太空科學研究所

Linear Algebra
Lecture 7a (Chap. 6)
Eigenvalue and Eigenvector
-- A Case Study

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Motivation

- 古典物理問題中有很多對稱矩陣 symmetric matrix ($A=A^T$)。例如：轉動慣量與壓力張量。
  - Symmetric matrices (with real elements) are diagonalizable.
- 近代物理問題中有很多 Hermitian matrix

  ($A=(A^T)^*$, where $(A^T)^*$ is the complex conjugate of A^T)
  - Hermitian matrices are diagonalizable.
- 本講將以 對稱矩陣 當作範例，介紹如何求 eigenvalues and eigenvectors.

Review: How to make a symmetric matrix

- If A is a square matrix, then $(A+A^T)/2$ is a symmetric matrix.
- For any matrix (square or not) R, the RR^T is a symmetric matrix.
- A diagonal matrix D is a symmetric matrix
- For any matrix (square or not) R, the RDR^T is a symmetric matrix.

將矩陣A分為A^{sy} +A^{asy} 就像把向量分成兩的分量
將對稱矩陣A^{sy}進一步分成三部份的連乘積(好似分解因式)
可"方便出題"或求A^n之值(資訊或電子迴路的設計)

Making a symmetric matrix: A=UDU^T

$$\text{Let } A = UDU^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -5 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 8 & 0 & -2 \\ -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 8 \end{bmatrix} = A$$

Note: U的每一個column vector互相垂直，但長度不為1。

Find the eigenvalues and eigenvectors of a matrix A

- Find the eigenvalues and eigenvectors of A

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 8 & 0 & -2 \\ -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 8 \end{bmatrix} \qquad \text{Find } \lambda \text{ and } \mathbf{x} \text{ such that } A\mathbf{x} = \lambda\mathbf{x}$$

- where λ is the eigenvalue of A and **x** is the corresponding eigenvector.

The eigenvalue l of A satisfies det(A-lI) = 0

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 8 & 0 & -2 \\ -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 8 \end{bmatrix}$$

$$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow (A - \lambda\mathbf{1})\mathbf{x} = 0$$

$$\text{For nonzero } \mathbf{x} \Rightarrow \det(A - \lambda\mathbf{1}) = 0 \ \Rightarrow$$

$$\det\begin{bmatrix} 1-\lambda & 0 & -3 & 0 \\ 0 & 8-\lambda & 0 & -2 \\ -3 & 0 & 1-\lambda & 0 \\ 0 & -2 & 0 & 8-\lambda \end{bmatrix} = 0$$

$\mathbf{x}$ 有非零解，表示矩陣 $A-\lambda\mathbf{1}$ 中的四個column vectors只能張(span)出一個dimension等於或小於3的空間，這樣才能在此四度空間中，找到另一個子空間，使該子空間中的向量 $\mathbf{x}$ 能同時垂直於 $A-\lambda\mathbf{1}$ 中的四個column vectors。也因此，we can conclude that the 4 column vectors在四度空間中所張出來的四度空間體積（ $=\left|\det(A-\lambda\mathbf{1})\right|$）必為零。

Find the eigenvalues

$$\det(A-\lambda\mathbf{1})=0 \Rightarrow \det\begin{bmatrix} 1-\lambda & 0 & -3 & 0 \\ 0 & 8-\lambda & 0 & -2 \\ -3 & 0 & 1-\lambda & 0 \\ 0 & -2 & 0 & 8-\lambda \end{bmatrix}$$

$$=(1-\lambda)\det\begin{bmatrix} 8-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ -2 & 0 & 8-\lambda \end{bmatrix} - 3\det\begin{bmatrix} 0 & -3 & 0 \\ 8-\lambda & 0 & -2 \\ -2 & 0 & 8-\lambda \end{bmatrix}$$

$$=[(1-\lambda)^2-9][(8-\lambda)^2-4]$$

$$=[(1-\lambda)+3][(1-\lambda)-3][(8-\lambda)+2][(8-\lambda)-2]=0$$

$$\Rightarrow \lambda_1=4, \quad \lambda_2=-2, \quad \lambda_3=10, \quad \lambda_4=6$$

Find the corresponding eigenvectors of $\lambda_1=4$

$$(A-\lambda\mathbf{1})\mathbf{x}=0 \Rightarrow \begin{bmatrix} 1-\lambda & 0 & -3 & 0 \\ 0 & 8-\lambda & 0 & -2 \\ -3 & 0 & 1-\lambda & 0 \\ 0 & -2 & 0 & 8-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=0$$

For $\lambda_1 = 4$

$$\begin{bmatrix} 1-4 & 0 & -3 & 0 \\ 0 & 8-4 & 0 & -2 \\ -3 & 0 & 1-4 & 0 \\ 0 & -2 & 0 & 8-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} -3 & 0 & -3 & 0 \\ 0 & 4 & 0 & -2 \\ -3 & 0 & -3 & 0 \\ 0 & -2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}=0$$

$$\Rightarrow \mathbf{e}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

Find the corresponding eigenvectors $\lambda_2 = -2$

$$(A-\lambda\mathbf{1})\mathbf{x}=0 \Rightarrow \begin{bmatrix} 1-\lambda & 0 & -3 & 0 \\ 0 & 8-\lambda & 0 & -2 \\ -3 & 0 & 1-\lambda & 0 \\ 0 & -2 & 0 & 8-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

