding eigenvector $\mathbf{u}$ increases with time (unstable).

It $\gamma_r = 0$, then the magnitude of the corresponding eigenvector $\mathbf{u}$ is a stable solution.

Eigen Mode Solutions of Differential Equations (Case 2)

Solve $\dfrac{d^2y}{dt^2}+b\dfrac{dy}{dt}+ky=0$

Method 1:

Let $y=c_1e^{\gamma t} \Rightarrow \boxed{\gamma^2+b\gamma+k=0}$

Method 2:

Let $u_1=y$ and $u_2=\dfrac{dy}{dt} \Rightarrow \dfrac{d}{dt}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}0 & 1\\-k & -b\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$

Let $u_1=c_1e^{\gamma t},\quad u_2=c_2e^{\gamma t} \Rightarrow \gamma\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}0 & 1\\-k & -b\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$

Let $A=\begin{bmatrix}0 & 1\\-k & -b\end{bmatrix}, \mathbf{u}=\begin{bmatrix}u_1\\u_2\end{bmatrix} \Rightarrow A\mathbf{u}=\gamma\mathbf{u} \Rightarrow \det(A-\gamma I)=0 \Rightarrow \det\begin{bmatrix}-\gamma & 1\\-k & -b-\gamma\end{bmatrix}=\boxed{\gamma^2+b\gamma+k=0}$

$\Rightarrow$ Solution $\gamma=\gamma_r+i\gamma_i$ is the eigenvalue of matrix A.

Eigen Mode Solutions of Differential Equations

(Case 3)

For $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, $\mathbf{k} = k(\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta)$, where θ is the angle between $\mathbf{B}_0$ and $\mathbf{k}$.

The dispersion relation of the MHD wave mode is

$$\begin{bmatrix} \frac{\omega^2}{k^2} - (C_{A0}^2 + C_{S0}^2 \sin^2\theta) & 0 & -C_{S0}^2\cos\theta\sin\theta \\ 0 & \frac{\omega^2}{k^2} - C_{A0}^2\cos^2\theta & 0 \\ -C_{S0}^2\cos\theta\sin\theta & 0 & \frac{\omega^2}{k^2} - C_{S0}^2\cos^2\theta \end{bmatrix} \begin{bmatrix} \tilde{V}_{1x} \\ \tilde{V}_{1y} \\ \tilde{V}_{1z} \end{bmatrix} = 0 \quad \text{where } C_{A0}^2 = \frac{B_0^2}{\mu\rho_0},\ C_{S0}^2 = \frac{\gamma p_0}{\rho_0}$$

$$\text{Let } A = \begin{bmatrix} C_{A0}^2 + C_{S0}^2\sin^2\theta & 0 & C_{S0}^2\cos\theta\sin\theta \\ 0 & C_{A0}^2\cos^2\theta & 0 \\ C_{S0}^2\cos\theta\sin\theta & 0 & C_{S0}^2\cos^2\theta \end{bmatrix}, \mathbf{u} = \begin{bmatrix} \tilde{V}_{1x} \\ \tilde{V}_{1y} \\ \tilde{V}_{1z} \end{bmatrix} \Rightarrow A\mathbf{u} = \frac{\omega^2}{k^2}\mathbf{u} \Rightarrow \det(A - \frac{\omega^2}{k^2} I) = 0$$

$\Rightarrow$ The eigenvalues $\frac{\omega^2}{k^2}$ are the fast - mode, Alfven - mode, slow - mode wave phase speeds.

The corresponding eigenvectors (which perpendicular to each other) are the eigen modes of the *MHD* plasma.

How to obtain the dispersion relation of the MHD wave modes

$$\frac{\omega^2}{k^2}\tilde{\mathbf{V}}_1 = C_{A0}^2(\hat{\mathbf{B}}_0\times)(\hat{\mathbf{k}}\times)(\hat{\mathbf{k}}\times)(\hat{\mathbf{B}}_0\times)\cdot\tilde{\mathbf{V}}_1 + C_{S0}^2(\hat{\mathbf{k}}\hat{\mathbf{k}})\cdot\tilde{\mathbf{V}}_1$$

where $(\hat{\mathbf{B}}_0\times)(\hat{\mathbf{k}}\times)(\hat{\mathbf{k}}\times)(\hat{\mathbf{B}}_0\times) =$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\cos\theta & 0 \\ +\cos\theta & 0 & -\sin\theta \\ 0 & +\sin\theta & 0 \end{bmatrix}\begin{bmatrix} 0 & -\cos\theta & 0 \\ +\cos\theta & 0 & -\sin\theta \\ 0 & +\sin\theta & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\cos\theta & 0 & \sin\theta \\ 0 & -\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\cos\theta & 0 & 0 \\ 0 & -\cos\theta & 0 \\ \sin\theta & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\mathbf{k}}\hat{\mathbf{k}} = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}\begin{bmatrix} \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \sin^2\theta & 0 & \cos\theta\sin\theta \\ 0 & 0 & 0 \\ \cos\theta\sin\theta & 0 & \cos^2\theta \end{bmatrix}$$

$$\Rightarrow \frac{\omega^2}{k^2}\tilde{\mathbf{V}}_1 = \begin{bmatrix} C_{A0}^2 + C_{S0}^2\sin^2\theta & 0 & C_{S0}^2\cos\theta\sin\theta \\ 0 & C_{A0}^2\cos^2\theta & 0 \\ C_{S0}^2\cos\theta\sin\theta & 0 & C_{S0}^2\cos^2\theta \end{bmatrix}\tilde{\mathbf{V}}_1$$

True or False?

If λ is an eigenvalue of matrix A, and β is an eigenvalue of matrix B, then $\lambda\beta$ is an eigenvalue of matrix AB. True or False?

If it is True, Prove it. If it is False, explain why.

If λ is an eigenvalue of matrix A, and β is an eigenvalue of matrix B, then $\lambda + \beta$ is an eigenvalue of matrix $A + B$. True or False?

If it is True, Prove it. If it is False, explain why.

Conditions for common eigenvectors and orthogonal matrix S

Matrix A and matrix B have the same eigenvectors
if and only if $AB = BA$

Let A be a symmetric matrix ($A = A^T$), then $AA^T = A^T A$.
Let A be an asymmetric matrix ($A = -A^T$), then $AA^T = A^T A$.

A real matrix has perpendicular eigenvectors if and only if $AA^T = A^T A$