For $\lambda_2 = -2$

$$\begin{bmatrix} 1+2 & 0 & -3 & 0 \\ 0 & 8+2 & 0 & -2 \\ -3 & 0 & 1+2 & 0 \\ 0 & -2 & 0 & 8+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 & 0 \\ 0 & 10 & 0 & -2 \\ -3 & 0 & 3 & 0 \\ 0 & -2 & 0 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow \mathbf{e}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

Find the corresponding eigenvectors $\lambda_3 = 10$

$$(A - \lambda \mathbf{1})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 0 & -3 & 0 \\ 0 & 8-\lambda & 0 & -2 \\ -3 & 0 & 1-\lambda & 0 \\ 0 & -2 & 0 & 8-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

For $\lambda_3 = 10$

$$\begin{bmatrix} 1-10 & 0 & -3 & 0 \\ 0 & 8-10 & 0 & -2 \\ -3 & 0 & 1-10 & 0 \\ 0 & -2 & 0 & 8-10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9 & 0 & -3 & 0 \\ 0 & -2 & 0 & -2 \\ -3 & 0 & -9 & 0 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 0$$

$$\Rightarrow \mathbf{e}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

Find the corresponding eigenvectors $\lambda_4=6$

$$(A-\lambda\mathbf{1})\mathbf{x}=0 \Rightarrow \begin{bmatrix} 1-\lambda & 0 & -3 & 0 \\ 0 & 8-\lambda & 0 & -2 \\ -3 & 0 & 1-\lambda & 0 \\ 0 & -2 & 0 & 8-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=0$$

For $\lambda_4=6$

$$\begin{bmatrix} 1-6 & 0 & -3 & 0 \\ 0 & 8-6 & 0 & -2 \\ -3 & 0 & 1-6 & 0 \\ 0 & -2 & 0 & 8-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} -5 & 0 & -3 & 0 \\ 0 & 2 & 0 & -2 \\ -3 & 0 & -5 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}=0$$

$$\Rightarrow \mathbf{e}_4=\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Change of Basis

- Verify your results by changing the basis to the eigenvectors coordinate system. In that coordinate system, the matrix A should be a diagonal matrix with diagonal elements equal to the corresponding eigenvalues.

Review Change of Basis

When we change the basis from $B = \{\mathbf{e}_1, \ \mathbf{e}_2, \ \mathbf{e}_3\}$ to $B^* = \{\mathbf{e}_1^*, \ \mathbf{e}_2^*, \ \mathbf{e}_3^*\}$ the representation of **P** also changes

$$\begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix}^{-1}$$

$$\Rightarrow \quad \text{if } \mathbf{e}_i^{*T}\mathbf{e}_j^* = \delta_{ij} \quad \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix} = \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Diagonalized the Symmetric Matrix

$$[A]_{B^*} = \begin{bmatrix} \leftarrow \mathbf{e}_1^T \rightarrow \\ \leftarrow \mathbf{e}_2^T \rightarrow \\ \leftarrow \mathbf{e}_3^T \rightarrow \\ \leftarrow \mathbf{e}_4^T \rightarrow \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 8 & 0 & -2 \\ -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 8 \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} \qquad (\because \mathbf{e}_i^T \mathbf{e}_j = \delta_{ij})$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 8 & 0 & -2 \\ -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \quad ? \quad \end{bmatrix}$$

Your home work

Applications (in Physics)

- What are the principle axes of an inertial tensor?

  Your home work

- By definition:

  Toque =(inertial tensor) dot (angular acceleration)

  $$\tau = \mathbf{I} \cdot \alpha$$

- Under what condition the toque will be parallel to the angular acceleration?

  Your home work

Applications (in Math)

- Given

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 8 & 0 & -2 \\ -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 8 \end{bmatrix}$$

- Find $A^{123} = ?$

Your home work

Application to ODE & PDE (大科三、四空組同學)

- Determine the dispersion relation of the MHD waves with $\mathbf{B}_0 = B_0\mathbf{z}$ and wave number $\mathbf{k}$ in the x-z plane
- What is the eigenvector of each MHD wave mode?
- What is the corresponding eigenvalue?

大科三、四空組同學, Your home works!

Selected Problems

- Section 6.1
  - Problems 1~12, 22, 30 (pp. 283-287)
- Section 6.2
  - Problems 1~8, 21~23, 27 (pp.298-301)
- Section 6.3
  - Problems 1~9 (pp. 315-316)
- Section 6.4
  - Problems 1~9 (pp. 326-327)

課本上有些題目太難了，這裡只挑了些基本、簡單的習題！試試你的功力吧！難題等你長大了再回來做！